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AN INFINITE COMPANION MATRIX

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Abstract: An explicit formula is obtained for the entries of the powers of the companion matrix of a polynomial P in terms of the roots of P.

Key words: Eigenvalue, companion matrix, recurrence relation, critical exponent.

AMS: Primary 15
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1. Introduction. In the course of his investigations of the connection between the norms of powers of operators and the spectral radius the present author introduced, [2], for each polynomial P, an infinite matrix T^{∞} whose columns are the solutions of the recurrence relation with characteristic polynomial P and initial conditions

$$\begin{matrix} 1, & 0, & 0, & \dots & 0 \\ 0, & 1, & 0, & \dots & 0 \\ 0, & 0, & 1, & \dots & 0 \\ - & - & - & & - \end{matrix}$$

The problem considered in [2] was the following: to find, among all contractions A on n-dimensional Hilbert space whose spectral radius does not exceed a given number $p<1$, the operator for which $|A^n|$ assumes its maximum.

The main result of [2] was that this maximum is assumed for the restriction of the (backward) shift operator S to the subspace Ker $(S - p)^n$ of ℓ^2, the space of all square summable sequences of complex numbers. For the proof it was necessary to express the solution of the recurrence relation in terms of the roots of P and it was essential that the polynomials in the roots of P which appear in T have coefficients whose sign depends only on the column index (with the exception of the first n rows). The present author proved this for the first column and formulated the general case as a conjecture. At the author's request the late Professor V. Knichal supplied a proof which, unfortunately, was never published nor recorded. Since recent investigations require even more precise information the author proposed this as a problem in the functional analysis seminar. Three independent solutions were given almost simultaneously by N.J. Young, Z. Dostál and the author.

2. <u>The matrix</u> T^{∞}. We introduce the following notation:

$$E_i\,(x_1,\dots,x_n) = x_1^{e_1}\,x_2^{e_2}\,\dots\,x_n^{e_n}$$

the sum being taken over all sequences e_j with $0 \leq e_j \leq 1$ and $\sum e_j = i$

$$h_i\,(x_1,\dots,x_n) = x_1^{e_1}\,x_2^{e_2}\,\dots\,x_n^{e_n}$$

the sum being taken over all sequences of exponents e_j with $e_j \geq 0$ and $\sum e_j = i$. Now let $\alpha_1,\dots,\alpha_n$ be given complex numbers.

We write

$$P(z) = (z - \alpha_1)\dots(z - \alpha_n) = a_n z^n + a_{n-1}z^{n-1} + \dots + a_o$$

$$Q(z) = z^n P(\tfrac{1}{z}) = (1 - \alpha_1 z)\dots(1 - \alpha_n z) = a_o z^n + a_1 z^{n-1} + \dots + a_n$$

where $a_n = 1$, $a_i = (-1)^{n-i} E_{n-i} (\alpha_1, \dots, \alpha_n)$ and

$$P_r(z) = (z - \alpha_1)\dots(z - \alpha_r) \qquad 1 \leq r \leq n$$

$$T = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_o & -\alpha_1 & -\alpha_2 & \dots & -\alpha_n \end{pmatrix}$$

The matrix T^{∞} corresponding to the polynomial P was defined [2] as follows (we change the numbering of the indices slightly). The matrix T^{∞} has n columns numbered 0, 1,..., n - 1 and an infinite number of rows 0,1,2,... . The j-th column is defined to be the solution of the recurrence relation

$$x_{r+n} + a_{n-1} x_{r+n-1} + \dots + a_o x_r = 0$$

with the initial condition

$$x_o = 0,\ x_1 = 0, \dots x_j = 1, \dots x_{n-1} = 0$$

We have seen that T^{∞} possesses the following simple property: given any m = 0,1,..., the matrix consisting of the n consecutive rows of T starting with the m-th row is exactly T^m. Another useful property of T is the following:

for any $r \geq n$ the power T^r may be expressed in terms of $T^o, T^1, \dots, T^{n-1}$ as follows

$$T^r = t_{ro} + t_{r1} T + t_{r2} T^2 + \dots + t_{r,n-1} T^{n-1}.$$

If we assume that all $|\alpha_i| < 1$ we can identify the columns of T with certain H^2 functions as follows

$$f_j(z) = \sum_{r=0}^{\infty} t_{rj} z^r$$

To obtain explicit expressions for the coefficients t_{rj} in terms of the α's we observe [3] that the definition of f_j may be expressed as follows. The requirement that the sequence t_{rk}, $r = 0,1,\dots$ be a solution of the recurrence relation is equivalent to the requirement that

$$(*) \qquad P(S)\, f_k = 0$$

where S is the backward shift operator on H^2

$$(S\, p)(z) = \frac{1}{z}\,(p(z) - p(0))$$

The initial condition for f_k may be replaced by the requirement that $f_k(z) = z^k + z^n g_k$ for some $g_k \in H^2$. We have seen [3] that $(*)$ implies $f_k(z) = \frac{w_k(z)}{Q(z)}$ for a suitable polynomial w_k of degree $\leq n - 1$. Rewriting the condition for f_k in the form

$$\frac{w_k(z)}{z^n} = \frac{Q(z)}{z^n}\, z^k + Q(z)\, g_k(z) =$$

$$= z^k\, P(\tfrac{1}{z}) + Q(z)\, g_k(z)$$

we see that - to meet this requirement - we must set

$$w_k(z) = a_n z^k + a_{n-1} z^{k+1} + \dots + a_{k+1} z^{n-1}$$

so that

$$f_k(z) = \frac{z^k}{Q(z)} \sum_{j=0}^{n-k-1} a_{n-j} z^j$$

Define $F(y,z) = \sum_{t=0}^{n-1} f_t(z) y^t$. It follows that

$$Q(z) F(y,z) = \sum_{t=0}^{n-1} w_t(z) y^t = \sum_{t=0}^{n-1} (yz)^t \sum_{j=0}^{n-t-1} a_{n-j} z^j =$$

$$= \sum_{j=0}^{n-1} a_{n-j} z^j \sum_{t=0}^{n-j-1} (yz)^t = \sum_{j=0}^{n-1} a_{n-j} z^j \frac{1 - (yz)^{n-j}}{1 - yz}$$

whence

$$(1 - yz) Q(z) F(y,z) = \sum_{j=0}^{n-1} a_{n-j} (z^j - z^n y^{n-j}) =$$

$$= Q(z) - z^n P(y).$$

Now use the formula

$$p_1 p_2 \dots p_n - q_1 q_2 \dots q_n = (p_1 - q_1) p_2 \dots p_n +$$

$$+ q_1(p_2 - q_2)p_3 \dots p_n + q_1 q_2 (p_3 - q_3)p_4 \dots p_n + q_1 q_2 \dots$$

$\dots q_{n-1}(p_n - q_n)$ for $p_j = 1 - \alpha_j z$ and $q_j = z(y - \alpha_j)$ so that $p_j - q_j = 1 - yz$. Hence $Q(z)F(y,z) = p_2 \dots p_n + q_1 p_3 \dots p_m +$ $q_1 q_2 p_4 \dots p_n + q_1 \dots q_{n-2} p_n + q_1 \dots q_{n-1}$ and

$$F(y,z) = \frac{1}{p_1} + \frac{q_1}{p_1 p_2} + \frac{q_1 q_2}{p_1 p_2} + \frac{q_1 q_2}{p_1 p_2 p_3} + \dots + \frac{q_1 \dots q_{n-1}}{p_1 \dots p_n} =$$

$$= \frac{1}{1 - \alpha_1 z} + \frac{z P_1(y)}{(1 - \alpha_1 z)(1 - \alpha_2 z)} +$$

$$+ \frac{z^2 P_2(y)}{(1 - \alpha_1 z)(1 - \alpha_2 z)(1 - \alpha_3 z)} + \dots$$

$$+ \frac{z^{n-1} P_{n-1}(y)}{(1 - \alpha_1 z)(1 - \alpha_2 z) \dots (1 - \alpha_n z)}$$

where $P_r(y) = (y - \alpha_1) \dots (y - \alpha_r) =$

$$= \sum_{k=0}^{r} (-1)^{r-k} E_{r-k}(\alpha_1 \dots \alpha_r)\, y^k$$

Since f_k is the coefficient of y^k in this sum we obtain

$$f_k = \sum_{n-1 \geq r \geq k} \frac{z^r}{p_1 \dots p_{r+1}} (-1)^{r-k} E_{r-k}(\alpha_1 \dots \alpha_r) =$$

$$= \frac{z^k}{p_1 \dots p_{k+1}} - \frac{z^{k+1}}{p_1 \dots p_{k+2}} E_1(\alpha_1 \dots \alpha_{k+1}) +$$

$$+ \frac{z^{k+2}}{p_1 \dots p_{k+3}} E_2(\alpha_1 \dots \alpha_{k+2}) + \dots + (-1)^{n-1-k} \frac{z^{n-1}}{p_1 \dots p_n}$$

$$E_{n-1-k}(\alpha_1 \dots \alpha_{n-1})$$

To unify the formulas it will be convenient to define the binomial coefficient $\binom{a}{b}$ to be zero if $a < b$.

The rest of the paper is purely combinatorial. We shall need the following lemma.

(2,1) For each pair of integers $0 \leq j \leq q - 1$ the following relation holds.

$$\binom{q-1}{j} - \binom{q}{j} + \binom{q}{j-1} - \binom{q}{j-2} + \dots + (-1)^{k+j+1} \binom{q}{k} + \dots +$$

$(- 1)^{j+1} \binom{q}{0} = 0.$

Proof. Denote the expression on the left hand side of the above equation by $x(q,j)$. We shall use the well-known fact that the binomial coefficients satisfy the following relation

$$\binom{a}{a} + \binom{a}{b+1} = \binom{a+1}{b+1} \quad \text{for } 0 \leq b \leq a - 1$$

Now consider the first two terms in the expression for $x(q,j)$. Since

$$\binom{q-1}{j} - \binom{q}{j} = -\binom{q-1}{j-1}$$

we easily obtain the relation $x(q,j) = -x(q,j - 1)$. Since $x(q,0) = 0$ the lemma is proved.

Now let $j (0 \leq j \leq n - 1)$ be fixed. Set

$$g_t = (-1)^{t-j} \frac{E_{t-j}(\alpha_1 \ldots \alpha_t)}{p_1 \cdots p_{t+1}} \text{ for } t \geq j$$

so that $f_j = z^j g_j + z^{j+1} g_{j+1} + \ldots + z^{n-1} g_{n-1}$

and

$t_{rj} = g_{j,r-j} + g_{j+1,r-j-1} + \cdots g_{n-1,r-n+1}$,

g_{ts} being the coefficient of z^s in the expression of g_t.

We have

$g_{t,r-t} = (-1)^{t-j} E_{t-j}(\alpha_1, \ldots, \alpha_t) h_{r-t}(\alpha_1, \ldots, \alpha_{t+1}) =$

$$= (-1)^{t-j} \sum \eta(e_1,\dots,e_{t+1}) \alpha_1^{e_1} \dots \alpha_{t+1}^{e_{t+1}}$$

the sum being taken over all sequences of exponents $e_1, \dots, e_{t+1}$ whose sum equals $r - j$; all coefficients η are nonnegative integers. Summing the contributions from the different g_t we see that

$$t_{rj} = \sum c_{jr}(m)m$$

the sum being extended over all monomials of the form $\alpha_1^{e_1} \dots \alpha_n^{e_n}$ with $\sum e = r - j$. Our main result is

(2,2) For each $0 \leqq j \leqq n - 1$ and $r \geqq n$

$$c_{jr}(e_1, \dots, e_n) = (-1)^{n-j-1} \binom{q-1}{n-j-1}$$

where q is the number of positive elements in the sequence $a_i, \dots, e_n$.

Proof. Let us first observe that the coefficients c do not change if we replace the sequence $e_1 \dots e_n$ by any permutation of the e's. Hence given n, $r \geqq n$, $0 \leqq j \leqq n$ and $1 \leqq q \leqq n$ we may limit ourselves to the evaluation of $c_{jr}(e_1, \dots, e_q, 0, \dots 0)$ where the first q entries are all positive.

Consider a fixed t, $j \leqq t \leqq n - 1$ and determine the coefficient with which the term

$$\alpha_1^{e_1} \dots \alpha_q^{e_q} z^{r-t}$$

appears in the expansion of g_t. Since all $e_1, \dots, e_q$ are to be positive, we must have $t + 1 \geqq q$ (the denominator of g_t being $p_1 \ldots p_{t+1}$) and $t - j \leqq q$ (the numerator of g_t being $E_{t-j}(\alpha_1 \ldots \alpha_t)$) otherwise we either have too few α's or too many. Hence contributions can only be expected from the g_t with

$$\max(j, q-1) \leqq t \leqq \min(q+j, n-1)$$

Consider a fixed $t \geqq j$. The contribution to $\alpha_1^{e_1} \ldots \alpha_q^{e_q}$ from g_t is clearly

$$C(t) = \begin{cases} (-1)^{t-j} \begin{pmatrix} q \\ t-j \end{pmatrix} & \text{if } t \geqq q \\ (-1)^{q-1-j} \begin{pmatrix} q-1 \\ t-j \end{pmatrix} = (-1)^{q-1-j} \begin{pmatrix} q-1 \\ j \end{pmatrix} & \text{if } t = q-1 \end{cases}$$

and, of course, zero if $t < q - 1$. We have thus

$$c_{jr} = \sum_{\max(j,q-1)}^{\min(q+j,n-1)} C(t)$$

To compute c_{jr} we shall distinguish two cases.

(1) Consider first the case $j \geqq q - 1$. We find first that $C(j) = 1$ so that

$$\sum_{t=j}^{\min(q+j,n-1)} C(t) = \sum_{t=j}^{\min(q+j,n-1)} (-1)^{t-j} \begin{pmatrix} q \\ t-j \end{pmatrix}$$

$$= \sum_{s=0}^{\min(q,n-1-j)} (-1)^s \begin{pmatrix} q \\ s \end{pmatrix}$$

The last sum is zero if $q \leqq n - j - 1$ or, using lemma (2,1),

$$(-1)^{n-j-1}\binom{q-1}{n-j-1}.$$

if $n - j - 1 < q$

(2) if $j < q - 1$, we have

$$\sum_{t=q-1}^{\min(q+j,n-1)} C(t) = (-1)^{q-1-j}\binom{q-1}{j} +$$

$$+ \sum_{t=q}^{\min(q+j,n-1)} (-1)^{t-j}\binom{q}{t-j}$$

$$= (-1)^{q-1-j}\binom{q-1}{j} + \sum_{t=q}^{\min(q+j,n-1)} (-1)^{t-j}\binom{q}{q+j-t}$$

$$= (-1)^{q-1-j}\binom{q-1}{j} + \sum_{s=0}^{\min(j,n-q-1)} (-1)^{s+q-j}\binom{q}{j-s}$$

The last sum is zero if $j \leq n - q - 1$ by (2,1). If $n - q -$ $- 1 < j$ the last sum - again by (2,1) - equals

$$-\sum_{s=n-q}^{j} (-1)^{q+s-j}\binom{q}{j-s} = -(-1)^{n-j}\binom{q-1}{j-(n-q)} =$$

$$= (-1)^{n-j-1}\binom{q-1}{n-j-1}$$

The proof is complete.

3. We conclude by stating another formula which yields the qualitative statement about the signs of the elements of T^{∞} immediately. The function

$$G(z,y) = \sum_{k=0}^{n-1} (f_k(z) - z^k)\, y^k = F(z,y) - \frac{1 - y^n z^n}{1 - yz} =$$

$$= \frac{z^n}{(1 - yz)\, Q(z)} (y^n Q(z) - P(y))$$

may be transformed (using again the formula for the difference of two products) to the following form

$$G(z,y) = z^n \left(\frac{\alpha_1 y^{n-1}}{p_1} + \frac{\alpha_2 y^{n-2} P_1(y)}{p_1 p_2} + \right.$$

$$\left. + \frac{\alpha_3 y^{n-3} P_2(y)}{p_1 p_2 p_3} + \dots + \frac{\alpha_n P_{n-1}(y)}{p_1 \cdots p_n} \right)$$

whence

$$(-1)^{n-1} G(z,-y) = z^n \left(\frac{\alpha_1 y^{n-1}}{p_1} + \frac{\alpha_2 y^{n-2} (y + \alpha_1)}{p_1 p_2} + \right.$$

$$+ \frac{\alpha_3 y^{n-3}(y + \alpha_1)(y + \alpha_2)}{p_1 p_2 p_3} + \dots$$

$$\left. \dots + \frac{\alpha_n (y + \alpha_1)\dots(y + \alpha_{n-1})}{p_1 \cdots p_n} \right)$$

and all coefficients of the expansion of the right hand side are clearly nonnegative.

R e f e r e n c e s

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