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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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# GAUSSIAN MEASURES AND THE DENSITY THEOREM

D. PREISS

Abstract: It is shown that there is a Gaussian probability measure $\gamma$ in a separable Hilbert space H and a Borel set $M \subset H$ with $\gamma(M) < 1$ such that
$$\lim_{r \to 0_+} \frac{\gamma(B(x,r) \cap M)}{\gamma(B(x,r))} = 1$$
almost everywhere.

Key words: Gaussian measures, Density Theorem.

Classification: 28A15, 28C20

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If $\mu$ is an arbitrary locally finite Borel measure in a finitely dimensional Banach space B then the following form of the Density Theorem holds. Whenever $M \subset B$ is a Borel set, then

$$\lim_{r \to 0_+} \frac{\mu(B(x,r) \cap M)}{\mu(B(x,r))} = \chi_M(x) \qquad \mu\text{ - a.e.}$$

(See e.g. [B],[M],[F, p. 147-150].) If the assumption of finite dimensionality is dropped, the situation becomes different. It was noted in [P1] that the Vitali Theorem need not hold if B is a separable Hilbert space and $\mu$ is a Gaussian measure. In [P2] it was shown that there is a probability measure $\mu$ in a separable Hilbert space and a set $M \subset H$ with $\mu(M) < 1$ such that

$$\lim_{r \to 0_+} \frac{\mu(B(x,r) \cap M)}{\mu(B(x,r))} = 1 \qquad \mu\text{ - a.e.}$$

The method used in this example can be generalized to other spaces. This generalization shows that the validity of the Density Theorem for all finite Borel measures in a complete separable metric space is equivalent to some notion of finite dimensionality of the space. (This result is now prepared for publication.) These results, however, do not say anything about the question of validity of the Density Theorem for Gaussian measures. Here we shall fill this gap by proving the following result.

Theorem. There is a Gaussian probability measure $\gamma$ in a separable Hilbert space H and a Borel set $M \subset H$ with $\gamma(M) < 1$ such that

$$\lim_{r \to 0_+} \frac{\gamma(B(x,r) \cap M)}{\gamma(B(x,r))} = 1 \quad \gamma \text{ - almost everywhere.}$$

($B(x,r)$ denotes the ball with center x and radius r.)

Note that this implies that the Density Theorem for Gaussian measures in a Hilbert space does not hold even in the sense of Mattila. (See [MA] where general criteria for the validity of generalized Differentiation Theorems are given.)

Proof of the theorem. We shall use the following notation:

$\mathbb{R}$ denotes the real line,

if $x \in \mathbb{R}^n$ then $|x|$ denotes the Euclidean norm of x,

$K_n(x,r)$ denotes the ball with center x and radius r,

the Lebesgue measure and the k-dimensional Hausdorff measure in $\mathbb{R}^n$ are denoted by $\mathcal{L}^n$ and $\mathcal{H}^k$, respectively.

$\gamma_n$ is the Gaussian measure in $\mathbb{R}^n$ defined by

$$\gamma_n(A) = (2\pi)^{-\frac{n}{2}} \int_A e^{-\frac{1}{2}|x|^2} \, d\mathcal{L}^n(x).$$

Before we start with the construction, we shall prove some estimates of measures of balls.

First we deduce a simple estimate of integrals of logaritmically concave functions.

(1) If $f$ is a positive function on $\langle t_o,t_2\rangle \subset \mathbb{R}$, $t_1 \in \langle t_o,t_2)$ and if $\log f$ is concave on $\langle t_o,t_2)$ and $f(t_1) < f(t_o)$ then

$$\int_{t_1}^{t_2} f(t)dt \leq \left(\frac{f(t_o)}{f(t_1)} - 1\right)^{-1} \int_{t_o}^{t_1} f(t)dt.$$

Proof.

$$\log f(t) \leq \log f(t_o) + \frac{t - t_o}{t_1 - t_o} \log \frac{f(t_1)}{f(t_o)} \text{ for } t \in (t_1,t_2),$$

$$\log f(t) \geq \log f(t_o) + \frac{t - t_o}{t_1 - t_o} \log \frac{f(t_1)}{f(t_o)} \text{ for } t \in (t_o,t_1),$$

hence

$$\left(\int_{t_1}^{t_2} f(t)dt\right)\left(\int_{t_o}^{t_1} f(t)dt\right)^{-1} \leq$$

$$\leq \left(\int_{t_1}^{\infty}\left(\frac{f(t_1)}{f(t_o)}\right)^{\frac{t}{t_1-t_o}} dt\right)\left(\int_{t_o}^{t_1}\left(\frac{f(t_1)}{f(t_o)}\right)^{\frac{t}{t_1-t_o}} dt\right)^{-1} = \left(\frac{f(t_o)}{f(t_1)} - 1\right)^{-1}.$$

Next we prove the main estimate of the measure of an intersection of a ball with a set of small measure.

(2) If $a \in (0,\frac{1}{2})$ then there are $s \in (0,1)$ and $c_1 > 0$ possessing the following property:

Whenever $r \in \langle a,\frac{1}{2}\rangle$ and $x \in \mathbb{R}^n$ with $(1 - c_1)n^{\frac{1}{2}} \leq |x| \leq \leq (1 + c_1)n^{\frac{1}{2}}$ then

$$\gamma_n(K_n(x,rn^{\frac{1}{2}}) \setminus K_n(0,sn^{\frac{1}{2}})) \leq (e^{c_1 n} - 1)^{-1}\gamma_n(K_n(x,rn^{\frac{1}{2}})).$$

**Proof.** There are $\varepsilon > 0$, $\delta < 1$ and $s_0, s \in (1-a,1)$, $s_0 < s$, such that

$$0 < \frac{2(c^2+r^2) - (s^2+s_0^2)}{4c^2r^2 - (c^2+r^2-s_0^2)^2} \leq \delta \text{ whenever } c \in \langle 1-\varepsilon, 1+\varepsilon\rangle$$

and $r \in \langle a, \frac{1}{2}\rangle$. Choose $c_1 \in (0,\frac{1}{2})$ such that $c_1 + 1 - a < s_0$, $c_1 < \varepsilon$, $1 + a - c_1 > s$, $\frac{1}{2} + \frac{1}{4(1-c_1)} < \frac{4}{5}$ and

$$\frac{5}{3}\left(e^{\frac{1}{2}n(s-s_0)(1-\delta)} - 1\right)^{-1} \leq (e^{c_1 n} - 1)^{-1} \text{ for each natural } n.$$

Since $\gamma_n$ is invariant with respect to rotations it suffices to prove the assertion for $x = (0,0,\dots,0,cn^{\frac{1}{2}})$ where $1 - c_1 \leq$ $\leq c \leq 1 + c_1$. Then (see [F, Theorem 3.2.12])

$$\gamma_n(K_n(x, rn^{\frac{1}{2}}) \setminus K_n(0, sn^{\frac{1}{2}})) =$$

$$= (2\pi)^{-\frac{n}{2}} \int_{sn^{\frac{1}{2}}}^{(c+r)n^{\frac{1}{2}}} e^{-\frac{1}{2}t^2} \mathcal{H}^{n-1}(\{y \in \mathbb{R}^n; |y| = t,\ |y - x| \leq rn^{\frac{1}{2}}\})dt =$$

$$= \tfrac{1}{2}(2\pi)^{-\frac{n}{2}} \int_{sn^{\frac{1}{2}}}^{(c+r)n^{\frac{1}{2}}} e^{-\frac{1}{2}t^2} \mathcal{H}^{n-1}(\{y \in \mathbb{R}^n; |y| = t, \sum_{i=1}^{n-1} y_i^2 \leq r^2 n -$$

$$- \left(\frac{c^2 n + r^2 n - t^2}{2cn^{1/2}}\right)^2\})dt.$$

Since

$$\tau^2 = r^2 n - \left(\frac{c^2 n + r^2 n - t^2}{2cn^{1/2}}\right)^2 = \frac{(t^2 - (c-r)^2 n)((c+r)^2 n - t^2)}{4c^2 n} \leq$$

$\leq t^2\left(\frac{c+r}{2c}\right)^2 \leq t^2\left(\frac{1}{2} + \frac{r}{2(1-c_1)}\right)^2 \leq (\frac{4}{5}t)^2$, we have

$$\tfrac{1}{2}\mathcal{H}^{n-1}(\{y \in \mathbb{R}^n; |y| = t, \sum_{i=1}^{n-1} y_i^2 \leq \tau^2\}) =$$

$$= \int_{K_{n-1}(0,\tau)} \left(1 - \frac{\sum_{i=1}^{n-1} y_i^2}{t^2}\right)^{-1/2} d\mathcal{L}^{n-1}(y) \leq \frac{5}{3}\mathcal{L}^{n-1}(K_{n-1}(0,\tau)).$$

Hence

$$\gamma_n(K_n(x,rn^{\frac{1}{2}}) \setminus K_n(0,sn^{\frac{1}{2}})) \leq \tfrac{5}{3}(2\pi)^{-\frac{n}{2}} \mathcal{L}^{n-1}(K_{n-1}(0,1)) \int_{sn^{\frac{1}{2}}}^{(c+r)n^{\frac{1}{2}}} f(t)dt,$$

where $f(t) = e^{-\frac{1}{2}t^2}\left(r^2n - \left(\frac{c^2n + r^2n - t^2}{2cn^{1/2}}\right)^2\right)^{\frac{n-1}{2}}$ for $t \in ((c - r)n^{\frac{1}{2}}, (c + r)n^{\frac{1}{2}})$.

Since log $f$ is concave and

$$\log \frac{f(sn^{\frac{1}{2}})}{f(s_o n^{1/2})} = \frac{n}{2}(s_o^2 - s^2) + \frac{n-1}{2} \log \frac{4c^2r^2 - (c^2 + r^2 - s^2)^2}{4c^2r^2 - (c^2 + r^2 - s_o^2)^2} \leq$$

$$\leq \frac{n}{2}(s_o^2 - s^2) + \frac{n-1}{2}\frac{(c^2 + r^2 - s_o^2)^2 - (c^2 + r^2 - s^2)^2}{4c^2r^2 - (c^2 + r^2 - s_o^2)^2} =$$

$$= \frac{n}{2}(s_o^2 - s^2) + \frac{n-1}{2}(s^2 - s_o^2)\frac{2(c^2 + r^2) - (s_o^2 + s^2)}{4c^2r^2 - (c^2 + r^2 - s_o^2)^2} \leq$$

$$\leq \frac{n}{2}(s_o^2 - s^2) + \frac{n}{2}(s^2 - s_o^2)\delta \leq -\frac{1}{2}n(s - s_o)(1 - \delta),$$

we may use (1) to obtain

$$\int_{sn^{\frac{1}{2}}}^{(c+r)n^{\frac{1}{2}}} f(t)dt \leq \left(e^{\frac{1}{2}n(s-s_o)(1-\delta)} - 1\right)^{-1} \int_{(c-r)n^{\frac{1}{2}}}^{(c+r)n^{\frac{1}{2}}} f(t)dt.$$

Hence

$$\gamma_n(K_n(x,rn^{\frac{1}{2}}) \setminus K_n(0,sn^{\frac{1}{2}})) \leq (e^{c_1 n} - 1)^{-1}(2\pi)^{-\frac{n}{2}} \mathcal{L}^{n-1}(K_{n-1}$$

$$(0,1)) \int_{(c-r)n^{\frac{1}{2}}}^{(c+r)n^{\frac{1}{2}}} f(t)dt \leq (e^{c_1 n} - 1)^{-1}(2\pi)^{-\frac{n}{2}} \int_{(c-r)n^{\frac{1}{2}}}^{(c+r)n^{\frac{1}{2}}} e^{-\frac{1}{2}t^2}$$

$$\mathcal{H}^{n-1}\{y \in \mathbb{R}^n; |y| = t, |y - x| \leq rn^{\frac{1}{2}}\}\, dt =$$

$$= (e^{c_1 n} - 1)^{-1}\, \gamma_n(K_n(x,rn^{\frac{1}{2}})).$$

We shall need also the following estimate of the "typical radius" of a ball.

3) If $\tau > 0$ and $q > 1$ then there are $\omega > 0$ and $c_2 > 0$ possessing the following property:

Whenever $2 \leq n \leq m \leq qn$ are natural numbers, $\|\cdot\|$ is a norm in $\mathbb{R}^m$ such that $\tau \|\cdot\| \geq |\cdot|$,

$z_1 \in \mathbb{R}^n$ such that $\frac{1}{2} n^{\frac{1}{2}} \leq |z_1| \leq \frac{3}{2} n^{\frac{1}{2}}$,

$z_2 \in \mathbb{R}^m$ such that $\frac{1}{2} m^{\frac{1}{2}} \leq |z_2| \leq \frac{3}{2} m^{\frac{1}{2}}$,

$s \in (0, \frac{1}{2} n^{\frac{1}{2}}\rangle$, and

$$A_\alpha = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m;\ |x_1 - z_1|^2 + \|x_2 - z_2\|^2 \leq s^2,$$
$$|x_1 - z_1| \leq \alpha s\},$$

then

$$\gamma_{n+m}(A_\omega) \leq (e^{c_2 n} - 1)^{-1}\ \gamma_{n+m}(A_1).$$

<u>Proof</u>. Let $\widetilde{B}(z_2, t)$ denote the ball in $(\mathbb{R}^m, \|\cdot\|)$ with center $z_2$ and radius $t$. If $0 < \omega < \omega_0 \leq 1$ then

$$\frac{\gamma_{n+m}(A_\omega)}{\gamma_{n+m}(A_{\omega_0})} = \frac{\int_{B_n(z_1, \omega s)} \gamma_m(\widetilde{B}(z_2, (s^2 - |x_1 - z_1|^2)^{\frac{1}{2}}))\, d\gamma_m(x_1)}{\int_{B_n(z_1, \omega_0 s)} \gamma_m(\widetilde{B}(z_2, (s^2 - |x_1 - z_1|^2)^{\frac{1}{2}}))\, d\gamma_n(x_1)} \leq$$

$$\leq e^{-\frac{1}{2}(|z_1| - \omega_0 s)^2 + \frac{1}{2}(|z_1| + \omega_0 s)^2}$$

$$\frac{\int_{B_n(z_1, \omega s)} \int_{\widetilde{B}(z_2, (s^2 - |x_1 - z_1|^2)^{\frac{1}{2}})} e^{-\frac{1}{2}|x_2|^2}\, d\mathcal{L}^m(x_2)\, d\mathcal{L}^n(x_1)}{\int_{B_n(z_1, \omega_0 s)} \int_{\widetilde{B}(z_2, (s^2 - |x_1 - z_1|^2)^{\frac{1}{2}})} e^{-\frac{1}{2}|x_2|^2}\, d\mathcal{L}^m(x_2)\, d\mathcal{L}^n(x_1)} =$$

$$= e^{2|z_1|\omega_0 s}$$

$$\cdot \frac{\int_{B_n(z_1, \omega s)} \int_{\widetilde{B}(z_2, s)} (s^2 - |x_1 - z_1|^2)^{\frac{m}{2}} e^{-\frac{1}{2}|z_2 + \frac{(s^2 - |x_1 - z_1|^2)^{\frac{1}{2}}}{s}(x_2 - z_2)|^2}\, d\mathcal{L}^m(x_2)\, d\mathcal{L}^n(x_1)}{\int_{B_n(z_1, \omega_0 s)} \int_{\widetilde{B}(z_2, s)} (s^2 - |x_1 - z_1|^2)^{\frac{m}{2}} e^{-\frac{1}{2}|z_2 + \frac{(s^2 - |x_1 - z_1|^2)^{\frac{1}{2}}}{s}(x_2 - z_2))|}\, d\mathcal{L}^m(x_2)\, d\mathcal{L}^n(x_1)}.$$

Since, for $0 \leq \tilde{s} \leq s$ and $x_2 \in \tilde{B}(z_2,s)$, it holds

$$\left| |z_2 + \frac{\tilde{s}}{s}(x_2 - z_2)|^2 - |x_2|^2 \right| \leq |x_2 - z_2| \frac{s-\tilde{s}}{s} (|z_2 + \frac{\tilde{s}}{s}(x_2 - z_2)| + |x_2|) \leq 2\tau(s-\tilde{s})(|z_2| + \tau s),$$

we obtain

$$\frac{\gamma_{n+m}(A_\omega)}{\gamma_{n+m}(A_{\omega_0})} \leq e^{2|z_1|\omega_0 s + 2\tau(1-\omega_0)^{\frac{1}{2}})s(|z_2|+\tau s)}$$

$$\frac{\int_{B_n(z_1,\omega s)} (s^2 - |x_1 - z_1|^2)^{\frac{m}{2}} \int_{\tilde{B}(z_2,s)} e^{-\frac{1}{2}|x_2|^2} \, d\mathcal{L}^m(x_2) d\mathcal{L}^n(x_1)}{\int_{B_n(z_1,\omega_0 s)} (s^2 - |x_1 - z_1|^2)^{\frac{m}{2}} \int_{\tilde{B}(z_2,s)} e^{-\frac{1}{2}|x_2|^2} \, d\mathcal{L}^m(x_2) d\mathcal{L}^n(x_1)} \leq$$

$$\leq e^{2|z_1|\omega_0 s + 2\tau(1-(1-\omega_0^2)^{\frac{1}{2}})s(|z_2|+\tau s)} \quad \frac{\int_0^\omega (1-t^2)^{\frac{m}{2}} \, t^{n-1} \, dt}{\int_0^{\omega_0} (1-t^2)^{\frac{m}{2}} \, t^{n-1} \, dt}$$

Moreover,

$$\log \frac{(1-\omega^2)^{\frac{m}{2}}\omega^{n-1}}{(1-\omega_0^2)^{\frac{m}{2}}\omega_0^{n-1}} = \frac{m}{2}\log\frac{1-\omega^2}{1-\omega_0^2} + \frac{n-1}{2}\log\frac{\omega^2}{\omega_0^2} \leq \frac{m}{2}\,\frac{\omega_0^2-\omega^2}{1-\omega_0^2} +$$

$$+ \frac{n-1}{2}\,\frac{\omega^2-\omega_0^2}{\omega_0^2} = -n\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)\left(\frac{n-1}{2n} - \frac{m}{2n}\,\frac{\omega_0^2}{1-\omega_0^2}\right) \leq$$

$$\leq -n\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)\left(\frac{1}{4} - \frac{q}{2}\,\frac{\omega_0^2}{1-\omega_0^2}\right).$$ Hence, if $\omega_0$ is sufficiently small and $\omega = \frac{1}{2}\omega_0$, we may use (1) with $t_0 = \omega_0$, $t_2 = 0$ to infer

$$\frac{\gamma_{n+m}(A_\omega)}{\gamma_{n+m}(A_{\omega_0})} \leq e^{\frac{3}{2}(\omega_0 + \tau(1-(1-\omega_0^2)^{\frac{1}{2}})(q^{\frac{1}{2}}+\tau)n} \left(e^{\frac{1}{12}n} - 1\right)^{-1}.$$

From this inequality we see that, to prove the assertion, it suffices to choose $\omega_0 > 0$ such that

$$\frac{1}{4} - \frac{q}{2}\frac{\omega_0^2}{1-\omega_0^2} > \frac{1}{9} \text{ and } \frac{3}{2}\,(\omega_0 + \tau(1-(1-\omega_0^2)^{\frac{1}{2}})(q^{\frac{1}{2}} + \tau) < \frac{1}{24}$$

and to put $\omega = \frac{1}{2}\omega_0$ and to find $c_2 > 0$ such that

$$\frac{e^{\frac{1}{24}n}}{e^{\frac{1}{12}n} - 1} \leq (e^{c_2 n} - 1)^{-1}.$$

Finally we show the following simple well-known estimate of Gaussian measure (see e.g. [Š]).

(4) There is $c_3 > 0$ such that

$$\gamma_n\{x \in \mathbb{R}^n;\ ||x| - n^{\frac{1}{2}}| \geq \alpha\} \leq \frac{c_3}{\alpha^2} \text{ and}$$

$$\gamma_n\{x \in \mathbb{R}^n;\ |\langle x,z\rangle| \geq \alpha\} \leq \frac{c_3}{\alpha^2}$$

for each natural number n, each $\alpha > 0$ and each $z \in \mathbb{R}^n$ with $|z| = 1$.

Proof. Follows from the Chebyshev inequality and from the fact that $\int (|x| - n^{\frac{1}{2}})^2 d\gamma_n(x)$ and $\int (\langle x,z\rangle)^2 d\gamma_n(x)$ are bounded by some absolute constant.

Now we shall construct the desired example.

Put $n_k = 10^k$ and choose $\sigma > 0$ such that $2(\sigma + \sum_{k=1}^{\infty} 10^{-k}) \leq \frac{1}{4}$. For $\tau = 10$ and $q = 11$ find $\omega$ and $c_2$ according to (3). Put $a = \omega\sqrt{\frac{\sigma}{10}}$ and find s and $c_1 < \frac{1}{2}$ according to (2). Let $k_0$ be so large that $\frac{c_3}{(1-s)^2}\sum_{k=k_0}^{\infty} 10^{-k} < 1$.

Let $X = \prod_{k=k_0}^{\infty} \mathbb{R}^{n_k}$, $\gamma = \bigotimes_{k=k_0}^{\infty} \gamma_{n_k}$. Then $H = \{ x \in X;$ $\sum_{k=k_0}^{\infty} 10^{-2k} |x_k|^2 < \infty \}$ is a Hilbert space with the norm $\|x\| = \left( \sum_{k=k_0}^{\infty} 10^{-2k} |x_k|^2 \right)^{\frac{1}{2}}$ and $\gamma(H) = 1$ (see [G]).

If $z \in H$ and $r > 0$ denote

$B_k(z,r) = \{ x \in H; \sum_{p=k_0}^{k} 10^{-2p} |x_p - z_p|^2 \leq r^2 \}$,

$B^k(z,r) = \{ x \in H; \sum_{p=k}^{\infty} 10^{-2p} |x_p - z_p|^2 \leq r^2 \}$, and

$B(z,r) = B^{k_0}(z,r)$.

Let $M = \{ x \in H;\ |x_k| \leq s n_k^{\frac{1}{2}}$ for some $k \geq k_0 \}$. Then, according to (4), $\gamma(M) \leq \sum_{k=k_0}^{\infty} \frac{c_3}{(1-s)^2 n_k} < 1$. We shall show that

$$\lim_{r \to 0_+} \frac{\gamma(M \cap B(z,r))}{\gamma(B(z,r))} = 1$$

for $\gamma$ almost every $z \in H$.

Let $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2} c_1$ and $4(2\varepsilon + \varepsilon^2) \sum_{p=1}^{\infty} 10^{-p} \leq$ $\leq \sigma$. Denote $\varepsilon_k = \max \left( \frac{2c_3}{\varepsilon^2} \sum_{p=k}^{\infty} 10^{-p},\ (e^{c_1 n_k} - 1)^{-1},\ (e^{c_2 n_k} - 1)^{-1} \right)$. If $E = \bigcup_{k=k_0}^{\infty} \{ z \in H;\ (1-\varepsilon) n_p^{\frac{1}{2}} \leq |z_p| \leq (1+\varepsilon) n_p^{\frac{1}{2}}$ for all $p \geq k \}$ then $\gamma(E) = 1$ according to (4) and the Borel-Canteli lemma.

Let $z \in E$. There is $k_1 \geq k_0$ such that $(1 - \frac{1}{2} c_1)^2 n_k \leq$ $\leq |z_k|^2 \leq (1 + \frac{1}{2} c_1)^2 n_k$ for all $k \geq k_1$. Hence there is $k_2 > k_1$ such that

$$(5) \quad (1 - c_1)^2 \sum_{\substack{p=k_0 \\ p \neq k}}^{k+1} n_p \leq \sum_{\substack{p=k_0 \\ p \neq k}}^{k+1} |z_p|^2 \leq (1 + c_1)^2 \sum_{\substack{p=k_0 \\ p \neq k}}^{k+1} n_p$$

for all $k \geq k_2$. We may also assume that $\varepsilon_{k_2} < 1$.

Let $r > 0$. If $r$ is sufficiently small then

$$2\sigma\, 10^{-(k+1)} + 2\sum_{p=k+2}^{\infty} 10^{-p} \leq r^2 \leq 2\sigma\, 10^{-k} + 2\sum_{p=k+1}^{\infty} 10^{-p}$$

for some $k \geq k_2$. Put

$$r_\alpha = (r^2 - \alpha^2 \sigma\, 10^{-(k+1)})^{\frac{1}{2}} \text{ for } \alpha \in \langle 0,1 \rangle,$$

$$r(x) = (r^2 - \sum_{p=k+2}^{\infty} 10^{-2p} |x_p - z_p|^2)^{\frac{1}{2}} \text{ for } x \in B^{k+2}(z,r),$$

$$a(x) = (\sum_{\substack{p=k_0 \\ p \neq k}}^{\infty} 10^{-2p} |x_p - z_p|^2)^{\frac{1}{2}}, \text{ and}$$

$$b(x) = (10^k (r^2 - a^2(x)))^{\frac{1}{2}} \text{ for } x \in H,\ a(x) \leq r.$$

First note that if $|x_p| \leq (1+\varepsilon) n_p^{\frac{1}{2}}$ and $|\langle x_p, z_p \rangle| \leq$ $\leq \varepsilon\, n_p^{\frac{1}{2}} |z_p|$ for all $p \geq k+2$ then $\sum_{p=k+2}^{\infty} 10^{-2p} |x_p - z_p|^2 \leq$

$$\leq 2\sum_{p=k+2}^{\infty} 10^{-p} + 4(2\varepsilon + \varepsilon^2) \sum_{p=k+2}^{\infty} 10^{-p} \leq r^2 - \sigma\, 10^{-(k+1)} = r_1^2.$$

Hence (4) implies

$$\gamma(X \setminus B^{k+2}(z,r_1)) \leq \frac{2\, c_3}{\varepsilon^2} \sum_{p=k+2}^{\infty} 10^{-p} \leq \varepsilon_{k+2}.$$

Consequently, $\gamma(B(z,r)) = \int_{B^{k+2}(z,r)} \gamma(B_{k+1}(z, r(x)))\, d\gamma(x) \leq$

$$\leq \int_{B^{k+2}(z,r_1)} \gamma(B_{k+1}(z,r(x)))\, d\gamma(x) + \gamma(X \setminus B^{k+2}(z,r_1)) \cdot$$

$$\cdot\, \gamma(B_{k+1}(z, (\sigma\, 10^{-(k+1)})^{\frac{1}{2}})) \leq \int_{B^{k+2}(z,r_1)} \gamma(B_{k+1}(z,r(x)))\, d\gamma(x) +$$

$$+ \frac{\varepsilon_{k+2}}{1 - \varepsilon_{k+2}}\, \gamma(B^{k+2}(z,r_1)) \cdot \gamma(B_{k+1}(z, (\sigma\, 10^{-(k+1)})^{\frac{1}{2}})) \leq$$

$$\leq \left(1 + \frac{\varepsilon_{k+2}}{1 - \varepsilon_{k+2}}\right) \int_{B^{k+2}(z,r_1)} \gamma(B_{k+1}(z,r(x)))\, d\gamma(x) =$$

$$= (1 - \varepsilon_{k+2})^{-1}\, \gamma(B(z,r) \cap B^{k+2}(z,r_1)).$$

If $m = \sum_{\substack{j=k_0 \\ j \neq k}}^{k+1} n_j$ then $n_k \leq m \leq q n_k$ and $\tau^2 \sum_{\substack{j=k_0 \\ j \neq k}}^{k+1} 10^{-2j+2k} |x_j|^2 \geq$ $\geq \sum_{\substack{j=k_0 \\ j \neq k}}^{k+1} |x_j|^2$. Hence, according to (3) (its assumptions follow from (5)), $\gamma_{n_k+m}\{y \in \prod_{j=k_0}^{k+1} \mathbb{R}^{n_j}, \sum_{j=k_0}^{k+1} 10^{-2j+2k}|y_j - z_j|^2 \leq s^2,$ $|y_k - z_k| \leq \omega s\} \leq (e^{c_2 n_k} - 1)^{-1} \gamma_{n_k+m}\{y \in \prod_{j=k_0}^{k+1} \mathbb{R}^{n_j},$ $\sum_{j=k_0}^{k+1} 10^{-2j+2k}|y_j - z_j|^2 \leq s^2\}$ for each $s \in (0, \frac{1}{2} n_k^{\frac{1}{2}}\rangle$. Since $10^{2k} r^2(x) \leq 10^{2k} r^2 \leq \frac{1}{4} 10^k = (\frac{1}{2} n_k^{\frac{1}{2}})^2$ for $x \in B(z,r)$, it follows that

$$\gamma\{y \in B_{k+1}(z,r(x)); |y_k - z_k| \leq 10^k \omega r(x)\} \leq \varepsilon_k \gamma(B_{k+1}(z,r(x))).$$

Moreover, for $x \in B(z,r) \cap B^{k+2}(z,r_1)$ we have $10^{-(k+1)} \sigma \leq$ $\leq r^2(x)$. Hence,

$$\gamma\{y \in B_{k+1}(z,r(x)); |y_k - z_k| \leq 10^k \omega (\sigma 10^{-(k+1)})^{\frac{1}{2}}\} \leq$$
$$\leq \varepsilon_k \gamma(B_{k+1}(z,r(x)))$$ and

$$\gamma(B(z,r) \cap B^{k+2}(z,r_1)) = \int_{B^{k+2}(z,r_1)} \gamma(B_{k+1}(z,r(x))) d\gamma(x) \leq$$
$$\leq (1 - \varepsilon_k)^{-1} \int_{B^{k+2}(z,r_1)} \gamma\{y \in B_{k+1}(z,r(x)); |y_k - z_k| \geq$$
$$\geq 10^k \omega (\sigma 10^{-(k+1)})^{\frac{1}{2}}\} d\gamma(x) = (1 - \varepsilon_k)^{-1} \gamma\{x \in B(z,r); a(x) \leq$$
$$\leq r_\omega\} = (1 - \varepsilon_k)^{-1} \int_{\{x \in H; a(x) \leq r_\omega\}} \gamma\{y \in H; 10^{-2k} |y_k - z_k|^2 \leq$$
$$\leq r^2 - a^2(x)\} d\gamma(x) = (1 - \varepsilon_k)^{-1} \int_{\{x \in H; a(x) \leq r_\omega\}} \gamma_{n_k}(K_{n_k}(z_k,$$
$$(b(x) n_k^{\frac{1}{2}})))d\gamma(x).$$

Since $b(x) \leq (10^k r_\omega^2)^{\frac{1}{2}} \leq \frac{1}{2}$ and since $b(x) \geq (10^k(r^2 - r^2))^{\frac{1}{2}} =$

$= \omega \sqrt{\frac{\sigma}{10}} = a$, we infer from (2) that

$$\int_{\{x \in H; a(x) \le r_\omega\}} \gamma_{n_k}(K_{n_k}(z_k, b(x)\, n_k^{\frac{1}{2}}))\, d\gamma(x) \le$$

$$\le (1 - \varepsilon_k)^{-1} \int_{\{x \in H; a(x) \le r_\omega\}} \gamma_{n_k}(K_{n_k}(z_k, b(x)\, n_k^{\frac{1}{2}}) \cap$$

$$\cap K_{n_k}(0, s n_k^{\frac{1}{2}}))\, d\gamma(x) \le (1 - \varepsilon_k)^{-1} \gamma(B(z,r) \cap M).$$

Combining these estimates we obtain

$$\gamma(B(z,r) \cap M) \ge (1 - \varepsilon_{k+2})(1 - \varepsilon_k)^2 \gamma B(z,r).$$

If $r$ is small then $k$ is large, hence $\varepsilon_k$, $\varepsilon_{k+2}$ are small. This implies that

$$\lim_{r \to 0_+} \frac{\gamma(B(z,r) \cap M)}{\gamma(B(z,r))} = 1.$$

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