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EXPANSIVE COLLECTIONS OF CONTINUA

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Abstract: Let X be a continuum. A collection $\mathcal{F}$, of proper subcontinua of X is said to be expansive provided that if $F \in \mathcal{F}$ and G is a proper subcontinuum of X such that $F \subset G$, then $G \in \mathcal{F}$. In this paper such collections of subcontinua are studied. In particular, if X is the union of the members of $\mathcal{F}$ then conditions are given which imply that X can be written as the union of two members of $\mathcal{F}$.

Key words and phrases: Continuum, expansive collections, indecomposable, irreducible, non-separating subcontinua.

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In this paper certain collections of proper subcontinua of a continuum X are studied. In particular, those collections which "expan" with respect to set inclusion are investigated and properties of such collections are developed. If X is the union of the subcontinua from such a collection, then conditions are given which imply that X is the union of exactly two subcontinua from the collection.

Throughout this paper the continuum X is a compact connected metric space. The continuum X is said to be decomposable if it is the union of two proper subcontinua; otherwise, the continuum is indecomposable. If K is a proper subcontinuum of X, then K is non-separating in X means that

$X - K$ is connected. The continuum X is irreducible if there are two points p and q in X such that no proper subcontinuum of X contains both p and q. If A is a subset of X, then the closure of A in X will be denoted by $\overline{A}$. For terms and notation used but not defined herein, the reader is referred to [3].

Definition: A collection, $\mathcal{F}$, of proper subcontinua of X is said to be expansive provided if $F \in \mathcal{F}$ and G is a proper subcontinuum of X such that $F \subset G$, then $G \in \mathcal{F}$.

Let $S \subset X$, $S \neq \phi$, and $\mathcal{F}$ be the collection of all proper subcontinua of X that contain S. Then $\mathcal{F}$ is an expansive collection.

A proper subcontinuum K of X is said to be a terminal continuum provided if A and B are proper subcontinua of X such that $X = A \cup B$ and $A \cap K \neq \phi \neq B \cap K$ then $X = A \cup K$ or $X = B \cup K$ [1]. The terminal subcontinua of X form an expansive collection of non-separating subcontinua of X.

If $\mathcal{F}$ is an expansive collection of subcontinua of X, then we shall let $\mathcal{F}^* = \cup\{F \mid F \in \mathcal{F}\}$.

It is easily seen that $\mathcal{F}^*$ is dense in X and is non-separating in X. Moreover, if $X - \mathcal{F}^*$ is a non-empty subcontinuum then $X - \mathcal{F}^*$ does not separate any subcontinuum of X. But when is $X - \mathcal{F}^*$ a continuum? The following theorem provides an answer.

Theorem 1. Suppose $\mathcal{F}$ is an expansive collection of proper subcontinua of X. Then $X - \mathcal{F}^*$ is a continuum if and only if the only subcontinua that intersect both $\mathcal{F}^*$ and $X - \mathcal{F}^*$ are decomposable.

Proof: Assume that $X - \mathcal{F}^*$ is not a continuum. Then $X - \mathcal{F}^* \subsetneq \overline{X - \mathcal{F}^*}$. Thus $\overline{X - \mathcal{F}^*}$ is a continuum which intersects both $\mathcal{F}^*$ and $X - \mathcal{F}^*$, hence is decomposable. Let A and B be proper subcontinua such that $\overline{X - \mathcal{F}^*} = A \cup B$. It follows that $X - \mathcal{F}^* \not\subset A$ and $X - \mathcal{F}^* \not\subset B$. Now either A or B intersects $\mathcal{F}^*$ so without loss of generality assume that $A \cap \cap \mathcal{F}^* \neq \phi$. Let $F \in \mathcal{F}$ such that $A \cap F \neq \phi$. Then $A \cup F$ is a continuum.

If $A \cup F = X$, then $X - \mathcal{F}^* \subset X - F$. Since $X - F \subset A$ this would imply that $X - \mathcal{F}^* \subset A$ which is not the case. Thus $A \cup F$ must be a proper subcontinuum of X. Since $F \subset A \cup F$ then $A \cup F \in \in \mathcal{F}$ and it follows that $A \subset \mathcal{F}^*$. This implies that $X - - \mathcal{F}^* \subset B$ which is a contradiction. Therefore $X - \mathcal{F}^*$ is, in fact, a continuum.

Now suppose that $X - \mathcal{F}^*$ is a continuum but that K is an indecomposable subcontinuum which intersects $\mathcal{F}^*$ and $X - - \mathcal{F}^*$. Let $F_\alpha \in \mathcal{F}$ such that $K \cap F_\alpha \neq \phi$, then $K \cup F_\alpha$ is a subcontinuum of X which contains F_α. Since $K \not\subset \mathcal{F}^*$, it follows that $K \cup F_\alpha \notin \mathcal{F}$ which implies that $X = K \cup F_\alpha$. Note that $X - \mathcal{F}^* \subset X - F_\alpha \subset K$. Let C be the composant of K which contains $X - \mathcal{F}^*$. Since $X - \mathcal{F}^* \subsetneq C$, there is a $F_\beta \in \in \mathcal{F}$ such that $C \cap F_\beta \neq \phi$. Let I be a subcontinuum contained in C such that $I \cap (X - \mathcal{F}^*) \neq \phi \neq I \cap F_\beta$. Now $I \cup F_\beta$ is a subcontinuum which contains F_β but $I \cup F_\beta \notin \mathcal{F}$. Since $\mathcal{F}$ is an expansive collection of proper subcontinua of X, it follows that $X = I \cup F_\beta$. Thus $K - I \subset F_\beta$ which implies that $K \subset F_\beta$. Since $X - \mathcal{F}^* \subset K$, then $X - \mathcal{F}^* \subset F_\beta$ which is a contradiction. Therefore the only subcontinua that intersect both $\mathcal{F}^*$ and $X - \mathcal{F}^*$ are decomposable.

Obviously, $X - \mathcal{F}^*$ is a continuum if the continuum X is hereditarily decomposable. It also holds with a somewhat weaker condition on X.

Corollary 1: Suppose X is a continuum such that each indecomposable subcontinuum has void interior and $\mathcal{F}$ is an expansive collection of proper subcontinua of X. Then $X - \mathcal{F}^*$ is a continuum.

Proof: Let K be a continuum which intersects both $\mathcal{F}^*$ and $X - \mathcal{F}^*$. Let $F \in \mathcal{F}$ such that $K \cap F \neq \phi$. Then $K \cup F$ is a continuum which contains F but $K \cup F \notin \mathcal{F}$. Thus $X = K \cup F$ and $X - F$ is an open set contained in K. By hypothesis K is decomposable and the corollary follows from the previous theorem.

Continua which satisfy the hypothesis of the following theorem are equivalent to the "type A" continua of Thomas [2].

Theorem 2. Let X be an irreducible continuum such that each indecomposable subcontinuum has void interior. If $X = \mathcal{F}^*$, then there exists $F_\alpha \in \mathcal{F}$ and $F_\beta \in \mathcal{F}$ such that $X = F_\alpha \cup F_\beta$.

Proof: Suppose $\{p,q\} \subset X$ such that X is irreducible between p and q. Let $\mathcal{F}_p = \{F \in \mathcal{F} \mid p \in F\}$ and $\mathcal{F}_q = \{F \in \mathcal{F} \mid q \in F\}$. Then $\mathcal{F}_p$ and $\mathcal{F}_q$ are non-empty expansive collections of subcontinua and, according to Corollary 1, $X - \mathcal{F}^*_p$ and $X - \mathcal{F}^*_q$ are continua. Note that $p \in X - \mathcal{F}^*_q$ and $q \in X - \mathcal{F}^*_p$.

Case 1. If $\mathcal{F}^*_p \cap \mathcal{F}^*_q \neq \phi$, let $F_\alpha \in \mathcal{F}_p$ and $F_\beta \in \mathcal{F}_q$ such that $F_\alpha \cap F_\beta \neq \phi$. Since $F_\alpha \cup F_\beta$ is a subcontinuum containing $\{p,q\}$, it follows that $X = F_\alpha \cup F_\beta$.

Case 2. If $\mathcal{F}^*_p \cap \mathcal{F}^*_q = \phi$. Then $X = (X - \mathcal{F}^*_p) \cup \cup (X - \mathcal{F}^*_q)$. Let $F \in \mathcal{F}$ such that $p \in F$. Then $F \cap \mathcal{F}^*_q = \phi$ which implies that $F \subset X - \mathcal{F}^*_q$. Since $X - \mathcal{F}^*_q$ is a proper subcontinuum of X it follows that $X - \mathcal{F}^*_q \in \mathcal{F}$.

Likewise $X - \mathcal{F}^*_p \in \mathcal{F}$. So $X = F_\alpha \cup F_\beta$ where $F_\alpha = = X - \mathcal{F}^*_q$ and $F_\beta = X - \mathcal{F}^*_p$.

In the remaining portion of this paper we shall assume that the expansive collection $\mathcal{F}$ of subcontinua of X has the property that if $F_\alpha \in \mathcal{F}$ and $F_\beta \in \mathcal{F}$ then $F_\alpha \cap F_\beta \neq \phi$.

<u>Lemma 1:</u> There is a countable subcollection $\mathcal{F}_\sigma$, of $\mathcal{F}$ such that $\mathcal{F}^* = \mathcal{F}^*_\sigma$.

Proof: Case 1: If there exist $F_1 \in \mathcal{F}$ and $F_2 \in \mathcal{F}$ such that $X = F_1 \cup F_2$, let $\mathcal{F}_\sigma = \{F_1, F_2\}$. Then $\mathcal{F}^* = F_1 \cup F_2 = \mathcal{F}^*_\sigma$.

Case 2. Suppose that X is not the union of two members of $\mathcal{F}$. Choose a $F_\alpha \in \mathcal{F}$ and let $\{V_i\}_{i=1}^\infty$ be a countable basis for $X - F_\alpha$. For each positive integer i, let L_i be the component of $X - V_i$ which contains F_α. Since $\mathcal{F}$ is an expansive collection then each $L_i \in \mathcal{F}$. Let $\mathcal{F}_\sigma = \{L_i\}_{i=1}^\infty$. Then $\mathcal{F}_\sigma \subset \mathcal{F}$ which implies that $\mathcal{F}^*_\sigma \subset \mathcal{F}^*$.

Suppose $x \in \mathcal{F}^*$. Let $F_\beta \in \mathcal{F}$ such that $x \in F_\beta$. Now by hypothesis $F_\alpha \cup F_\beta$ is a subcontinuum of X. Since $F_\alpha \cup F_\beta \neq \neq X$, then $W = X - (F_\alpha \cup F_\beta)$ is an open subset of $X - F_\alpha$. There is a positive integer i such that $V_i \subset W$ which implies that $F_\alpha \cup F_\beta \subset X - V_i$. Thus $F_\alpha \cup F_\beta \subset L_i$ so $x \in F_\beta \subset L_i \subset \subset \mathcal{F}^*_\sigma$. Therefore $\mathcal{F}^* \subset \mathcal{F}^*_\sigma$ and it follows that $\mathcal{F}^*_\sigma = \mathcal{F}^*$.

<u>Theorem 3:</u> [1] Let $\mathcal{F}$ be an expansive collection of non-separating subcontinua of X. If $\mathcal{F}^* = X$, then there are

$F_\alpha \in \mathcal{F}$ and $F_\beta \in \mathcal{F}$ such that $X = F_\alpha \cup F_\beta$.

Proof: Suppose to the contrary that $\mathcal{F}^* = X$ but X is not the union of any two numbers of $\mathcal{F}$. Then it follows that X is not the union of any finite subcollection of $\mathcal{F}$. Let $\mathcal{F}_6 = \{F_i\}_{i=1}^{\infty}$ be a countable subcollection of $\mathcal{F}$ such that $\mathcal{F}_6^* = \mathcal{F}^*$. For each positive integer n let $N_n = \bigcup_{i=1}^{n} F_i$. Then $\{N_i\}_{i=1}^{\infty}$ is an increasing sequence of proper subcontinua of X with $\bigcup_{i=1}^{\infty} N_i = X$.

<u>Assertion:</u> For each integer i, there is a $j > i$ such that $\overline{X - N_j} \subsetneqq \overline{X - N_i}$. For if not, then there is an i such that for all $j > i$ $\overline{X - N_i} = \overline{X - N_j}$. Thus $\{X - N_j \mid j > i\}$ is a countable collection of open sets, each dense in $\overline{X - N_i}$. According to the Baire Category Theorem, there is a point $p \in \bigcap_{j>i}(X - N_j) = X - \bigcup_{j>i} N_j$. But this would imply that $p \in X - \bigcup_{i=1}^{\infty} N_i$ which is a contradiction. Therefore the assertion holds.

Thus we may obtain a subsequence $\{N_i'\}_{i=1}^{\infty}$ of the sequence N such that for each i, $N_i' \subset N_{i+1}'$ while $\overline{X - N_{i+1}'} \subsetneqq \overline{X - N_i'}$. Clearly $X = \bigcup_{i=1}^{\infty} N_i'$. Since $\{\overline{X - N_i'}\}_{i=1}^{\infty}$ is a decreasing sequence of compact sets, there is an $x \in \bigcap_{i=1}^{\infty} (\overline{X - N_i'})$. Let j be a positive integer such that $x \in N_j'$. Then $x \in \overline{X - N_{j+1}'}$ and $[N_j' \cup \overline{X - N_{j+1}'}]$ is a proper subcontinuum of X. Since $[N_j' \cup \overline{X - N_{j+1}'}]$ does not separate X, it must be a member of $\mathcal{F}$. Thus N_{j+1}' and $[N_j' \cup \overline{X - N_{j+1}'}]$ are two members of $\mathcal{F}$ whose union is X. This contradiction establishes the theorem.

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[1] Theorem 3 was suggested by J.B. Fugate.

R e f e r e n c e s

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