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Josef Mlček

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COMPACTNESS AND HOMOGENEITY OF SATURATED STRUCTURES II

J. MLČEK

Abstract: We apply the criterion of homogeneity which is presented in the part I of this article, to the study of a homogeneity of the saturated models of the real (algebraically resp.) closed fields and Presbourgher arithmetic. We deduce from the homogeneity of models in question that the mentioned theories are complete. We investigate the problems of undefinability in Presbourgher arithmetic. We obtain, e.g., the assertion that the set of all primes of a given model of Peano arithmetic is not definable in the "additive part" of this model.

Key words: Saturated model, homogeneity, real closed fields, Presbourgher arithmetic, undefinability.

Classification: 03C50, 03C65

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§ 0. Introduction. We have proved, in § 3 of the part I of this article, a criterion of homogeneity for the saturated structures (and some corollaries, following from this homogeneity, too).

To show an applicability of this criterion, we shall study, using the criterion mentioned above, the homogeneity of saturated models of real closed fields, algebraically closed fields and Presbourgher arithmetic. We shall discuss some problems of definability in models of Presbourgher arithmetic, too.

§ 1. We shall formulate, at first, a criterion of homogeneity for a certain class of saturated models for a language

$L = \langle \{f_i\}_\omega , \{c_i\}_\omega , \leq \rangle$, where f_i is an n_i-place function symbol, c_i is a constant, $i \in \omega$, and $\leq$ is a binary relation symbol. We denote by $Tm^{(k)}$ the class of all terms of L with exactly k variables.

Let us introduce now one notion. Given a model $A \models L$ and $C \subseteq |A|$, we denote by Def_C^A the set

$\{a \in |A|;$ a is definable in A by a formula of $L(C)\}$.

Theorem. Let $\mathfrak{M}$ be a class of saturated models for L with the following properties:

(i) $A \in \mathfrak{M} \Rightarrow A \models \leq$ is linear dense ordering without endpoints,

(ii) if $A \in \mathfrak{M}$ and $f(x), g(x) \in Tm^{(1)}(A)$ then

$$A \models (\forall x<y)((f(x) \leq g(x) \,\&\, f(y) \geq g(y)) \rightarrow (\exists z)(x< z< y) \,\&\, \& f(z) = g(z)).$$

Let $^-$ be a closure on $\omega_{\mathfrak{M}}$ in $\mathfrak{M}$. Assume further that

(iii) $^-$ respects $\langle \omega_{\mathfrak{M}}, At \rangle$-similarities,

(iv) if $A \in \mathfrak{M}$, $C \in \omega_{\mathfrak{M}}^A$ is closed and $f(x), g(x) \in Tm^{(1)}(C)$ then

$$(\forall a \in |A|)(A \models f(a) = g(a) \Rightarrow a \in C)$$

hold.

Then (1) $\mathfrak{M}$ is $\langle \omega_{\mathfrak{M}}, At \rangle$-homogeneous.

(2) $A \in \mathfrak{M} \,\&\, C \in \omega_{\mathfrak{M}}^A \Rightarrow Def_C^A \subseteq \bar{C}$.

(3) $A \in \mathfrak{M} \,\&\, C \in \omega_{\mathfrak{M}}^A \,\&\, C$ is closed $\Rightarrow Def_C^A = C$.

Proof. It suffices to prove only the following statements (a),(b):

(a) $^-$ is $\langle \neg(At), 0, \neg\{x \leq y,\ x = y\} \rangle$-stable,

(b) At, $\{x \leq y,\ x = y\}$ are conjugated by $^-$.

(Recall: $\neg\Psi = \Psi \cup \{\neg\psi ;\ \psi \in \Psi\}$.)

(a) Suppose G is a closed At-similarity of two models from $\mathfrak{M}$. We deduce from (i) that G respects types over $\{x \leq y,\ x = y\}^{(1)}$.

(b) Suppose $A \in \mathfrak{M}$ and let $C \in \omega_{\mathfrak{M}}^A$ be closed. Put

$\nabla = \neg\{x \leq y,\ x = y\}$. To prove $\overset{\nabla, A}{\underset{C}{=}} = \overset{At, A}{\underset{C}{=}}$, we must only clear up the inclusion $\subseteq$. Suppose

$$c \overset{\nabla, A}{\underset{C}{=}} d,\ d,\ c < d \text{ and } f(x),\ g(x) \in Tm^{(1)}(C).$$

Assume that there exists e such that $A \models c < e < d \,\&\, f(e) = g(e)$. Thus, $e \in C$, and $c \overset{\nabla, A}{\underset{C}{\neq}} d$, which is a contradiction. We deduce from this (by using the assumption (ii)) that $A \models f(c) \leq g(c) \leftrightarrow$ $\leftrightarrow f(d) \leq g(d)$ which is required. It remains to prove that C is dense in $|A|$ w.r.t. $\overset{\nabla, A}{\underset{C}{=}}$. But this fact is easy and (1) is proved.

The proof of (2),(3) follows from the statement:

if $A \in \mathfrak{M}$ and $C \in \omega_{\mathfrak{M}}$ is a closed subset of $|A|$ then $|[a]_{\overset{\nabla, A}{\underset{C}{=}}}| \geq 2$ holds for every $a \in |A| - C$. (See [4] in § 3 of the part I of this article.) Suppose $a \in |A| - C$ and $[a]_{\overset{\nabla, A}{\underset{C}{=}}} =$ $= \bigcap [c_n, d_n]$ (where $[c,d] = \{x \in |A|;\ A \models c \leq x \leq d\}$) with some $c_n, d_n \in C$, $c_n \leq c_{n+1} < d_{n+1} \leq d_n$, $n \in \omega$. Suppose $|[a]_{\overset{\nabla, A}{\underset{C}{=}}}| = 1$. Then there exists m such that $\{a\} = [a]_{\overset{\nabla, A}{\underset{C}{=}}} = [c_m, d_m]$. But $c_m < d_m$, which is a contradiction.

We shall use the last theorem to show a homogeneity of the class of all saturated models of the theory of real closed fields (RCF).

We can see this theory in the language $L = \langle +, \cdot, 0, 1, \leq \rangle$ and we have $RCF \vdash x \leq y \leftarrow \exists z)(z^2 = y - x)$. Writing At we mean here the class of all atomic formulas of this language L.

<u>Theorem</u>. The class $\mathfrak{M}$ of all saturated models of real closed fields is $\langle \omega_{\mathfrak{M}}, At \rangle$-homogeneous.

Proof. If $A \models RCF$ and $C \subseteq |A|$ we put

$$\bar{C} = \{x \in |A|;\ x \text{ is algebraic over } C_0\},$$

where C_0 is the smallest subfield of A which contains C. The following statements are well known in field theory.

Let $A, B \models RCF$. Then

(1) $A \models \leq$ is linear dense ordering without endpoints,

(2) $f(x), g(x) \in Tm^{(1)}(A) \Rightarrow A \models (\forall x < y)((f(x) \leq g(x) \,\&\, f(y) \geq g(y) \rightarrow (\exists z)(x < z < y \,\&\, f(z) = g(z))$.

(3) Let G be an At-similarity between A, B. Then G can be extended to an isomorphism between $A/\overline{dom(G)}$ and $B/\overline{rng(G)}$ (w.r.t. L).

(4) If $C \subseteq |A|$ then $\bar{\bar{C}} = C$. Suppose $C \subseteq |A|$ is at most countable. Then $\bar{C}$ is countable.

Using these facts and the previous theorem, we can conclude that $\mathfrak{M}$ is $\langle \omega_{\mathfrak{M}}, At \rangle$-homogeneous.

Corollary. (1) The theory of real closed fields is complete.

(2) $(\forall \varphi \in L^{(k)})(\exists \psi \in bool(At))(RCF \vdash \varphi \leftrightarrow \psi)$.

Let us investigate a homogeneity of saturated models of the theory of algebraically closed fields (ACF).

Theorem. The class $\mathfrak{M}$ of all saturated models of algebraically closed fields is $\langle \omega_{\mathfrak{M}}, At \rangle$-homogeneous.

Proof. We want to use the criterion of homogeneity. If $A \models ACF$ and $C \subseteq |A|$, let $\bar{C}$ denote the same as in the previous proof. We have $\bar{\bar{C}} = \bar{C}$ and assuming C countable, we obtain $\bar{C}$ countable, too. The following statement is well known in field theory: Suppose $A_i \models ACF$ and let T_i be a subfield of A_i, $i = 0,1$, G an isomorphism of T_0 and T_1. Then G can be extended to an isomorphism of $\bar{T}_0$ and $\bar{T}_1$. We deduce from the fact presented that $^{-}$ is a closure on $\omega_{\mathfrak{M}}$ in $\mathfrak{M}$ which respects At. (Not yet that an $\langle \omega_{\mathfrak{M}}, At \rangle$-similarity H can be uniquely extended to an isomorphism of $dom(H)_0$ and $rng(H)_0$, where X_0 has the same meaning as in the previous proof.) Put $\Phi_0 = \neg(At)$, $\Phi_1 = 0$ and $\nabla = \{x = y, x \neq y\}$. Then the presumptions of the criterion of homogeneity are satisfied and, consequently, $\mathfrak{M}$ is $\langle \omega_{\mathfrak{M}}, At \rangle$-homogeneous.

§ 2. A homogeneity of models of Presbourgher arithmetic. By Presbourgher arithmetic we mean the theory in $\langle +,1,0 \rangle$ with the following axioms: $x \neq 0 \rightarrow (\exists y)(x = y + 1)$, $x + 0 = x$, $x + 1 \neq 0$, $x + z = y + z \rightarrow x = y$, + is commutative and associative, $(\forall x,y)(\exists z)(x + z = y \vee y + z = x)$ and the schema $\{(\forall x)(\exists y)(x = n.y \vee x = n.y + 1 \vee \ldots \vee x = n.y + n - 1;\ n \geq 1\}$. Here and further on, we write, for a fixed $n \geq 1$, the abbreviation $n.y$ for the expression $y + y + \ldots + y$, n-times.

Let PrA denote again the above theory, extended on the definition

$$x < y \leftrightarrow (\exists z)(z \neq 0 \& x + z = y).$$

Thus, PrA is formulated in the language $L^+ = \langle +, <, 0, 1 \rangle$ and we have PrA $\vdash$ $<$ is discrete linear ordering with 0 and without the greatest element.

Let us denote At^+ the class of all atomic formulas of L^+. Writing

$$x \equiv_m n$$

we mean the formula $(\exists y)(x = m.y + n)$. We put yet

$$Kon = \{x \equiv_m n;\ m \geq 1 \& n \geq 0\}.$$

Proposition. Let $\mathfrak{M}$ be a class of saturated models of

PrA. Then $\mathfrak{M}$ is $\langle \omega_{\mathfrak{M}}, At^{+} \cup Kon \rangle$-homogeneous.

<u>Proof</u>. Put $\Phi_0 = \neg(At^{+})$, $\Phi_1 = \neg Kon$, $\nabla = \neg\{x<y, x = y\}$. Suppose $A \models PrA$ and let $X \subseteq |A|$. We define

$$\overline{X}^{+} = \{ a \in |A| ; (\exists t, \tilde{t} \in Tm^{(1)}(L^{+}(X)))(A \models t(a) = \tilde{t}(a))\}.$$

It is clear that $\overline{}^{+}$ is a closure on $\omega_{\mathfrak{M}}$ in $\mathfrak{M}$ which respects $\Phi_0 \cup \Phi_1$-similarities. It is not difficult to prove that Φ_0 and ∇ are conjugated by $\overline{}^{+}$. Thus, to obtain our statement by using the criterion of homogeneity, we must prove that every $\overline{}^{+}$-closed $\Phi_0 \cup \Phi_1$-similarity G between two models $A, B \in \in \mathfrak{M}$ respects $\Phi_1 \cup \nabla$-types. Let us denote $S = dom(G)$ and suppose $\tau \subseteq (\Phi_1 \cup \nabla)^{(1)}(S)$ is a finite type in A. Since a formula $\neg(x \equiv_n m)$ is equivalent in PrA to a formula of the form $\bigvee_{i=1,\dots,\ell} x \equiv_{n_i} m_i$, we can suppose that $\tau \subseteq (Kon \cup \nabla)^{(1)}(S)$. Assume that $a \in |A|$ realizes τ in A. If $a \in S$ then G(a) realizes τ^{G} in B. Suppose $a \notin S$. Let $\{c_i\}$ be a numbering of $\{c \in S ; A \models c < < a\}$ and let $\{d_j\}$ be a numbering of $\{d \in S ; A \models a < d\}$. We have

$$[a]_{\underset{\overline{S}}{\nabla, A}} = \bigcap_{i,j} [c_i, d_j],$$

where $[c_i, d_j] = \{b \in |A| ; A \models c_i \leq b \leq d_j\}$ if $\{d_j\} \neq 0$ and $[c_i, d_j] = = \{b \in |A| ; A \models c_i \leq b\}$ if $\{d_j\} = 0$. Every interval $[c_i, d_j]$ is infinite, thus, $\bigcap_{i,j}[c_i, d_j]$ contains an infinite interval. We deduce that the intersection $\bigcap_{i,j}[G(c_i), G(d_j)]$ obtains an infinite interval J, too. It is not difficult to prove

<u>Lemma</u>. Let $M \models PrA$ be a saturated model, I an infinite interval in M and suppose that the system

$$(*) \qquad x \equiv_{m_i} n_i, \; i = 1, \dots, \ell$$

has a solution in ω . Then there exists $b \in I$ such that

$$M \models \bigwedge_{i=1,\dots,\ell} b \equiv_{m_i} n_i.$$

(We can find $b_0 \in I$ such that $M \models b_0 \equiv_m 0$ holds for every $m \geq 1$ and, assuming $I = [\alpha, \beta]$, the interval $[b_0, \beta]$ is infinite. If $k \in \omega$ is a solution of $(*)$ then $b = b_0 + k$ is that one which we are looking for.)

Now, suppose that $(*)$ is $\tau \cap \mathrm{Kon}$. Let $b \in J$ be a solution of $(*)$ in B. It is clear that b realizes τ^G in B.

Before deriving some corollaries of this proposition, let us denote PA^+ the Peano additive arithmetic. An explicit way of giving the theory PA^+ is the following: PA^+ is the theory in $\langle +, 0, 1 \rangle$ with the axioms $x + 0 = x$, $x + 1 = 1 + x$, $(x + y) + 1 = x + (y + 1)$, $x + 1 \neq 0$, $x \neq 0 \rightarrow (\exists y)(x = y + 1)$, $x + 1 = y + 1 \rightarrow x = y$, $(\forall x, y)(\exists z)(x + z = y \vee y + z = x)$ and with the schema of induction.

It is not difficult to see that PA^+ is stronger than PrA.

<u>Corollaries</u>. (1) PrA and PA^+ are equivalent.

(2) PA^+ is a complete theory.

(3) $(\forall \varphi \in L^+)(\exists \psi \in \mathrm{bool}(At^+ \cup \mathrm{Kon}))\ PA^+ \vdash \varphi \leftrightarrow \psi$.

(4) The class $\mathcal{M}$ of all saturated models of PA^+ is $\langle \sigma_{\mathcal{M}}, At^+ \cup \mathrm{Kon} \rangle$-homogeneous.

<u>Proof</u>. (1), (2) and (3) follow immediately from the previous proposition.

(4) We use the criterion of homogeneity. Put $\Phi_0 = \neg At^+$, $\Phi_1 = \neg \mathrm{Kon}$, $\nabla = \neg\{x < y, x = y\}$. We deduce from (1) that $\overline{\ \ }^{+}$ countably determines ∇. (It holds because every definable part of each model of PA^+ has the first element and, thus, the monad $[a]_{\underset{\overline{\overline{S}}}{\nabla, A}}$ of an element $a \in |A|$, where $A \models PA^+$ and $S \subseteq |A|$ is a $\sigma_{\{A\}}$-class, has the form $\bigcap_{\omega} [c_i, d_i]$ with some $c_i, d_i \in S$, $i \in \omega$.) Because every formula from Φ_1 has exactly one free variable,

the closure $\overline{}^{+}$ countably determines Φ_1, too. By using these facts, we can deduce similarly as in the previous proof that $\overline{}^{+}$ is $\langle \Phi_0, \Phi_1, \nabla \rangle$-stable closure on $\sigma_{\mathcal{M}}$ in $\mathcal{M}$. It is not difficult to prove that Φ_0 and ∇ are conjugated by $\overline{}^{+}$. Thus, the presumptions of the criterion of homogeneity are satisfied and (4) is proved.

§ 3. Undefinability. Let M be, here and down, a fixed saturated model of Peano arithmetic. Its restriction M^+ on the language L^+ is a saturated model of PrA, too. We shall write down $\overset{\Phi}{\underset{S}{\equiv}}$ instead of $\overset{\Phi, M^+}{\underset{S}{\equiv}}$

Note first that

$$M^+ \text{ is } \langle \sigma_M, At^+ \cup Kon \rangle\text{-homogeneous.}$$

It can be proved quite similarly as the point (4) of the previous corollary.

Criterion of undefinability. Suppose $S \subseteq |M|$ is a $\overline{}^{+}$-closed σ_M-class.

(1) If $F: |M| \to |M|$ is a function such that $F''S \cap (|A| - S) \neq 0$, then F is definable in no S-expansion of M^+.

(2) If $U \subseteq |M|$ is a set such that $(\exists a \in U - S)(\exists$ an infinite interval $I \subseteq [a]_{\underset{S}{\underline{\{x<y, x=y\}}}})(I \cap U = 0)$ then U is definable in no S-expansion of M^+.

Proof. We use the propositions [2] and [3] in § 3, part I.

(1) Suppose $a \in S$ and $F(a) \notin S$ hold. The class $[F(a)]_{\underset{S}{\underline{\{x<y, x=y\}}}}$ contains an infinite interval I. We have proved above that there exists $b \in I$ such that $M^+ \models F(a) \equiv_i b$ holds for each $i \geq 1$. Thus $|[F(a)]_{\underset{S}{\underline{At^+ \cup Kon}}} \cap (M - S)| \geq 2$.

(2) can be proved similarly.

Unary predicate undefinability. Let $U \subseteq |M|$. We say that U has i-property iff the following holds: if $I \subseteq |M|$ is an infinite interval then there exists an infinite interval $J \subseteq I$ such that $J \cap U = 0$.

Proposition. Suppose that $U \subseteq |M|$ has i-property and let X be a σ_M-class such that $(M - \omega) \cap (U - \overline{X}^+) \neq 0$ holds. Then U is definable in no $\overline{X}^+$-expansion of M^+.

Proof. Assume $a \in (M - \omega) \cap (U - \overline{X}^+)$. The class $[a]_{\underline{\underline{\{x<y, x=y\}}}}^{\overline{\overline{X}}^+}$ contains an infinite interval I. Thus, there exists an infinite interval $J \subseteq I$ such that $J \cap U = 0$. The required conclusion follows from the previous criterion.

Let us give some examples of sets which have i-property. We use the following notations: we put, for every $\xi \in |M|$, $\xi^{(M)} = \{\xi^{\alpha};\ \alpha \in |M|\}$ and $M^{(\xi)} = \{\alpha^{\xi};\ \alpha \in |M|\}$.

(1) If $1 < \xi \in |M|$ then both $\xi^{(M)}$ and $M^{(\xi)}$ have i-property and, consequently, they are not definable in M^+.

(2) The class $Prm^M = \{a \in |M|;\ M \models a \text{ is prime}\}$ has i-property. Thus, Prm^M is not definable in M^+.

Proof. Assume $I \subseteq |M|$ is an infinite interval, $I = [a,b]$. There exists $c \in [a, \frac{a+b}{2}]$ such that $M \models c \equiv_i 0$, $i \geq 1$. We have $Prm^M \cap [c + 2, c + n] = 0$, $n \geq 2$. Thus, there exists an $\eta \in |M| - \omega$ such that $Prm^M \cap [c + 2, c + \eta] = 0$.

(3) Assume that $S \subseteq |M|$ is a σ_M-class such that

$(\forall a \in S)(\forall n \geq 1)(M \models a \equiv_n 0)$.

Then the predicate $2^{(M)}$ is definable in no $\overline{S}^+$-expansion of M^+.

Proof. Suppose $\alpha \in \overline{S}^+$. Then there exist $c_i, d_i, m \in \omega$, $c \in Z$ and $\gamma_i, \delta_i \in S$, $i \leq k$ such that $m \cdot \alpha = \sum_k c_i \gamma_i - \sum_k d_i \delta_i + c$.

Thus $m|c$ and we have $\alpha = \beta + b$, where $b \in Z$ and $\beta \equiv_m 0$ holds for every $m \geq 1$. Suppose $\gamma \in |M| - \omega$ and let $2^{\gamma} = \alpha$ ($= \beta + b$). It is clear that $|b| = 2^m$ for some $m \in \omega$. We deduce from this that $2^m(2^{\gamma - m} \pm 1) = \beta$ holds. But $2^{m+1} | \beta$ and, consequently, $2 | 2^{\gamma - m} \pm 1$, which is a contradiction. Thus $(2^{(M)} \cap \overline{S}^+) \cap (M - \omega) = 0$ and the required statement follows from the previous proposition.

<u>Unary function undefinability</u>. Let us range $r, r_1, i \in \omega$ over standard rationals. Put

$$K = \{x;\ x \text{ is rational over } M \,\&\, x \geq 0\}.$$

We define, for each $x, y \in K$:

$$x \sim y \leftrightarrow (\exists m \in \omega)(\tfrac{1}{m} \cdot x < y < m \cdot x).$$

It is clear that $\sim$ is an equivalence on K and the class $\{[a]_{\sim} \cap |M|;\ a \in |M|\}$ is dense ordered by $<^M$ (i.e. assuming $a, b \in |M|$, $a <^M b$ and $a \not\sim b$, we can find $c \in |M|$ such that $a < c < b$ and $a \not\sim c$, $b \not\sim c$. The following properties of $\sim$ hold for every $x, y \in K$.

(a) $x \sim x' \,\&\, y \sim y' \rightarrow x + y \sim x' + y'$, (b) $x + y \sim \mathrm{Max}\{x, y\}$,

(c) $r \geq 1 \rightarrow r \cdot x \sim x$, (d) $x \not\sim y \rightarrow (x - y \geq 0 \rightarrow x - y \sim x)$.

To simplify the next formulas we put, for every $\delta \in |M|$,

$$\check{\delta} = \{\alpha \in |M|;\ \delta \leq \alpha\}.$$

Let $X \subseteq |M|$. We say that X is $\sim$ <u>-dispersed</u> iff $(x, y \in X \,\&\, x \neq y) \rightarrow x \not\sim y$ holds for every $x, y \in X$. X is said to be <u>almost</u> $\sim$ - <u>dispersed</u> iff there exists $\delta \in |M|$ such that $X \cap \check{\delta}$ is $\sim$ -dispersed.

Let us denote yet by $[X]_{\sim}$ the set $\bigcup\{[x]_{\sim} \cap |M|;\ x \in X\}$.

<u>Lemma</u>. Suppose that $X \subseteq |M|$ is almost $\sim$ -dispersed. Then there exists $\delta \in |M|$ such that

$(\forall \xi \geq \sigma)(X \cap \check{\xi}$ is $\sim$-dispersed $\& (\overline{X}^+ \cap \check{\xi}) \subseteq [X \cap \check{\xi}]_\sim)$.

This lemma follows immediately from the definitions and (d).

Let $F: |M| \to |M|$ be a function. F is called $\sim$ _-regular_ iff

(1) $a \in |M| - \omega \Rightarrow F(a) \not\sim a$,

(2) suppose that $I \subseteq |M|$ is an interval such that $(\exists x \in I)([x]_\sim \cap |M| \subseteq I)$ holds. Then there is no $\sim$-dispersed class $Y \in \mathfrak{D}^1_M$ such that $F''I \subseteq [Y]_\sim$.

Proposition. Assume that $F \in \mathfrak{D}^{\leq}_M$ is a $\sim$-regular increasing function and let $X \in \mathfrak{D}^1_M$ be an almost $\sim$-dispersed part of $|M|$. Then F is definable in no $\overline{X}^+$-expansion of M^+.

Proof. Note first the following: Let $\varphi(x_1,\dots,y_1,\dots) \in L(M)$. Then $(\forall x_1 \dots)(\exists m_1 \dots) M \models \varphi(x_1,\dots,m_1,\dots)$ iff $(\exists m_1 \dots) M \models (\forall x_1 \dots)(\exists y_1 \leq m_1 \dots)\varphi(x_1,\dots,y_1,\dots)$ holds. It follows immediately from the saturativity of M.

Choose $\sigma \in |M| - \omega$ such that $X \cap \check{\sigma}$ is $\sim$-dispersed and $\overline{X}^+ \cap \check{\sigma} \subseteq [X \cap \check{\sigma}]_\sim$. Let φ be the formula

$(\forall x,y \in X)((x,y > \sigma \,\&\, x < y \,\&\, [x,y] \cap X = \{x,y\}) \to$

$\to (\forall z \in [x,y])(x \not\sim z \not\sim y \to F(z) \in [X]_\sim \cap \check{\sigma}))$.

(We denote by $[x,y]$ the interval with endpoints x,y.)

Our aim is to prove that $M \models \neg\varphi$. Assume $M \models \varphi$. By using the first fact of this proof we can see that there exists m and n such that $M \models (\forall x,y \in X)((x,y > \sigma \,\&\, x < y \,\&\, [x,y] \cap X = \{x,y\}) \to (\exists \overline{v} \leq m, \overline{w} \leq n)(\forall z \in [x,y])(\exists v_z \leq \overline{v}, w \leq \overline{w})((v_z \cdot x < z \,\&\, \& \, z \cdot v_z < y) \to (\exists x \in X)(x < w \cdot F(z) \,\&\, F(z) < w \cdot x)))$. Let $x,y \in |M|$ be fixed, $x,y > \sigma$, $x < y$ and $[x,y] \cap X = \{x,y\}$. Choose $z \in [x,y]$ such that $m \cdot x < z \,\&\, m \cdot z < y$. Then $v_z \cdot x \leq m \cdot x < z$ and $v_z \cdot z \leq m \cdot z < y$. Thus, $F(z) \in [X]_\sim \cap \check{\sigma}$ holds.

The interval $[m \cdot x, \frac{1}{m} \cdot y]$ contains an element t such that

$[t]_\sim \subseteq [m\cdot x, \frac{1}{m}\cdot y]$.

We have just proved that $F''[m\cdot x, \frac{1}{m}\cdot y] \subseteq [X]_\sim \cap \check{\sigma}$, which is a contradiction with our assumption that F is $\sim$-regular. Thus, $M \models \neg \varphi$ is true.

Choose $x, y \in |M|$ such that $x, y \in X \cap \check{\sigma}$, $[x,y] \cap X = \{x,y\}$, $x < y$, and let $a \in [x,y]$ be such that $x \not\sim a \not\sim y$, $F(a) \notin [X]_\sim \cap \check{\sigma}$. We have $F(a) \notin [X \cup \{a\}]_\sim$ and, consequently, $F(a) \notin \overline{X \cup \{a\}}^+$. (Note that the relation $\overline{X \cup \{a\}}^+ \cap \check{\sigma} \subseteq [(X \cap \check{\sigma}) \cup \{a\}]_\sim$ follows from the fact that $(X \cap \check{\sigma}) \cup \{a\}$ is $\sim$-dispersed.) Now, the required statement follows immediately from the criterion of undefinability.

Examples. (1) If $\xi \in |M| - \omega$ then $\xi^{(M)}$ is $\sim$-dispersed.

(2) Every function x^n, $n \geq 2$, is $\sim$-regular.

(3) Every function n^x, $n \geq 2$, is $\sim$-regular.

Proof. (1) is quite clear. (2),(3): Let $n \geq 2$ be fixed. Conversely, suppose that there exist an infinite interval $[\alpha, \beta]$ in M and a class $Y \in \mathcal{D}^1_M$ such that Y is $\sim$-dispersed and $F''[\alpha, \beta] \subseteq [Y]_\sim$, where F is x^n or n^x.

(2) The monads $\{[a]_\sim \cap |M|; a \in [\alpha, \beta]\}$ are dense ordered by $<$ and $x \sim y \leftrightarrow x^n \sim y^n$ holds for every $x, y \in |M|$. But the monads $\{[a]_\sim \cap |M|; a \in Y\}$ are not dense ordered by $<$, which is a contradiction.

(3) Put, for every $x, y \in |M|$, $x \approx y \leftrightarrow |x - y| \in \omega$. Then $\approx$ is an equivalence on $|M|$ and the relation $x \approx y \leftrightarrow n^x \sim n^y$ holds for every $x, y \in |M|$. The monads $\{[a]_\approx : a \in [\alpha, \beta]\}$ are dense ordered by $<$, but the monads $\{[a]_\sim \cap |M; a \in Y\}$ are not, which is a contradiction.

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Matematický ústav University Karlovy, Sokolovská 83, 18600 Praha 8, Czechoslovakia

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