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LONG CHAINS IN RUDIN-FROLÍK ORDER

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Abstract: We construct chains of ultrafilters in Rudin-Frolík order of βN , order-isomorphic to $(2^{\aleph_0})^+$.

Key words: Ultrafilter, type of an ultrafilter, Rudin-Frolík order, independent family, stratified set.

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§ 0. Introduction. In Rudin-Frolík order of types of ultrafilters in βN (shortly RF) there are at most $2^{\aleph_0}$ predecessors of the type of an ultrafilter [3]. The cardinality of each branch in RF is at least $2^{\aleph_0}$. These two results yield that there are only two possibilities for the cardinality of branches in RF and these are $2^{\aleph_0}$ or $(2^{\aleph_0})^+$.

The main result of this paper is the following assertion.

THEOREM. In RF there exists a chain order-isomorphic to $(2^{\aleph_0})^+$. This chain is, of course, unbounded.

THEOREM is a particular solution of the problem: Which of two possible cardinalities of the branches in RF can occur?

Note, that the existence of a branch of the cardinality $2^{\aleph_0}$ is not proved.

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§ 1. **Basic concepts**. The notation and terminology used here can be found e.g. in [1], [3]. Let $p, q \in \beta N$, then $p \leq q$ in RF iff there exists a countable discrete set X such that $q = \Sigma(X, p)$. If G is a centered system of sets then (G) denotes the filter generated by this system. F refers to the Fréchet filter.

Definition 1.1: A set of filters $\{\mathcal{G}_{\beta,n} ; \beta \in \alpha \in On, n \in \omega\}$ is stratified iff

(1) the set $\{\mathcal{G}_{\beta,n} ; n \in \omega\}$ is discrete for each $\beta \in \alpha$, i.e. there exists a set $\{D_{\beta,n} \subseteq \omega ; n \in \omega\}$ satisfying $D_{\beta,k} \in \mathcal{G}_{\beta,k}$ for each $k \in \omega$ and $D_{\beta,k} \cap D_{\beta,\ell} = \emptyset$ for $k \neq \ell$,

(2) for each $\beta \in \alpha$, $n \in \omega$ and each $\gamma > \beta$, $\gamma < \alpha$ the set $\{\ell ; A \in \mathcal{G}_{\gamma,\ell}\}$ is infinite for all $A \in \mathcal{G}_{\beta,n}$.

Definition 1.2.: Let $\{\mathcal{G}_{\beta,n} ; \beta \in \alpha, n \in \omega\}$ be a stratified set of filters and C be its subset. We define

$C(0) = C$,

$C(\gamma) = \bigcup_{\delta < \gamma} C(\delta)$, if γ is limit,

$C(\gamma+1) = C(\gamma) \cup \{\mathcal{G}_{\beta,k} ; \exists \varepsilon > \beta, \exists A \in \mathcal{G}_{\beta,k}$ such that

$\{\mathcal{G}_{\varepsilon,\ell} ; A \in \mathcal{G}_{\varepsilon,\ell}\} \subseteq C(\gamma)$ modulo finite$\}$,

$\mathcal{C} = \bigcup_{\gamma < \omega_1} C(\gamma)$.

Definition 1.3: The system $\mathcal{A} = \{A_{\xi,j} ; \xi \in I, j \in J\}$, $I \neq \emptyset$, $J \neq \emptyset$ is called independent family with respect to (IF w.r.t.) the filter $\mathcal{F} \supseteq F$ iff

(1) $\bigcup_{j \in J} A_{\xi,j} = \omega$; $A_{\xi,k} \cap A_{\xi,\ell} = \emptyset$ for each $\xi \in I$, $k, \ell \in J$, $k \neq \ell$,

(2) $E \cap \bigcap_{\gamma \in r} A_{\gamma,j_\gamma} \neq \emptyset$ whenever $E \in \mathcal{F}$, $r \in [I]^{<\omega}$ and for each $\gamma \in r$, $j_\gamma \in J$.

§ 2. **Proof of THEOREM.** Theorems 2.1 and 2.2 imply immediately the assertion of THEOREM. Note, that Theorem 2.2 ensures not only the existence of the chain of the order-type $(2^{\aleph_0})^+$ but also the existance of a special system of such chains.

Theorem 2.1: Let $\{\mathcal{G}_{\beta,n} \supseteq F ;\ \beta \in \mathcal{L},\ n \in \omega\}$ be a stratified set of filters. Then there exists a stratified set of ultrafilters $\{q_{\beta,n} ;\ \beta \in \mathcal{L},\ n \in \omega\}$ satisfying $q_{\beta,n} \supseteq \mathcal{G}_{\beta,n}$.

The original proof of Theorem 2.1 needed the technique of the independent family of sets. Petr Simon has pointed out how to avoid the use of this technique what led to the simplification of the proof.

Proof. We shall construct the ultrafilters $q_{\beta,n} ;\ \beta \in \mathcal{L}$, $n \in \omega$ by the transfinite induction in $2^{\aleph_0}$ stages.

Let $\{B_\eta ;\ \eta \in 2^{\aleph_0}\}$ be the fixed enumeration of all subsets of ω . At the stage $\xi < 2^{\aleph_0}$ we construct filters $\mathcal{G}^{\xi}_{\beta,n} ;\ \beta \in \mathcal{L}$, $n \in \omega$ with the following properties:

1) $\mathcal{G}^{0}_{\beta,n} = \mathcal{G}_{\beta,n}$,

2) $\{\mathcal{G}^{\xi}_{\beta,n} ;\ \beta \in \mathcal{L},\ n \in \omega\}$ is a stratified set,

3) B_ξ or $\omega - B_\xi$ belongs to $\mathcal{G}^{\xi+1}_{\beta,n}$,

4) if ξ is a limit ordinal then $\mathcal{G}^{\xi}_{\beta,n} = \bigcup_{\eta<\xi} \mathcal{G}^{\eta}_{\beta,n}$.

Suppose that we are at the step $\xi+1$ and we have constructed the filters $\mathcal{G}^{\xi}_{\beta,n}$ for each $\beta \in \mathcal{L},\ n \in \omega$. Let $C = \{\mathcal{G}^{\xi}_{\beta,n} ;\ B_\xi \in \mathcal{G}^{\xi}_{\beta,n}\}$. Evidently, $\mathcal{G}^{\xi}_{\beta,k} \cup \{\omega - B_\xi\}$ is a centered system if $\mathcal{G}^{\xi}_{\beta,k} \notin \tilde{C}$. We show that $\mathcal{G}^{\xi}_{\beta,k} \cup \{B_\xi\}$ is a centered system if $\mathcal{G}^{\xi}_{\beta,k} \in \tilde{C}$. Suppose in order to get a contradiction that γ is the least ordinal number such that there exists a filter $\mathcal{G}^{\xi}_{\gamma,i} \in C(\gamma) - C$ and $\mathcal{G}^{\xi}_{\gamma,i} \cup \{B_\xi\}$ is not a centered system. This means that $\omega - B_\xi \in \mathcal{G}^\xi_{\gamma,i}$. Since the set $\{\mathcal{G}^\xi_{\beta,m}\,;\beta\in\alpha, m\in\omega\}$ is stratified there exists $\varepsilon > \delta$ such that the set $\{\mathcal{G}^\xi_{\varepsilon,e} \in C(r-1); \omega - B_\xi \in \mathcal{G}^\xi_{\varepsilon,e}\}$ is nonempty. This is a contradiction with the minimality of r.

Now, we are ready to define

$$\mathcal{G}^{\xi+1}_{\beta,m} = \begin{cases} (\mathcal{G}^\xi_{\beta,m} \cup \{B_\xi\}), & \text{if } \mathcal{G}^\xi_{\beta,m} \in \tilde{C}, \\ \\ (\mathcal{G}^\xi_{\beta,m} \cup \{\omega - B_\xi\}), & \text{otherwise.} \end{cases}$$

We prove that the set $\{\mathcal{G}^{\xi+1}_{\beta,m}\,;\beta\in\alpha, m\in\omega\}$ is a stratified set of filters.

The assumption that the set $\{\mathcal{G}^\xi_{\beta,m}\,;\beta\in\alpha, m\in\omega\}$ is stratified and definitions of C and $\tilde{C}$ yield that for each $A \in \mathcal{G}^\xi_{\beta,m}$ where $\mathcal{G}^\xi_{\beta,m} \in \tilde{C}$ and each $\varepsilon > \beta$ the set $\{\mathcal{G}^{\xi+1}_{\varepsilon,e}\,;\, A \cap B_\xi \in \mathcal{G}^{\xi+1}_{\varepsilon,e}\}$ is infinite.

If $\mathcal{G}^\xi_{\beta,m} \notin \tilde{C}$ then for each $\varepsilon > \beta$ and for each $A \in \mathcal{G}^\xi_{\beta,m}$ the set $\{\mathcal{G}^\xi_{\varepsilon,e}\,;\, A \in \mathcal{G}^\xi_{\varepsilon,e}$ and $\mathcal{G}^\xi_{\varepsilon,e} \notin \tilde{C}\}$ is infinite. Therefore, $\{\mathcal{G}^{\xi+1}_{\varepsilon,e}\,;\, A \cap \{\omega - B_\xi\} \in \mathcal{G}^{\xi+1}_{\varepsilon,e}\}$ is infinite, too.

One can easily see that we can proceed the induction for each $\xi < 2^{\aleph_0}$ and it does not depend on whether $\mathcal{G}^\xi_{\beta,m}$ is a filter or an ultrafilter.

Finally, we set $q_{\beta,m} = \bigcup_{\xi<2^{\aleph_0}} \mathcal{G}^\xi_{\beta,m}$. The condition 2) ensures that the set $\{q_{\beta,m}\,;\beta\in\alpha, m\in\omega\}$ is stratified and the condition 3) guarantees that $q_{\beta,m}$ is an ultrafilter.

q.e.d.

<u>Theorem 2.2</u>: There exists a set of ultrafilters $\{q^\alpha_{\beta,m}\,; m\in\omega,\ \alpha<(2^{\aleph_0})^+,\ \beta<\alpha\}$ such that

(1) $\{q^\alpha_{\beta,m}\,; m\in\omega\}$ is a discrete set of ultrafilters for each $\alpha < (2^{\aleph_0})^+$, $\beta < \alpha$,

(2) if $\beta < \delta < \alpha$, $k \in \omega$ then $q^{\alpha}_{\beta,k} = \Sigma(\{q^{\alpha}_{\delta,\ell} ; \ell \in \omega\}, q^{\delta}_{\beta,k})$, (hence $q^{\delta}_{\beta,k} < q^{\alpha}_{\beta,k}$).

<u>Proof</u>. We shall construct the proposed set of ultrafilters by the transfinite induction in $(2^{\aleph_0})^+$ stages. More precisely, at each stage $\alpha < (2^{\aleph_0})^+$ we construct all ultrafilters $q^{\alpha}_{\beta,n}$; $\beta \in \alpha$, $n \in \omega$.

The sets $\{q^{\alpha}_{\beta,n} ; n \in \omega\}$ have to be discrete for every $\alpha < (2^{\aleph_0})^+$ and $\beta < \alpha$. Therefore, define at first the system $\{\{D^{\alpha}_{\beta,n} ; n \in \omega\} ; \alpha < (2^{\aleph_0})^+, \beta < \alpha\}$ of partitions of ω such that $D^{\alpha}_{\beta,n}$ will belong to $q^{\alpha}_{\beta,n}$.

Let $\mathcal{A} = \{A_{\xi,n} ; \xi \in 2^{\aleph_0}, n \in \omega\}$ be IF w.r.t. F. Such an independent family exists (see e.g. [4]).

For each $\alpha < (2^{\aleph_0})^+$ take all partitions $\{\{A_{\xi,n} ; n \in \omega\} ; \xi \in |\alpha|\}$ and remumerate them in such a way that they will be ordered in the type α, i.e. we take a bijection φ from α onto $|\alpha|$ and for all $\beta < \alpha$ we set $D^{\alpha}_{\beta,n} = A_{\varphi(\beta),n}$.

Now, we can start the construction.

Let $\{q^{1}_{0,n} ; n \in \omega\}$ be an arbitrary set of the nontrivial ultrafilters satisfying $D^{1}_{0,\ell} \in q^{1}_{0,\ell}$.

Suppose that we have constructed the ultrafilters $q^{\beta}_{\delta,n}$ for each $\beta < \alpha$, $\delta < \beta$, $n \in \omega$ with required properties and we want to construct the ultrafilters $q^{\alpha}_{\beta,n}$. Firstly we define filters

$$\mathcal{G}_{\beta,k} = (F \cup \{D^{\alpha}_{\xi,m} ; \xi < \beta \ \& \ k \in D^{\beta}_{\xi,m}\} \cup D^{\alpha}_{\beta,k} \cup \cup \bigcup_{\beta < \xi < \alpha} \{\bigcup_{m \in B} D^{\alpha}_{\xi,m} ; B \in q^{\xi}_{\beta,k}\}) \text{ for each } \beta < \alpha, k \in \omega.$$

$\mathcal{G}_{\beta,k}$ is a filter because we use only the sets from IF w.r.t. F.

We prove now that the set of filters $\{\mathcal{G}_{\beta,n} ; \beta \in \mathcal{L}, n \in \omega\}$ is stratified.

It follows from the definition of $\mathcal{G}_{\beta,k}$ that the set $\{\mathcal{G}_{\beta,n} ; n \in \omega\}$ is discrete for each $\beta < \mathcal{L}$ and thus the first condition is fulfilled.

To verify the second condition suppose that $A \in \mathcal{G}_{\beta,k}$ and $\sigma > \beta$. We show that $\{\ell ; A \in \mathcal{G}_{\sigma,\ell}\} \in q^{\sigma}_{\beta,k}$. It is sufficient to consider the sets A satisfying $A \supseteq D^{\alpha}_{\xi_1,\ell} \cap D^{\alpha}_{\beta,k} \cap \bigcap_{i=2}^{4} \bigcup_{j \in B_i} D^{\alpha}_{\xi_i,j}$ where $\xi_1 < \beta < \xi_2 < \xi_3 = \sigma < \xi_4$ and $B_i \in q^{\xi_i}_{\beta,k}$. The proof for the sets from $\mathcal{G}_{\beta,k}$ of the other form as A is more complicated, only.

By the definition of $\mathcal{G}_{\gamma,m}$ the set $D^{\alpha}_{\xi_1,\ell}$ belongs to all $\mathcal{G}_{\gamma,m}$ such that $m \in D^{\sigma}_{\xi_1,\ell} \in q^{\sigma}_{\xi_1,\ell}$. It follows from the construction that the set $D^{\sigma}_{\xi_1,\ell}$ belongs to all $q^{\sigma}_{\beta,\varkappa}$ such that $\varkappa \in D^{\beta}_{\xi_1,\ell}$ and therefore $D^{\sigma}_{\xi_1,\ell} \in q^{\sigma}_{\beta,k}$, too.

The set $D^{\alpha}_{\beta,k}$ belongs to all $\mathcal{G}_{\sigma,m}$ such that $m \in D^{\sigma}_{\beta,k}$ and $D^{\sigma}_{\beta,k} \in q^{\sigma}_{\beta,k}$.

The set $\bigcup_{j \in B_2} D^{\alpha}_{\xi_2,j}$, where $B_2 \in q^{\xi_2}_{\beta,k}$, belongs to all $\mathcal{G}_{\sigma,m}$ such that $m \in \bigcup_{j \in B_2} D^{\sigma}_{\xi_2,j}$. The ultrafilter $q^{\sigma}_{\beta,k}$ was constructed in such a way that $\bigcup_{j \in B_2} D^{\sigma}_{\xi_2,j} \in q^{\sigma}_{\beta,k}$.

The set $\bigcup_{j \in B_3} D^{\alpha}_{\sigma,j}$ is an element of all $\mathcal{G}_{\sigma,m}$ such that $m \in B_3$ and $B_3 \in q^{\sigma}_{\beta,k}$.

The set B_4 is from $q^{\xi_4}_{\beta,k}$. It follows from the induction that $q^{\sigma}_{\beta,k} < q^{\xi_4}_{\beta,k}$ and therefore there exists a set $E \in q^{\sigma}_{\beta,k}$ and there exist sets $Z_\ell \in q^{\xi_4}_{\sigma,\ell}$ such that $B_4 = \bigcup_{\ell \in E} Z_\ell$. Then for each $\ell \in E$ the set $\bigcup_{j \in Z_\ell} D^{\alpha}_{\xi_4,j}$ belongs to $\mathcal{G}_{\sigma,\ell}$, hence

$\bigcup_{j\in B_4} D^{\alpha}_{\xi_4,j} \in \mathcal{G}_{\sigma,\ell}$. This means that $\bigcup_{j\in B_4} D^{\alpha}_{\xi_4,j} \in \mathcal{G}_{\sigma,m}$ for each $m \in E$.

From the reasons mentioned above the set A belongs to all $\mathcal{G}_{\sigma,m}$ such that $m \in D^{\sigma}_{\xi_1,\ell} \cap D^{\sigma}_{\beta,k} \cap \bigcup_{j\in B_2} D^{\sigma}_{\xi_2,j} \cap B_3 \cap E$. This set is from the ultrafilter $q^{\sigma}_{\beta,k}$. Hence, the validity of the second condition is proved.

It follows from Theorem 2.1 that there exists a stratified set of ultrafilters $\{q^{\alpha}_{\beta,m}\,;\,\beta\in\alpha,\, m\in\omega\}$ such that $q^{\alpha}_{\beta,m} \supseteq \mathcal{G}_{\beta,m}$.

Let $\beta<\sigma<\alpha$ and $k\in\omega$. By the definition of the filter $\mathcal{G}_{\beta,k}$ it holds true that $q^{\sigma}_{\beta,k} \subseteq \{\{\ell\,;\, A\in\mathcal{G}_{\sigma,\ell}\}\,;\, A\in\mathcal{G}_{\beta,k}\}$. This fact ensures that $q^{\sigma}_{\beta,k} < q^{\alpha}_{\beta,k}$.

q.e.d.

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