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TOPOLOGICAL PROPER SEPARATION THEOREMS

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Abstract: In the last 15 years new algebraic separation theorems have been found. The goal of this paper is to show that some of these theorems admit a topological version, too.

Key word: Separation theorem.

Classification: 15A03, 46B99

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1. Some notations. Let V and W be two subsets of a real linear space L. A linear functional f on L is said to separate V and W properly (or frankly) if there exists an $r \in \mathbb{R}$ and an $u \in V \cup W$ satisfying $f(V) \leq r \leq f(W)$ and $f(u) \neq r$.

Say that a set $\tilde{L}$ of linear functionals separates V and W properly if there exists an $f \in \tilde{L}$ separating V and W. This definition excludes the case $f(W) = f(V) = r$.

Let I be equal to $\{0,1,\dots,n\}$.

A family $\{V_i \mid i \in I\}$ of subsets of L is properly separated by a family $\{f_i \mid i \in I\}$ of linear functionals if there exists a family $\{\lambda_i \mid i \in I\}$ of real numbers, a $k \in I$, a $u \in V_k$, an $\ell \in \mathcal{L}(\bigcup_{i \in I} V_i)$, the affine hull of $\bigcup_{i \in I} V_i$, such that

(1) $f_k(u) < \lambda_k$,

(2) $f_i(V_i) \leq \lambda_i$ and $f_i(\ell) = \lambda_i$ whenever $i \in I$,

(3) $\sum_{i \in I} f_i = 0$ and $\sum_{i \in I} \lambda_i = 0$.

In this situation we also say that $\widetilde{L}^I$ <u>separates</u> $\{V_i | i \in I\}$ <u>properly</u> if $f_i \in \widetilde{L}$ $(i \in I)$.

A point $v \in V$ is said to belong to the <u>core of</u> V <u>with respect to a linear subspace</u> Y <u>of</u> L if for each $y \in Y$ there exists a positive ε such that $v + \sigma y \in V$ for each $0 \leq \sigma \leq \varepsilon$. This set is denoted by ${}^{i(Y)}V$ (see [9], p. 36).

Let c(V) be the <u>core</u> of V with respect to L.

The core of V with respect to the linear subspace parallel to the affine hull of V is called the <u>intrinsic core of</u> V and is denoted by ic (V). The <u>linear hull</u> <u>(affine hull)</u> of V is denoted by Span (V) ($\mathcal{L}(V)$).

Now let V be a subset of the topological linear space L. int V denotes the interior of V, intv V denotes the interior of V with respect to the minimal flat $F \supset V$.

Let iint V (intrinsic interior) denote the interior of V with respect to the minimal closed flat $F \supset V$. Then iint $V \subset$ intv V and if iint $V \neq \emptyset$, then iint V = intv V; intv $V \subset$ ic V and if intv $V \neq \emptyset$ is convex then intv V = ic V.

Let I be the set $\{0,1,\ldots,n\}$.

Let <u>L^*</u> denote the set of all linear functionals on L and if L is a topological vector space, let <u>L'</u> denote the set of all continuous linear functionals on L.

2. <u>Separation and finite deficiency</u>:

2.1. <u>Theorem</u> (see Bair, Jongmans [2], p. 475). Let V and W be two convex subsets of a real linear space L. Let W and the intrinsic core of V be nonempty and disjoint. If the deficiency of V with respect to Span $(V \cup W)$ is finite then L^*

separates V and W properly.

[4], p. 263, [6], p. 11 have similar results. A continuous version of this theorem is mentioned in [4], p. 240 and p. 253.

The following Lemma is well known.

2.2. Lemma: Let H be a closed flat of a topological linear space L with finite deficiency. A linear functional f on L is continuous if and only if f is continuous on H.

2.3. Theorem (a similar result is Lempio [6], pp. 31-32). Let V and W be two convex subsets of a locally convex linear space L. Let W and the intrinsic interior of V be nonempty and disjoint. If the deficiency of V with respect to Span $(V \cup W)$ is finite then L' separates V and W properly. This is a consequence of Theorem 2.5.

If Span $(V \cup W)$ has finite deficiency with respect to L, the theorem 2.3 is correct for any topological linear space. Now look at the case of a finite number of sets V_i:

2.4. Theorem (see Bair [1], p. 13). Let V_0 be a convex nonempty set; let $\{V_i \mid i \in I \setminus \{0\}\}$ be a family of convex sets with nonempty intrinsic cores and of finite deficiency with respect to Span $\bigcup_{i \in I} V_i$. If

$$V_0 \cap \bigcap_{j=1}^{m} ic\,(V_j) = \emptyset$$

then $\{V_i \mid i \in I\}$ can be separated by L^{*I} properly.

A continuous version of this theorem is

2.5. Theorem. Let V_0 be a convex nonempty set. $\{V_i \mid i \in I \setminus \{0\}\}$ be a family of convex sets with nonempty intrinsic interiors and finite deficiency with respect to Span $\bigcup_{i\in I} V_i$. If

$$V_0 \cap \bigcap_{j=1}^{m} \operatorname{iint}(V_j) = \emptyset$$

then $\{V_i \mid i \in I\}$ can be separated by L'^I properly.

Proof: Since $\operatorname{ic}(V_i) = \operatorname{iint}(V_i)$ $(i \in I \setminus \{0\})$, by Theorem 2.4 there are $\{f_i' \mid i \in I\}$ separating $\{V_i \mid i \in I\}$ properly. Because $\operatorname{iint} V_i$ is nonempty, f_i' is continuous on $\mathcal{L}(V_i)$ and $\mathcal{L}(V_i)$ is closed $(i \in I \setminus \{0\})$. Hence f_i' is continuous on Span $\bigcup_{j\in I} V_j$ by Lemma 2.2 and by extending f_i' continuous to L we obtain continuous linear functions $f_i (i \in I \setminus \{0\})$. Define

$$f_0 := -\sum_{i=1}^{m} f_i.$$

f_0 is continuous, too.
On Span $\bigcup_{i\in I} V_i$ we have $f_0 = f_0'$ because $\sum_{i\in I} f_i' = 0$.

3. Extension Theorems:

3.1. Proposition: Let A and B be two closed flats of a Banach space L and f' be a linear functional continuous on A and B. If Span $(A \cup B)$ has finite deficiency in a closed subspace H then there exists a continuous linear functional f on L satisfying $f'(\ell) = f(\ell)$ for all $\ell \in H$.

Proof: Define $H_A :=$ Span A. $\mathcal{L}(A)$ has deficiency 0 or 1 with respect to H_A. Let G be a subspace of H such that Span $(A \cup B) + G = H$ and Span $(A \cup B) \cap G = \{0\}$. Define $H_B = G +$ + Span B. Because G is finitedimensional, B has finite deficiency in H_B. Now we have $H = H_A + H_B$. Lemma 2.2 proves that f' is continuous on H_A and H_B. Define

$O_A := \{h \in H_A \mid f'(h) > 0\}$ and

$O_B := \{h \in H_B \mid f'(h) > 0\}.$

They are open with respect to H_A and H_B respectively. Because H, H_A, H_B are closed, H, H_A, H_B, $H_A \times H_B$ are Banach spaces. ($H_A \times H_B$ has the product topology.) $h(\ell_1, \ell_2) := \ell_1 + \ell_2$ is a continuous linear surjective functional $h: H_A \times H_B \longrightarrow H_A + H_B = H$.

The theorem of Banach-Schauder (see e.g. [5], p. 170(2)) shows that h is open. That is why $O_A + O_B$ is open with respect to H. Since $f'(O_A + O_B) > 0$, f' is continuous on H. Because f' is continuous on H, we obtain a continuous linear functional f (by extending f' on L) with the required property.

3.2. <u>Lemma</u>: Let $f, f_0', \ldots, f_n'$ $(n \geq 0)$ be linear functionals satisfying

$$f = \sum_{i=0}^{n} f_i'.$$

Let $H_0, \ldots, H_n$ be subspaces of L and

$$\bigcap_{i=1}^{n} H_i \subset H_0 \quad \Big(\bigcap_{i=1}^{0} H_i =: L\Big).$$

Let f_0 be a linear functional satisfying $f_0(h) = f_0'(h)$ for all $h \in H_0$. Then there exist $f_1, \ldots, f_n$ such that $f_i(h) = f_i'(h)$ for all $h \in H_i$ $(i \in \{1, \ldots, n\})$ and

$$f = \sum_{i=0}^{n} f_i.$$

<u>Proof</u>: For n = 0 the conclusion is easy.

Now assume that the lemma is correct for n and assume the case of n + 1. On the subspace H_{n+1} holds

$$H_{n+1} \cap H_0 \subset \bigcap_{i=1}^{n} (H_{n+1} \cap H_i)$$

By the assumption we find $f_1, \ldots, f_n$ defined on H_{n+1} satisfying $f_i(h) = f_i'(h)$ for all $h \in H_{n+1} \cap H_i$ and

$$f - f'_{n+1} = \sum_{i=0}^{m} f_i \text{ on } H_{n+1}.$$

The condition $f_i = f'_i$ on H_i extends f_i on $H_{n+1} + H_i$ $(i \in \{1,\dots,n\})$. Let $f_i (i \in \{1,\dots,n\})$ be defined on the whole of L by extension. Now define

$$f_{n+1} = f - \sum_{i=0}^{m} f_i.$$

Now holds $f_{n+1} = f'_{n+1}$ on H_{n+1} because

$$f - f'_{n+1} = \sum_{i=0}^{m} f'_i \text{ on } H_{n+1}.$$

3.3. <u>Lemma</u> : Let $H_i \subset L$ $(i \in \{0,\dots,n\})$ $(n>0)$ be subspaces of L. If for each $i,j \in \{0,\dots,n\}$ $(i \neq j)$ there exists a closed subspace G_{ij} such that $H_i + H_j$ has finite deficiency with respect to G_{ij} then there exists a closed subspace G such that

$$H_0 + \bigcap_{i=1}^{m} H_i$$

has finite deficiency with respect to G. We may choose

$$G = \bigcap_{i=1}^{m} G_{0i}.$$

<u>**Proof**</u>: 1) Let M, H, G be subspaces of L. If $H \subset G$ has finite deficiency in G then $M \cap H$ has finite deficiency in $M \cap G$: Let B_0 be a basis of $M \cap H$ and $B_0 \cup B_2$ a basis of $M \cap G$. We have to show that the number of elements of B_2 is finite. Since $H \cap (M \cap G) = M \cap H$, there is a basis $B_0 \cup B_1$ of H satisfying $B_0 \cup B_1 \cup B_2$ is a basis of $(M \cap G) + H = (M \cap G) + (H \cap G) = (M + H) \cap G$. Let $B_0 \cup B_1 \cup B_2 \cup B_3$ be a basis of G. Because H has finite deficiency in G, the number of elements of $B_2 \cup B_3$ is finite.

2) If n = 1 there is nothing to prove. Now assume that the lemma is correct for n and assume the case of n + 1. Let

G' be a closed subspace such that

$$H_0 + \bigcap_{i=1}^{n} H_i \text{ has finite deficiency with respect to } G'.$$

Then

$$H_0 + \bigcap_{i=1}^{n+1} H_i = (H_0 + H_{n+1}) \cap (H_0 + \bigcap_{i=1}^{n} H_i)$$

has finite deficiency in $(H_0 + H_{n+1}) \cap G'$ since 1). And since 1) $(H_0 + H_{n+1}) \cap G'$ has finite deficiency in $G_{0,n+1} \cap G' =: G$.

3.4. Proposition: Let $f_0', f_1', \dots, f_n'$ be linear functionals on a Banach space L such that

$$\sum_{i=0}^{n} f_i'$$

is equal to a continuous functional f. Let f_i' be continuous on the closed flat F_i $(i \in \{0,\dots,n\})$ and assume that for all $i \neq j$ $(j \in \{0,\dots,n\})$ there exists a closed subspace G_{ij} such that $F_i + F_j$ has finite deficiency with respect to G_{ij}. Then there exist continuous linear functionals $f_0, \dots, f_n$ satisfying

$$\sum_{i=0}^{n} f_i = f$$

and $f_i'(h) = f_i(h)$ for all $h \in F_i$ $(i \in \{0,\dots,n\})$.

Proof (by induction with respect to n). The case of $n = 0$ is easy. Let n be larger than zero and assume that the conclusion is true for $m < n$. By Lemma 2.2 f_0' is continuous on Span F_0. Because

$$f_0' = f - \sum_{i=1}^{n} f_i', \ f_0' \text{ is continuous on } \bigcap_{i=1}^{n} \text{Span } F_i.$$

By Lemma 3.3 there exists a subspace G such that

$$\text{Span } F_0 + \bigcap_{i=1}^{n} \text{Span } F_i$$

has finite deficiency with respect to G. Then by Proposition 3.1 f_0' is continuous on G. Define $f_0(g)$ equal to $f_0'(g)$ for all $g \in G$ and let f_0 be a continuous linear functional on L (extension theorem). By Lemma 3.2 there exist $f_1,\dots,f_n$ satisfying

$$\sum_{i=0}^{m} f_i = f \quad \text{and} \quad f_i(h) = f_i'(h)$$

for all $h \in \operatorname{Span} F_i$. The assumption of our induction completes the proof.

4. Symmetric separation theorems. A similar result to the next theorem 4.1 you find in Klee [4], p. 253 and a proof of this result in Lempio [6], p. 11.

4.1. Theorem (see Bair, Jongmans [2], p. 475). Let V and W be two convex subsets of a real linear space L. Let the intrinsic core of V and the intrinsic core of W be both nonempty. L^* separates properly V and W if and only if $(\text{ic } V) \cap (\text{ic } W) = \emptyset$.

Theorem 2.3 is a word by word translation of the theorem 2.1 in the topological situation. Such a translation of the theorem 4.1 is not correct. It is correct in a locally convex linear space if and only if the sum $H_1 + H_2$ of any two closed linear subspaces H_1 and H_2 is closed itself. This is fulfilled for the strong topology. It is not fulfilled for Hilbert spaces. We are able to prove the following result.

4.2. Theorem: Let V and W be convex subsets of a Banach space L. Let the intrinsic interior of both sets V and W be nonempty. If there exists a closed subspace H in which $\operatorname{Span}(V \cup W)$ has finite deficiency then L' separates properly the sets V and W if and only if $\text{iint } V \cap \text{iint } W = \emptyset$.

This is a consequence of Theorem 4.4. An elementary proof is:

Proof: "$\Leftarrow$" By Theorem 4.1 we obtain a linear functional $f' \in L^*$ separating V and W. f' is continuous on $\mathcal{L}(V)$ and $\mathcal{L}(W)$ since iint V and iint W are nonempty. By Proposition 3.1 we obtain a continuous linear functional f on L with the required properties.

" $\Rightarrow$ " Now let L' separate V and W. Then there exist $f \in L'$, $r \in \mathbb{R}$, $u \in V \cup W$ such that $f(V) \leq r \leq f(W)$ and $f(u) \neq r$. Assume $u \in V$. Hence $f(u) < r$. That is why f (iint V) $< r$. Because $f(W) \geq r$, (iint V) $\cap$ (iint W) $= \emptyset$.

Theorem 4.1 leads to a separation theorem for finite families which is due to Vlach [11].
We quote a version of Baire. Note that there is an interesting symmetric proof of Vangeldère [10], p. 157.

4.3. <u>Theorem</u> (see Baire [1], p. 13). If a family $\{V_i \mid i \in I\}$ $(I = \{1,\dots,n\})$ of subsets of a real linear space L satisfies the conditions

(a) V_i is convex for each $i \in I$,

(b) ic (V_i) is nonempty for each $i \in I$,

then the family $\{V_i \mid i \in I\}$ can be separated properly by L^{*I} if and only if

$$\bigcap_{i \in I} \text{ic } V_i = \emptyset.$$

A continuous version of this theorem is:

4.4. <u>Theorem</u>: If a family $\{V_i \mid i \in I\}$ $(I = \{0,\dots,n\})$ of subsets of a Banach space L satisfies the conditions

(a) V_i is convex for each $i \in I$,

(b) iint (V_i) is nonempty for each $i \in I$,

(c) for all $i,j \in I$ $(i \neq j)$ there exists a closed subspace G_{ij} such that Span $(V_i \cup V_j)$ has finite deficiency in G_{ij}, then the family $\{V_i | i \in I\}$ can be separated properly by L'^I if and only if

$$\bigcap_{i \in I} \operatorname{iint} V_i \text{ is empty.}$$

This is a consequence of Theorem 4.6. We look now at a somewhat more general situation.

Let T_i, S_i be subspaces of L. Vangeldère [10], p. 148 defines that $\{T_i | i \in I\}$ $(I = \{0,\dots,n\}\ (n \geq 1))$ has the <u>property of intersection relative to</u> $\{S_i | i \in I\}$ if

$$\bigcap_{i \in I} (s_i - T_i) \neq \emptyset \text{ for all } s_i \in S_i \text{ and } i \in I.$$

Let S_i now be equal to the subspace of L parallel to $\mathcal{L}(V_i)$. Vangeldère proves the

4.5. <u>Theorem</u> (see Vangeldère [10], p. 157). Let $\{T_j | j \in I\}$ be a family subspaces of L $(I = \{0,\dots,n\})$ having the property of intersection relative to $\{S_j | j \in I\}$. If $^{i(T_j)}V_j \neq \emptyset$ for all $j \in I$, the family $\{V_j | j \in I\}$ can be separated properly by L^{*I} if and only if

$$\bigcap_{j \in I} {}^{i(T_j)}V_j = \emptyset.$$

This theorem is more general as all other nontopological separation theorems in this paper. Define now

$$\operatorname{in}^{(T)}V = \{v \in V | v \in \operatorname{iint} [(v + T) \cap V]\}.$$

A continuous version of Theorem 4.5 is:

4.6. <u>Theorem</u>: Let $\{T_j | j \in I\}$ be a family of closed subspaces of the Banach space L having the property of intersection relative to $\{S_j | j \in I\}$. For all $i,j \in I$ $(i \neq j)$ let exist a closed subspace G_{ij} such that $T_i + T_j$ has finite deficiency in G_{ij}, and that $V_i \cup V_j \subset G_{ij}$.

If ${}^{\mathrm{in}(T_j)}V_j \neq \emptyset$ for all $j \in I$, the family $\{V_j \mid j \in I\}$ can be separated properly by L'^I if and only if

$$\bigcap_{j \in I} {}^{\mathrm{in}(T_j)}V_j = \emptyset.$$

Proof: "$\Leftarrow$" By Theorem 4.5 there exists a family $\{f_i' \in L^* \mid i \in I\}$ of linear functionals separating $\{V_i \mid i \in I\}$ properly. Since ${}^{\mathrm{in}(T_i)}V_i \neq \emptyset$, f_i' is continuous on $\mathcal{L}({}^{\mathrm{in}(T_i)}V_i)$. That is why f_i' is continuous on T_i. Because

$$f_i' = -\sum_{j \neq i} f_j', f_i' \text{ is continuous on } \bigcap_{j \neq i} T_j.$$

Since Lemma 3.3 $T_i + \bigcap_{j \neq i} T_j$ has finite deficiency in $\bigcap_{j \neq i} G_{ij}$. Now since Proposition 3.1 f_i' is continuous on $\bigcap_{j \neq i} G_{ij}$. Define $F_i := \bigcap_{j \neq i} G_{ij}$. $F_i + F_k$ has finite deficiency in G_{ik} because

$$T_i \subset \bigcap_{j \neq i} G_{ij} = F_i \quad \text{and} \quad T_k \subset F_k.$$

By Proposition 3.4 there exist continuous $f_0, \ldots, f_n$ satisfying

$$\sum_{i \in I} f_i = 0$$

and

$$f_i'(h) = f_i(h) \text{ for all } h \in F_i \supset V_i.$$

"$\Rightarrow$" is proved by Theorem 4.5.

If we restrict ourselves to Banach spaces, this theorem 4.6 is more general as all other topological separation theorems in this paper.

Especially Theorems 2.5 and 4.4 are consequences of Theorem 4.6 in the case of Banach spaces.

We are interested in hearing about the following

Problems:

a) It would be useful to ask if there are continuous separation theorems for other definitions of separation. See Deumlich, Elster, Nehse [3], p. 276.

b) We think, it could be that the assumption of Banach spaces in the theorems of chapter 4 is too strong.

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