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PURELY FINITELY GENERATED ABELIAN GROUPS

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Abstract: In this note a new structural description of purely finitely generated abelian groups is presented. This criterion enables us to show that the class of purely finitely generated groups is closed under pure subgroups and that this class is contained in the class of all factor-splitting torsionfree abelian groups. As an application a theorem concerning the splitting of pure subgroups is generalized.

Key words: Purely finitely generated group, factor-splitting group, p-rank, generalized regular subgroup, splitting group.

Classification: 20K15

By the word "group" we shall always mean an additively written abelian group. The symbol π will denote the set of all primes. If H is a torsionfree group, M a subset of H and $\pi' \subseteq \pi$ then $\langle M \rangle_{\pi'}^{G}$ denotes the π'-pure closure of M in H. $Z_{\pi'}$ will denote the group of rationals with denominators prime to every $p \in \pi'$. Any maximal linearly independent set of elements of a torsionfree group H is called a basis. Other notations and terminology is essentially that as in [8].

1. Purely finitely generated groups. Recall [3] that a torsionfree group H is said to be purely finitely generated if it contains elements $h_1, h_2, \dots, h_m$ such that $H = \sum_{i=1}^{m} \langle h_i \rangle_{\pi}^{H}$.

Lemma 1: Let H be a completely decomposable torsionfree group of finite rank n. Then there is a decomposition $\pi = \bigcup_{i=1}^{k} \pi_i$ of the set π such that for each i=1,2,...,k the group $H \otimes Z_{\pi_i}$ is completely decomposable with ordered type set.

Proof: Let $H = \sum_{i=1}^{n}{}^{\oplus} J_i$ be a complete decomposition of H, $h_i \in J_i$, i=1,2,...,n. For any permutation $\varphi \in S_n$ define π_{φ} to be the set of all primes p with $h_p^G(h_{\varphi(1)}) \geq$ $\geq h_p^G(h_{\varphi(2)}) \geq \dots \geq h_p^G(h_{\varphi(n)})$. The group $H \otimes Z_{\pi_{\varphi}}$ is obviously completely decomposable with ordered type set for each $\varphi \in S_n$.

Theorem 2: If H is a torsionfree group of finite rank n then the following conditions are equivalent:

(i) H is purely finitely generated.

(ii) There exists a decomposition $\pi = \bigcup_{i=1}^{k} \pi_i$ of the set π such that the group $H \otimes Z_{\pi_i}$ is completely decomposable for each i=1,2,...,k.

(iii) There exists a decomposition $\pi = \bigcup_{i=1}^{k} \pi_i$ of the set π such that for each i=1,2,...,k the group $H \otimes Z_{\pi_i}$ is completely decomposable with ordered type set.

(iv) There exists a pair-wise disjoint decomposition $\pi = \bigcup_{i=1}^{k} \pi_i$ of the set π such that for each i=1,2,...,k the group $H \otimes Z_{\pi_i}$ is completely decomposable with ordered type set.

Proof: (i) implies (ii). Let H be a purely finitely ordered group of rank n. By [3; Lemma 5] to every prime p there exists a linearly independent subset $\{h_1^{(p)}, h_2^{(p)}, \ldots$ $\ldots, h_{\ell_p}^{(p)}\}$ of the set $\{h_1, h_2, \ldots, h_m\}$ such that $\langle h_1, h_2, \ldots$ $\ldots, h_m\rangle_p^H = \sum_{i=1}^{\ell_p} \langle h_i\rangle_p^H + \sum_{i=1}^{m} \langle h_i\rangle$. For each linearly independent subset $S = \{h_{i_1}, h_{i_2}, \ldots, h_{i_k}\}$ of $\{h_1, h_2, \ldots, h_m\}$ we denote by π_S the set of all primes p for which $\langle h_1, h_2, \ldots, h_m\rangle_p^H = \sum_{j=1}^{k} \langle h_{i_j}\rangle_p^H + \sum_{i=1}^{m} \langle h_i\rangle$. By the preceding part we clearly have $\pi = \bigcup_S \pi_S$ and this union is obviously finite. To finish the proof of the implication in question it suffices to show that $H \otimes Z_{\pi_S}$ is a completely decomposable group. Assuming a suitable enumeration of the elements $h_1, h_2, \ldots, h_m$ we can suppose that $S = \{h_1, h_2, \ldots, h_k\}$ and that $\{h_1, h_2, \ldots, h_n\}$ is a basis of H. Moreover, taking suitable multiples of h_i's, $i=1,2,\ldots,m$, we can suppose that $h_{n+1}, \ldots$ $\ldots, h_m \in \langle h_1, h_2, \ldots, h_n\rangle$. Then for each prime $p \in \pi_S$ we have $\langle h_1, h_2, \ldots, h_m\rangle_p^H = \sum_{i=1}^{k} \langle h_i\rangle_p^H + \sum_{i=1}^{m} \langle h_i\rangle = \sum_{i=1}^{k}{}^{\oplus} \langle h_i\rangle_p^H \oplus$ $\oplus \sum_{i=k+1}^{m}{}^{\oplus} \langle h_i\rangle$.

Now it is easily seen that $\langle h_1, h_2, \ldots, h_m\rangle_{\pi_S}^H =$ $= \sum_{i=1}^{k}{}^{\oplus} \langle h_i\rangle_{\pi_S}^H \oplus \sum_{i=k+1}^{m}{}^{\oplus} \langle h_i\rangle$ and consequently the group $H \otimes Z_{\pi_S} = \sum_{i=1}^{k}{}^{\oplus} (\langle h_i\rangle_{\pi_S}^H \otimes Z_{\pi_S}) \oplus \sum_{i=k+1}^{m}{}^{\oplus} (\langle h_i\rangle \otimes Z_{\pi_S})$ is completely decomposable.

(ii) implies (iii). Suppose that $\pi = \bigcup_{i=1}^{k} \pi_i$ and that the group $H \otimes Z_{\pi_i}$ is completely decomposable for each $i=1,2,\ldots,k$. By Lemma 1 for each $i=1,2,\ldots,k$ the set π decomposes into $\pi = \bigcup_{j=1}^{k_i} \pi_j^{(i)}$ in such a way that the group $H \otimes Z_{\pi_i} \otimes Z_{\pi_j^{(i)}}$ is completely decomposable with ordered type set for each $j=1,2,\dots,k_i$. Now the assertion follows easily from the simple fact that $Z_{\pi'} \otimes Z_{\pi''} \cong Z_{\pi' \cap \pi''}$ for all subsets π', π'' of π.

(iii) implies (iv). For each $i=1,2,\dots,k$ put $\bar{\pi}_i = \pi_i \setminus (\bigcup_{j=1}^{i-1} \pi_j)$. The group $H \otimes Z_{\bar{\pi}_i}$, $i=1,2,\dots,k$ is obviously completely decomposable with ordered type set and the union $\pi = \bigcup_{i=1}^{k} \bar{\pi}_i$ is clearly pair-wise disjoint.

(iv) implies (i). Suppose that the set π decomposes into a pair-wise disjoint union $\pi = \bigcup_{i=1}^{k} \pi_i$ in such a way that for each $i=1,2,\dots,k$ the group $H \otimes Z_{\pi_i}$ is completely decomposable with ordered type set. Then for each $i=1,2,\dots$ $\dots,k$ there are elements $h_1^{(i)}, h_2^{(i)}, \dots, h_n^{(i)}$ in H such that $H_i = H \otimes Z_{\pi_i} = \sum_{j=1}^{m}{}^{\oplus} \langle h_j^{(i)} \otimes 1 \rangle_{\pi}^{H_i}$. We are going to show that $H = \sum_{i=1}^{k} \sum_{j=1}^{n} \langle h_j^{(i)} \rangle_{\pi}^{H}$. Let $h \in H$ be an arbitrary element. For each $i=1,2,\dots,k$ the set $\{h_1^{(i)}, h_2^{(i)}, \dots, h_n^{(i)}\}$ is obviously a basis of H so that there is a positive integer α such that $\alpha h \in \langle h_1^{(i)}, h_2^{(i)}, \dots, h_n^{(i)} \rangle$, $\alpha h = \sum_{j=1}^{m} \lambda_j^{(i)} h_j^{(i)}$, for each $i=1,2,\dots,k$. Write α in the form $\alpha = r_i s_i$, $i=1,2,\dots,k$, where r_i is divisible by the primes from π_i only and $(s_i, p) = 1$ for each $p \in \pi_i$. Now $r_i h \otimes 1 = \alpha h \otimes 1/s_i = \sum_{j=1}^{m} \lambda_j^{(i)} h_j^{(i)} \otimes 1/s_i = \sum_{j=1}^{m} \lambda_j^{(i)} (h_j^{(i)} \otimes 1/s_i)$. From [9; § 60 Ex. 9(a)] it easily follows that $r_i \mid \lambda_j^{(i)} h_j^{(i)}$ for each $j=1,2,\dots,n$ and consequently $s_i h \in \sum_{j=1}^{m} \langle h_j^{(i)} \rangle_{\pi_i}^{H}$. However, $(s_1, s_2, \dots, s_k) = 1$ in view of the disjointness of

$\bigcup_{i=1}^{k} \pi_i$. Thus $\sum_{i=1}^{k} s_i \alpha_i = 1$ for suitable integers α_1, $\alpha_2, \ldots, \alpha_k$ and consequently $h = \sum_{i=1}^{k} \alpha_i s_i h \in$ $\in \sum_{i=1}^{k} \sum_{j=1}^{m} \langle h_j^{(i)} \rangle_{\pi_i}^H \subseteq \sum_{i=1}^{k} \sum_{j=1}^{m} \langle h_j^{(i)} \rangle_{\pi}^H$.

Theorem 3: Every pure subgroup of a purely finitely generated torsionfree group is purely finitely generated.

Proof: Let S be a pure subgroup of a purely finitely generated group H. By Theorem 2(iii) there exists a decomposition $\pi = \bigcup_{i=1}^{k} \pi_i$ of the set π such that for each i=1,2,... ...,k the group $H \otimes Z_{\pi_i}$ is completely decomposable with ordered type set. By [9; Theorem 60.4] the group $S \otimes Z_{\pi_i}$ is isomorphic to a pure subgroup of $H \otimes Z_{\pi_i}$ for each i=1,2,... ...,k. Consequently, $S \otimes Z_{\pi_i}$ is a completely decomposable group with ordered type set by [1, Theorem 1] and it suffices to use Theorem 2.

Recall that a torsionfree group H is said to be factor-splitting if each homomorphic image of H splits.

Theorem 4: Every purely finitely generated torsionfree group is factor-splitting.

Proof: Let H be a purely finitely generated group. By Theorem 2(ii) there exists a decomposition $\pi = \bigcup_{i=1}^{k} \pi_i$ of the set π such that the group $H \otimes Z_{\pi_i}$ is completely decomposable for each i=1,2,...,k. Now for each i=1,2,...,k the group $H \otimes Z_{\pi_i}$ is factor-splitting by [7; Theorem 6] and it suffices to use [7; Lemma 5].

2. Splitting of pure subgroups. If H is a torsionfree group then the set of all elements h of H having infinite

p-height is a subgroup of H which will be denoted by $H[p^{\infty}]$. It is well-known (see [11]) that if H is a torsionfree group and F its free subgroup of the same rank then the number $r_p(H)$ of summands $C(p^{\infty})$ in H/F does not depend on the particular choice of F and this number is called the p-rank of H. A subgroup K of a torsionfree group H is called generalized regular if for every $g \in K$ the characteristics of g in K and in H differ only in finitely many places.

Now we shall formulate Conditions (α), (γ) (see [2]). A mixed group G with the torsion part T satisfies Condition (α) if to any $g \in G \setminus T$ there exists an integer m such that mg has in G the same type as g+T in G/T. We say that a mixed group G with the torsion part T satisfies Condition (γ) if it holds: If G/T contains a non-zero element of infinite p-height, then the p-primary component T_p of T is a direct sum of a divisible and a bounded group.

Lemma 5: Let $H = \{a_1, a_2, \dots\}$ be a torsionfree group of finite rank n and let $\{h_1, h_2, \dots, h_n\}$ be a basis of H, F = $= \sum_{i=1}^{n}{}^{\oplus} \langle h_i \rangle$. If for each m=1,2,... it is

$$H / \langle F \cup \sum_{i=1}^{m} \langle a_i \rangle_{\pi}^{H} \rangle \cong \sum_{p_i \in \pi_m}^{\oplus} T_{p_i},$$

where $T_{p_i} \neq 0$ and π_m is an infinite set of primes, then H contains a generalized regular subgroup K of rank n such that the factor-group H/K has infinitely many non-zero primary components.

Proof: In each set π_m, m=1,2,..., choose a prime p_m in such a way that all these primes are pair-wise different.

By hypothesis, to each m=1,2,... there exists a subgroup $K_m \supsetneqq \langle F \cup \sum_{i=1}^{m} \langle a_i \rangle_{\pi}^{H} \rangle$ of H such that $H/K_m \cong C(p_m^{k_m})$, where $k_m \in \{1,2,\dots,\infty\}$. If we put $K = \bigcap_{m=1}^{\infty} K_m$ then for each m=1, 2,... it is $K \subsetneqq K_m$ and the factor-group H/K_m is a homomorphic image of H/K. Consequently, the factor-group H/K has non-zero p_m-primary component for each m=1,2,... and it remains to show that K is a generalized regular subgroup of H.

If $g \in K$ is an arbitrary element then $g = a_m$ for some m=1,2,... . If the equation $p^k x = g$ is solvable in H then $x \in \langle g \rangle_{\pi}^{H} = \langle a_m \rangle_{\pi}^{H}$ and $x \in \bigcap_{i=m}^{\infty} K_i$ by the choice of K_m's. If $p \notin \{p_1, p_2, \dots, p_{m-1}\}$ then $x \in K_1 \cap K_2 \cap \dots \cap K_{m-1}$ and so $x \in K$. Thus the characteristics $\tau^K(g)$ and $\tau^H(g)$ can differ only on places corresponding to the primes $p_1, p_2, \dots, p_{m-1}$ and K is a generalized regular subgroup of H.

Lemma 6: Let H be torsionfree group of finite rank n. If the set π' of all primes p with $r(H[p^{\infty}]) < r_p(H)$ is infinite then H contains a generalized regular subgroup K of rank n such that the factor-group H/K has infinitely many non-zero primary components.

Proof: Let $\{h_1, h_2, \dots, h_n\}$ be a basis of H, $F = \sum_{i=1}^{n} {}^{\oplus} \langle h_i \rangle$ and $H = \{a_1, a_2, \dots\}$. With respect to Lemma 5 it suffices to show that for each m=1,2,... the factor-group $H/\langle F \cup \sum_{i=1}^{m} \langle a_i \rangle_{\pi}^{H} \rangle$ has infinitely many non-zero p-primary components. Proving indirectly suppose the existence of a positive integer m such that the factor-group $H/\langle F \cup \sum_{i=1}^{m} \langle a_i \rangle_{\pi}^{H} \rangle$ is π_1-primary where π_1 is a finite set of primes. Denoting $\pi_2 = \pi \setminus \pi_1$, the sequence

$$0 \to Z_{\pi_2} \otimes (F + \sum_{i=1}^{m} \langle a_i \rangle_{\pi}^{H}) \to Z_{\pi_2} \otimes H \to Z_{\pi_2} \otimes H/(F + \sum_{i=1}^{m} \langle a_i \rangle_{\pi}^{H}) = 0$$

is exact by [9; Theorem 60.6]. Thus the group $Z_{\pi_2} \otimes H$ is purely finitely generated and consequently $r_p(H) = r(H[p^{\infty}])$ for each $p \in \pi_2$ by [3; Proposition 1]. Then necessarily $\pi' \subseteq \pi_1$, which contradicts the hypothesis.

Now we are ready to prove the following generalization of [4; Theorem 5].

<u>Theorem 7</u>: The following conditions are equivalent for a torsionfree group H of finite rank n:

(i) If G is a mixed group with the torsion part T such that $G/T \cong H$ then every pure subgroup of G of rank n splits if and only if G satisfies Conditions (α), (γ).

(ii)(a) $r_p(H) = r(H[p^{\infty}])$ for almost all primes and for all primes p with $r(H[p^{\infty}]) = 0$,

(b) for every generalized regular subgroup K of H of the same rank n the factor-group H/K has only a finite number of non-zero primary components.

(iii) $r_p(H) = 0$ for each prime p with $r(H[p^{\infty}]) = 0$ and for every generalized regular subgroup K of H the torsion part of the factor-group H/K has only a finite number of non-zero primary components.

(iv) If G is a mixed group with the torsion part T such that $G/T \cong H$ then every pure subgroup of G splits if and only if G satisfies Conditions (α), (γ).

<u>Proof</u>: Conditions (i) and (ii) are equivalent by [4; Theorem 5].

(ii) implies (iii). The set $\pi' = \{p \in \pi \mid r_p(H) \neq r(H[p^{\infty}])\}$ is finite by the hypothesis and consequently

[3; Proposition 1] shows that the group $H \otimes Z_{\pi \setminus \pi'}$ is purely finitely generated. If $S = \langle K \rangle_{\pi}^{H}$ is the pure closure of K in H then $S \otimes Z_{\pi \setminus \pi'}$ is pure in $H \otimes Z_{\pi \setminus \pi'}$ by [9; Theorem 60.4] and so it is purely finitely generated by Theorem 3. By [9; Theorem 60.6] we have $S/K \otimes Z_{\pi \setminus \pi'} \cong (S \otimes Z_{\pi \setminus \pi'})/ /(K \otimes Z_{\pi \setminus \pi'})$ and hence the group $S/K \otimes Z_{\pi \setminus \pi'}$ has only a finite number of non-zero primary components by [3; Proposition 1], $K \otimes Z_{\pi \setminus \pi'}$ being generalized regular in $S \otimes Z_{\pi \setminus \pi'}$. Now it is obvious that the torsion part S/K of H/K has only a finite number of non-zero primary components.

(iii) implies (iv). Let G be a mixed group with the torsion part T such that $G/T \cong H$. If G satisfies Conditions (α), (γ) then every pure subgroup of G splits by Lemma 6 and [4; Lemma 2]. Conversely, if every pure subgroup of G splits then, especially, every pure subgroup of G of rank n splits and G satisfies Conditions (α), (γ) by Lemma 6 and [4; Theorem 5].

(iv) implies (ii). The proof is the same as that of the implication (i) implies (ii) in [4; Theorem 5].

References

[1] BICAN L.: Completely decomposable abelian groups any pure subgroup of which is completely decomposable, Czech. Math. J. 24(99)(1974), 176-191.

[2] BICAN L.: Mixed abelian groups of torsionfree rank one, Czech. Math. J. 20(95)(1970), 232-242.

[3] BICAN L.: Splitting in abelian groups, Czech. Math. J. 28(103)(1978), 356-364.

[4] BICAN L.: Splitting of pure subgroups, Comment. Math. Univ. Carolinae 17(1976), 241-249.

[5] BICAN L.: A note on mixed abelian groups, Czech. Math. J. 21(96)(1971), 413-417.

[6] BICAN L.: Factor-splitting abelian groups of rank two, Comment. Math. Univ. Carolinae 11(1970), 1-8.

[7] BICAN L.: Factor-splitting abelian groups of finite rank, Comment. Math. Univ. Carolinae 17(1976), 473-480.

[8] FUCHS L.: Abelian groups, Budapest, 1958.

[9] FUCHS L.: Infinite abelian groups I, Academic Press, 1970.

[10] MALCEV A.: Abelevy gruppy konečnogo ranga bez kručenija, Mat. Sb. 4(46)(1938), 45-68.

[11] PROCHÁZKA L.: O p-range abelevych grupp bez kručenija konečnogo ranga, Czech. Math. J. 12(87)(1962), 3-43.

[12] PROCHÁZKA L.: Zametka o rasčepljajemosti smešannych abelevych grupp, Czech. Math. J. 10(85)(1960), 479-492.

[13] PROCHÁZKA L.: O rasčepljajemosti faktor-grupp abelevych grupp bez kručenija konečnogo ranga, Czech. Math. J. 11(86)(1961), 521-557.

[14] PROCHÁZKA L.: Zametka o faktornorasčepljajemych abelevych gruppach, Čas. pěst. mat. 87(1962), 404-414.

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