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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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THE CONSISTENCY OF THE MEASURABILITY OF PROJECTIVE SEMISETS

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Abstract: The notion of projective classes (roughly speaking classes defined from internal objects and the predicate "to be an infinitely large number") is introduced. The consistency of the existence of a good approximation of projective semisets by sets is proved using Solovay's consistency of Lebesgue measurability of projective sets.

Key words: Projective class, projective set, Lebesgue measure, alternative set theory, nonstandard analysis, figure, totally disconnected indiscernibility equivalence.

Classification: Primary 03E70
Secondary 03E35

Introduction. In the nonstandard work we often treat the classes (relations) defined with the help of internal objects and the predicate "to be an infinitely large (or an infinitely small) number" As we prove in the paper, these classes are connected very closely with the classical projective sets and therefore we call them projective classes. We investigate the possibility of the approximation of these classes being moreover parts of *finite sets (projective semisets) by *finite sets. The approximation is closely related with the Lebesgue measurability as can be found also in the works of Loeb and Anderson. Using the Solovay's proof of the consistency of Lebesgue measurability of projective sets we obtain here the proof of the consistency of a good approximation for every projective semiset. The given result is interesting also from the point of view of nonstandard treatment with quantities.

This result has been proved in the author´s CSc-thesis published in Czech language in 1976.

As a suitable technical framework for our investigations we take the theory of semisets with designated classes from Balcar´s CSc-thesis. AST (by this we mean the special system of axioms described in [S1]) or some theories formalizing nonstandard analysis (e.g. this one of [N] or of [Č]) may be understood as some special cases of the used theory similarly as e.g. ZFC is a special case of a first order theory. The used theory suits also very well for work with nonstandard models of arithmetic. By our meaning the given results may be of some interest in the mentioned three branches of mathematics. From [V] we use also some notions which clear up the original text and have been arisen later.

§ 1. Preliminary considerations. We use three sorts of variables: x,y,... for sets , X ,Y,... for designated classes and classes. These three sorts of variables are subordinate in the given order: every set is a designated class and every designated class is a class. The fact that a class X is a designated class is expressed by Dsg(X).

If we restrict ourselves on sets and designated classes then our axioms are all the axioms of a special case of G.B. theory of sets. The axiom of infinity, regularity and choice are irrelevant for our considerations. Let us note that for every designated class X we have $(\forall x)(\exists y)(X \cap x = y)$.

For classes we use the following axioms.

1) $(\exists Y)(X \subseteq Y) \Rightarrow (\exists x)(X=x)$ (Sets)

2) $(\forall x)(x \in X \equiv x \in Y) \Rightarrow X=Y$ (Extensionality)

3) If $\varphi(x,X_1,\dots,X_n)$ is a formula in which only set variables are quantified (i.e. a normal formula), then

$(\forall X_1,\dots,X_n)(\exists Y)(x \in Y \equiv \varphi(x,X_1,\dots,X_n))$. (Scheme of existence)

Let us note that, due to results of Gödel, the last scheme can be exchanged by a finite number of special cases and that e.g. an ordered tuple $\langle x,y \rangle$ exists due to the first group of axioms (G.B. set theory). Theories of the given type have been introduced and investigated in [BB]. A special case of such a theory is AST, where the role of designated classes play e.g. Sd_V classes (or Sd_V^* classes). One can see that the theories of the given type are suitable for considerations about models, since sets and designated classes characterize internal objects and classes characterize external objects (maybe only some suitably chosen ones).

Notation: We use common notation. Moreover we use α, β, γ, ... for infinitely large natural numbers (introduced below) and σ, $\wp$, ς, ζ, ... for semisets (subclasses of sets). By Card(x) or $|x|$ we denote the cardinality of x (from the point of view of G.B. set theory); thus $|x| = \varkappa \equiv (\exists f)(f{:}x \leftrightarrow \varkappa)$. $|x|$ we use also for the absolute value in the context of the real numbers system. We use further N for the set or the designated class of natural numbers (i.e. ordinal numbers less than the first limit ordinal in the view point of G.B. set theory).

<u>Axiom FN</u>: $(\exists X \subsetneq N)(\forall n \in X)(n \subset X \;\&\; (\forall Y)(\exists y)(Y \cap n = y))$.

It appears that there is only one class X with the property described in the axiom and we denote this class by FN. The first part of the conjunction describes the completeness of FN and the second one may be understood as the standardness of elements of FN.

<u>Definition 1.1</u>: $IL(\alpha) \equiv \alpha \in N - FN$ (α is infinitely large)

<u>Definition 1.2</u>: $Fin(X) \equiv (\forall Y \subset X)(Y \in V)$

Here we are not consistent with the Tarski's definition.

Definition 1.3 $\mathrm{Count}(X) \equiv (\exists F)(F : X \leftrightarrow FN)$

Here we are also inconsistent with the commonly used definition. Our definitions of Fin(X) and Count(X) are consistent with the external meaning of these notions.

Axiom of prolongation: For every countable function F there is a set function f such that $F \subset f$.

Axiom of weak choice: $(\forall R, \mathrm{dom}(R)=FN)(\exists F, \mathrm{dom}(F)=FN)$ $(F \subset R \,\&\, F$ is a function).

Let us note that this axiom is not a consequence of the axiom of choice for designated classes, as the well ordering of V given by this axiom is the well one only with respect to designated classes.

Let us now remember some definitions and assertions from [CC].

Definition 1.4: $\mathrm{Dep}(X,Y) \equiv (\exists R, \mathrm{Dsg}(R))(X=R''Y)$

(X is dependent on Y)

$\mathrm{Dep}_d(X,Y) \equiv (\exists F, \mathrm{Dsg}(F))$ (F is a function $\& \; X=(F^{-1})''Y$)

(X is disjointly dependent on Y).

Both the notions are reflexive and transitive. Note that if X is a semiset then it is possible to assume the relations to be sets instead of designated classes.

Lemma 1.4: 1) $\mathrm{Dep}_d(X,Y) \Rightarrow \mathrm{Dep}(X,Y)$

2) $Z \neq 0 \Rightarrow (\forall X, \mathrm{Dsg}(X))\mathrm{Dep}(X,Z)$

3) $\mathrm{Dep}(X,Z) \,\&\, \mathrm{Dep}(Y,Z) \Rightarrow \mathrm{Dep}(X \cup Y, Z)$

4) $\mathrm{Dep}(X,Z) \Rightarrow (\forall A, \mathrm{Dsg}(A))\mathrm{Dep}(X-A,Z)$

$\mathrm{Dep}_d(X,Z) \Rightarrow (\forall A, \mathrm{Dsg}(A))\mathrm{Dep}_d(X-A,Z)$

5) $\mathrm{Dep}(X_1,Z_1) \,\&\, \mathrm{Dep}(X_2,Z_2) \Rightarrow \mathrm{Dep}(X_2 \times X_1, Z_1 \times Z_2) \,\&$

$\& \, Dep(X_1 \cap X_2, Z_1 \times Z_2)$

6) $Dep(dom(X),X) \,\&\, Dep(V \times X,X) \,\&\, Dep(Cnv(X),X) \,\&\, Dep(Cnv_3(X),X)$

7) $Dep(X,\sigma) \,\&\, a \supset \sigma \Rightarrow Dep_d(X,\mathcal{P}(a)-\mathcal{P}(a-\sigma))$

$Dep(\mathcal{P}(a)-\mathcal{P}(a-\sigma),\sigma)$

8) $Dep_d(X,Y) \,\&\, X \subset A \,\&\, Y \subset B \,\&\, Dsg(A) \,\&\, Dsg(B) \Rightarrow Dep_d(A-X,B-Y)$

9) $Dep(\mathcal{P}(X) \times \mathcal{P}(X),\mathcal{P}(X)) \,\&\, Dep(\mathcal{P}(\mathcal{P}(X)),\mathcal{P}(X))$

10) If $\sigma \subset a \,\&\, Dep(a-\sigma,\sigma)$ then $Dep_d(\sigma,\mathcal{P}(a-\sigma))$

Proof: All the mentioned assertions are proved by constructions of suitable designated relations of dependence. For point 7) let us note that $\mathcal{P}(a)-\mathcal{P}(a-\sigma)$ is the class of all subsets of a having a nonempty intersection with σ To prove 10) let us note that $Dep_d(a-\sigma,\mathcal{P}(a)-\mathcal{P}(a-\sigma))$ (see 7)) and remember 8).

In this paper we shall use only 5),6) and namely for proving the following theorem. The other assertions are given here for their importance in other branches and to make the reader more familiar with the important notion of dependence.

If we use further the words "let $\mathcal{S}$ be a system of classes", we mean by this (from the formal point of view) that we introduce a new sort of variables subordinal to the sort of classes. We use (not quite correctly) $X \in \mathcal{S}$ to denote that the class X is of this sort. A special case (and from a point of view the only one used in the paper) is the description of the sort by a formula $(X \in \mathcal{S} \equiv \mathcal{S}(X))$. Codable classes are another more special case (see [V]).

<u>Theorem 1.5</u>: If a nonempty system of classes $\mathcal{S}$ has the following properties

1) $\mathcal{S}$ is closed on the complement to the universal class V $(X \in \mathcal{S} \Rightarrow V-X \in \mathcal{S})$

2) $\mathcal{S}$ is closed on the cartesian product $(X,Y \in \mathcal{S} \Rightarrow X \times X \in \mathcal{S})$

3) $\mathcal{S}$ is closed on Dep($X \in \mathcal{S}$ & Dep(Y,X) $\Rightarrow Y \in \mathcal{S}$)
then $\mathcal{S}$ is closed on definition by normal formulas.

Proof: It is sufficient to prove that $\mathcal{S}$ is closed on Gödelian operations. We prove at first that all the designated classes are in $\mathcal{S}$. To prove this it suffices to prove that there is a non-empty class in $\mathcal{S}$. If $0 \in \mathcal{S}$ then $V-0=V \in \mathcal{S}$ by 1). If R is the designated relation such that $dom(R)=\{\langle x,x\rangle; x \in V\}$ and $R''\{\langle x,x\rangle\} = \{x\}$ (R is e.g. the first projection restricted on identity) then $X \cap Y = R''(X \times Y)$. We have also $X-Y = X \cap (V-Y)$. X^{-1} and $Cnv_3(X)$ can be obtained as images of X using suitable definable (thus designated) functions. Similarly $dom(X) = Pr_2''X$ where $Pr_2(\langle y,x\rangle)=x$. $E = \{\langle x,y\rangle; x \in y\}$ is designated and $X \upharpoonright Y = X \cap (V \times Y)$.

Now we define the notion of compactness for symmetric and reflexive relations.

<u>Definition 1.6</u>: Let R be reflexive and symmetric. R is said to be compact iff $(\forall x \subseteq dom(R))(\neg Fin(x) \Rightarrow (\exists y,z \in x)(y \neq z\ \&\ \langle y,z\rangle \in R))$.

Remember that Fin(x) denote $(\forall X \subseteq x)Set(X)$ and not e.g. the Tarski's definition.

Now we adapt the notion of an indiscernibility equivalence from [V]. Every indiscernibility equivalence from our paper satisfies properties in [V] but e.g. the relation $\doteq$ from [V] does not satisfy our properties if we interpret Dsg(X) as $Sd_V(X)$. If we interpret Dsg(X) as $X \in Sd_V^*$ (see [SV2]) then every indiscernibility equivalence from [V] is an indiscernibility equivalence in our sense but there are i.e. in our sense (being not real classes) which are not i.e. in the sense of [V].

<u>Definition 1.7</u>: A) Let G be a designated system of reflexi-

ve and symmetric relations (i.e. $Dsg(G) \& dom(G) \in N \& (\forall \alpha \in$ $\in dom(G))(G''\{\alpha\}$ is a reflexive and symmetric relation)).

G is said to be a generating system iff

1) $dom(G) \in N-FN \& (\forall \alpha, \beta \in dom(G))(dom(G''\{\alpha\}) = dom(G''\{\beta\}))$

2) $(\forall n \in FN)(G''\{n\}$ is compact)

3) $(\forall \alpha \in dom(G) \& \alpha \neq 0)(G''\{\alpha\} \circ G''\{\alpha\} \subseteq G''\{\alpha -1\}$

B) If G is a generating system then $\bigcap\{G''\{n\}; n \in FN\}$ is said to be an indiscernibility equivalence.

C) If for an indiscernibility equivalence there is a generating system consisting of equivalences, then this equivalence is said to be totally disconnected.

It is easy to see (using 3)) that every indiscernibility equivalence is really an equivalence. To get a better survey about indiscernibility equivalences and to get some coherence with the classical topology, we recommend the reader Chapter 3 of [V].

Theorem 1.8: Every indiscernibility equivalence is compact.

Proof: Let a be an infinite set and $a \subseteq dom(\doteq)$. Let G be a generating system. We have $(\forall n \in FN)(\exists x,y \in a)(x \neq y \& \langle x,y\rangle \in$ $\in G''\{n\})$. Using overspill we obtain the given property also for some $G''\{\alpha\}$ where $\alpha \in N-FN$ and so we obtain x,y such that $x \doteq y$.

Note that the axiom of prolongation is not used in the given proof. The proof from [V] of the existence of an infinite set of near elements in any infinite set can be also adapted.

If $\doteq$ is an indiscernibility equivalence then $dom(\doteq)/\doteq$ forms a compact separable (thus metrizable) topological space. If $\doteq$ is moreover totally disconnected then the given topological space is totally disconnected, too.

We now describe three ways how to understand this assertion. For $X \subseteq dom(\doteq)$ we define (cf. [V]) $\bar{X} = \{x \in dom(\doteq); (\forall y \subseteq$

$\subseteq \mathrm{dom}(\stackrel{\circ}{=}))((\forall t \in X)(\exists u \in y)(t \stackrel{\circ}{=} u) \Rightarrow (\exists u \in y)(x \stackrel{\circ}{=} u))$

1) If we work in nonstandard analysis then the closure $\bar{a}$ for $a \subseteq \mathrm{dom}(\stackrel{\circ}{=})/\stackrel{\circ}{=}$ we define as follows: Let $A \subseteq \mathrm{dom}(\stackrel{\circ}{=})$ be the preimage of a (the class of all elements of equivalence classes of $\stackrel{\circ}{=}$ being in a). Let now $\bar{A}$ be defined as above. We put then $\bar{a} = \bar{A}/\stackrel{\circ}{=}$.

2) If we work in a nonstandard model of arithmetic, where FN is understood as ω (standard natural numbers), then we understand our assertion in the metatheory in which we work with the nonstandard model analogously as in 1). For the compactness (and namely the completeness) we need the validity of the axiom of prolongation in the model (e.g. the saturatedness of the model is sufficient).

3) If we work in a fundamental mathematical theory based on nonstandardness (e.g. AST) then we understand $\mathrm{dom}(\stackrel{\circ}{=})/\stackrel{\circ}{=}$ as the structure of parts of $\mathrm{dom}(\stackrel{\circ}{=})$ saturated on $\stackrel{\circ}{=}$ $(x \in X \,\&\, x \stackrel{\circ}{=} y \Rightarrow y \in X)$ with the above defined operation of closure.

We shall not prove here the above mentioned properties of the space $\mathrm{dom}(\stackrel{\circ}{=})/\stackrel{\circ}{=}$. To the reader interested in it we recommend Ch. 3 from [V]. In the paper only closed bounded intervals of real numbers and compact separable totally disconnected spaces are of our interest. When using in the following text the notions set, function etc. for objects in the factor structure, we keep in mind notions from the classical point of view. Such objects need not be e.g. sets from the point of view of our theory.

Let us give two examples for the illustration. Let α be an infinitely large natural number.

1) Put $a = \{\beta/\alpha\,;\, \beta < \alpha\}$ - a set of rational numbers. Put $G''\{\sigma\} = \{\langle \beta_1/\alpha, \beta_2/\alpha \rangle;\, |\beta_1 - \beta_2|/\alpha \leq 1/\sigma\}$. G is a generating system for the indiscernibility equivalence $\doteq$ such that $x \doteq y$ iff x and y are infinitely near. If we suppose that for every $X \subseteq FN$ (i.e. in the point of view of nonstandard models of arithmetic

$X \subseteq \omega$ is a set from the metatheory) there is x such that X = = $FN \cap x$ (i.e. the standard system of the model is $\mathcal{P}(\omega)$) then $dom(\doteq)/\doteq$ is the closed interval [0,1] with the common topology.

2) Put $a = \{\beta/2^{\alpha}; \beta < 2^{\alpha}\}$. Put $G''\{\delta\} = \{\langle x, y\rangle; x, y \in a \,\&\, \& \lfloor x \cdot 2^{\delta} \rfloor = \lfloor y \cdot 2^{\delta} \rfloor\}$ where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. G is a generating system of a totally disconnected indiscernibility equivalence $\doteq$ such that $a/\doteq$ is Cantor´s discontinuum (if suitable properties hold).

In the following text we use common notions introduced on the corresponding factorspaces $dom(\doteq)/\doteq$ and namely the notions of projective sets and the Lebesgue measure. We use these notions for introducing a remarkable system of projective classes and for an orientation in this system and also for considerations on "an approximation" of semisets by sets and the connection of this approximation with the Lebesgue measure. In our considerations we use nontrivial results concerning the mentioned classical structures.

Let us now introduce notions and notations simplifying the duality - standard and nonstandard - of our language.

Definition 1.9: $Fig_{\doteq}(X) = (\doteq)''X$ (the figure of X)
$\mathcal{F}ig_{\doteq}(X) \equiv X = Fig_{\doteq}(X)$ (X is a figure).

We shall omit the subscript $\doteq$ if there is no danger of confusion. We shall use also the notation Fig(M) for the preimage of M in the factorization in spite of the fact that the factorization need not be done by choosing representants for the equivalence classes.

Definition 1.10: Let $\overset{1}{\doteq}, \overset{2}{\doteq}$ be indiscernibility equivalences. We define on $dom(\overset{1}{\doteq}) \times dom(\overset{2}{\doteq})$ the indiscernibility equivalence $\overset{1\times 2}{\doteq}$ by the description $\langle x_1, x_2\rangle \overset{1\times 2}{\doteq} \langle y_1, y_2\rangle \equiv x_1 \overset{1}{\doteq} y_1 \,\&\, x_2 \overset{2}{\doteq} y_2$.

We omit the easy proof of the correctness of the definition i.e. fiding out a suitable generating system.

Theorem 1.11: If $\overset{1}{=},\overset{2}{=}$ are totally disconnected then $\overset{1\times 2}{=}$ is totally disconnected.

The proof is obvious.

Theorem 1.12: If $X \subseteq \mathrm{dom}(\overset{\times}{=})$ & $\mathrm{Dsg}(X)$ then $\mathrm{Fig}(X)$ is closed.

Proof: Adapt the analogue in [V].

Definition 1.13: Let $\overset{1}{=},\overset{2}{=}$ be indiscernibility equivalences. Let F be a designated (partial) function from $\mathrm{dom}(\overset{1}{=})$ to $\mathrm{dom}(\overset{2}{=})$. F is said to be continuous iff $x \overset{1}{=} y \Longrightarrow F(x) \overset{2}{=} F(y)$.

Theorem 1.14: If Dsg(F) and F is continuous then $\mathrm{Fig}_{2\times 1}(F)/\overset{2\times 1}{=}$ is a continuous function w.r.t. the corresponding factor spaces.

Proof: The factor object $\mathrm{Fig}_{2\times 1}(F)/\overset{2\times 1}{=}$ is a function which is closed in the product space and the basic spaces are compact.

Theorem 1.15: If $\overset{1}{=},\overset{2}{=}$ are in.eq. then $\overset{1}{=}\cap\overset{2}{=}$ is an in.eq. If moreover $\overset{1}{=},\overset{2}{=}$ are totally disconnected then $\overset{1}{=}\cap\overset{2}{=}$ is also totally disconnected.

The proof is obvious.

Theorem 1.16: Let Dsg(F) and $\mathrm{rng}(F) \subseteq \mathrm{dom}(\overset{\times}{=})$. On the dom(F) let us define $x \overset{+}{=} y \equiv F(x) \overset{\times}{=} F(y)$. $\overset{+}{=}$ is an indiscernibility equivalence. In addition, if $\overset{\times}{=}$ is totally disconnected then $\overset{+}{=}$ is also totally disconnected.

Proof: Using the generating system for $\overset{\times}{=}$ and the disjointed relation F^{-1}, construct the generating system for $\overset{+}{=}$.

Theorem 1.17: If G is a generating system for a totally disconnected indiscernibility equivalence $\overset{\times}{=}$ on x then there is a function $f: \longleftrightarrow card(x)$ such that for every equivalence class b ($b=(G''\{\sigma\})''\{t\}$ & $t \in b$), f"b is a connected (w.r.t. the natural ordering of N) interval in card(x). Formally:

$(\forall \sigma \in dom(G))(\forall t \in x)(\forall \alpha, \beta, \gamma \in card(x))((\alpha < \beta < \gamma$ & & $\alpha, \gamma \in f''((G''\{\sigma\})''\{t\})) \Rightarrow \beta \in f''((G''\{\sigma\})''\{t\}))$

Proof: We construct f by the recursion based on the dom(G). Let $I_1^1, \ldots, I_k^1$ be the equivalence classes of $G''\{1\}$ and $f_i^1 : I_i^1 \longleftrightarrow$ $\longleftrightarrow |I_i^1|$. For $t \in I_i^1$ we put $f^1(t) = \sum_{j<i} |I_j^1| + f_i^1(t)$. Analogously we proceed with the construction of the automorphisms in the framework of the intervals $|I_i^n|$.

Theorem 1.18: If G is a generating system of a totally disconnected indiscernibility equivalence $\doteqdot$ on α consisting of partitions on intervals then there is a function f such that $f: \alpha \xrightarrow{1-1} \{\beta/\alpha ; 0 \leq \beta < 2\alpha\}$, $f(x) \doteq f(y) \Rightarrow x \doteqdot y$ and $Fig(rng(f))/\doteq$ is a closed set with the Lebesgue measure 1.

For the construction of f we adapt the idea from the classical construction of Cantor's discontinuum. Let us firstly define one technically useful notion. We put $b = \{\beta/\alpha ; 0 \leq \beta < 2\alpha\}$.

Definition 1.19: For $x \in b$ we denote by st(x) the standard real number $\mu(x)/\doteq$ (where $\mu(x) = Fig(\{x\}) = \doteq''\{x\}$).

Now we prove Th. 1.18.

Proof: Let I_j^i denote the intervals of the partition $G''\{i\}$. We define (for the construction of f) also the rational numbers ε_j^i, $\bar{\varepsilon}_j^i$ corresponding to the intervals I_j^i. We proceed by the recursion based on i. $\varepsilon_1^0 = 1$, $\bar{\varepsilon}_j^i = \varepsilon_j^i / 2\cdot((k \dot{-} 2)+1)$, $\varepsilon_p^{i+1} = \varepsilon_j^i/2k$, where k denotes the number of the subintervals of I_j^i, $\dot{-}$ denotes the natural subtraction ($x \dot{-} y = max(0, x-y)$) and I_p^{i+1} is a subinterval of I_j^i. If we divide intervals every time on two parts then

$\varepsilon^i_j = 2^{-2i}$. Note that we have $\sum_j \varepsilon^i_j = 2^{-i}$. We define (by recursion) intervals $\bar{I}^i_j \subseteq b$ corresponding to the intervals I^i_j. We put $\bar{I}^0_1 = b$. Put $\gamma = \mathrm{dom}(G)$. Suppose also that $G''\{0\} = \alpha \times \alpha$. For $i+1 < \gamma - 1$ we define $\bar{I}^{i+1}_j$ by the following manner: If I^{i+1}_j is the lower (in the ordering of α) subinterval of I^i_k then let $\bar{I}^{i+1}_j$ be the subinterval of $\bar{I}^i_k$ having the following properties: $\min(\bar{I}^i_k) = \min(\bar{I}^{i+1}_j)$ and $\mathrm{card}(\bar{I}^{i+1}_j) = \mathrm{card}(I^{i+1}_j) + \lfloor \varepsilon^{i+1} \cdot \alpha \rfloor$ (note that $\lfloor x \rfloor$ denote the largest integer $\leq x$). If I^{i+1}_j is the upper (in the ordering of α) subinterval (and different from the lower one) we proceed analogously, but we need the equality $\max(\bar{I}^i_k) = \max(\bar{I}^{i+1}_j)$ instead of the equality of min. In the other cases we need, except for the property $\mathrm{card}(\bar{I}^{i+1}_j) = \mathrm{card}(I^{i+1}_j) + \lfloor \varepsilon^{i+1}_j \cdot \alpha \rfloor$, the property $\mathrm{card}(\{x \in b;\ \max(\bar{I}^{i+1}_j) < x < \min(\bar{I}^{i+1}_{j+1})\}) = \lfloor \bar{\varepsilon}^i_k \cdot \alpha \rfloor$. For $i+1 = \gamma - 1$ we proceed analogously, we only adapt the property of cardinalities to $\mathrm{card}(\bar{I}^{\gamma-1}_j) = \mathrm{card}(I^{\gamma-1}_j)$. If for the given construction we have $j \neq k \Rightarrow \bar{I}^i_j \cap \bar{I}^i_k = 0$ then we put $\bar{b} = \bigcup_j \{\bar{I}^{\gamma-1}_j\}$ and let f be the order preserving isomorphism of α and $\bar{b}$. If the just mentioned property does not hold then we designate by γ_0 the smallest i for which the property does not hold and we construct $\bar{b}$ and f for $G \wedge \gamma_0$.

By the recursion based on FN it is possible now to prove the following properties. For $i,j \in FN$ we have $\mathrm{st}(\max(\bar{I}^i_j)) - \mathrm{st}(\min(\bar{I}^i_j)) = \mathrm{st}(\mathrm{card}(I^i_j)/\alpha) + \varepsilon^i_j$ and for $I^{i+1}_j, I^{i+1}_{j+1} \subseteq I^i_k$ we have $\mathrm{st}(\min(\bar{I}^{i+1}_{j+1})) - \mathrm{st}(\max(\bar{I}^{i+1}_j)) = \bar{\varepsilon}^i_k$. From these properties we obtain

1) $(\forall i,j \in FN)(\bar{I}^i_j \cap \bar{I}^i_{j+1} = 0$

2) $(\forall x,y \in \alpha)(f(x) \doteq f(y) \Rightarrow (\forall i,j \in FN)(x \in I^i_j \equiv y \in I^i_j))$

3) $\mathrm{Fig}(\mathrm{rng}(f))/\doteq$ has the Lebesgue measure 1, as $\sum_{i,j \in \omega} \bar{\varepsilon}^j_i = 1$

(Σ is understood in the standard sense as an infinite sum - we consider the factor structure). The proof of the theorem 1.18 is now an immediate consequence of the assertions 1),2),3).

§ 2. Projective Classes and Measurability. In this section we introduce classes connected very closely with the classical projective sets and that is why we call them projective classes. We prove that the system contains every designated class, the class FN and that it is closed on the definition by normal formulas. Let us note that the system is codable (see [V] for the definition).

Definition 2.1: 1) A figure Y in an indiscernibility equivalence $\stackrel{\asymp}{=}$ is said to be projective iff $Y/\stackrel{\asymp}{=}$ is projective in the classical sense (see e.g. [K]).

2) A class Y is said to be projective iff there is a totally disconnected indiscernibility equivalence $\stackrel{\asymp}{=}$ such that Y is a projective figure in $\stackrel{\asymp}{=}$.

Remark: The requirement for the indiscernibility equivalence $\stackrel{\asymp}{=}$ in 2) to be totally disconnected is superfluous as for every indiscernibility equivalence there is a finer totally disconnected indiscernibility equivalence. But the proof of this, not quite easy, fact exceeds the framework of the paper and we do not need this fact here.

Theorem 2.2: 1) Designated classes are projective.

2) FN is a projective class.

3) Projective classes are closed on the definitions by normal formulas.

Proof: 1) Let $\mathrm{Dsg}(X)$, put $\stackrel{\asymp}{=}\ = X \times X$.

2) For $\alpha \in N - FN$ we define $\stackrel{\asymp}{=}$ on α by the following manner: $\stackrel{\asymp}{=}\ = (Id \cap FN) \cup ((\alpha - FN) \times (\alpha - FN))$. Where $Id = \{\langle x,x\rangle ; x \in V\}$. $\stackrel{\asymp}{=}$ is a totally disconnected indiscernibility equivalence and for $\beta \in \alpha - FN$ we have $\mu(\beta) = \alpha - FN$. $\mu(\beta)/\stackrel{\asymp}{=}$ is a singleton and thus closed; this implies that $\mu(\beta)/\stackrel{\asymp}{=}$ is projective and the-

refore $FN = \alpha - (\alpha - FN)$ is projective, too.

3) This assertion can be proved using the following lemma and Theorem 1.5.

<u>Lemma 2.3</u>: 1) Let $\overset{1}{=} \subset \overset{2}{=}$ be indiscernibility equivalences and let $dom(\overset{1}{=}) = dom(\overset{2}{=})$. If $X \subseteq dom(\overset{2}{=})$ is a projective figure in $\overset{2}{=}$ then X is a projective figure in $\overset{1}{=}$.

2) If X is projective, then there is a totally disconnected indiscernibility equivalence $\overset{\asymp}{=}$ on V such that X is a projective figure in $\overset{\asymp}{=}$.

3) If X_1, X_2 are projective, then there is a totally disconnected indiscernibility equivalence $\overset{\asymp}{=}$ on V such that X_1, X_2 are projective figures in $\overset{\asymp}{=}$.

4) If X, Y are projective then $X \cap Y$, $X-Y$ are projective.

5) If X is projective then $V \times X$ and $X \times V$ are projective.

6) If X, Y are projective then $X \times Y$ is projective.

7) If X is projective then rng(X) is projective.

8) If X is projective and Dep(Y,X) then Y is projective.

Proof: 1) $Id \upharpoonright dom(\overset{1}{=})$ is a continuous function (see Th. 1.14) and $X = ((Id)^{-1})"X$. Hence (see [K]) X is projective also in $\overset{1}{=}$.

2) Let $\overset{+}{=}$ be a totally disconnected indiscernibility equivalence such that X is a projective figure in $\overset{+}{=}$. Put $\overset{\asymp}{=} \; = \; \overset{+}{=} \cup \cup (V - dom(\overset{+}{=}))^2$. $dom(\overset{+}{=})$ is a designated class being a figure, hence it is closed and thus X is projective figure also in $\overset{\asymp}{=}$ (see [K]).

3) Let $\overset{1}{=}, \overset{2}{=}$ be totally disconnected indiscernibility equivalences on V such that X_1, X_2 are projective figures in $\overset{1}{=}, \overset{2}{=}$ respectively. Put $\overset{\asymp}{=} \; = \; \overset{1}{=} \cap \overset{2}{=}$. Now use 1).

4) Use 3) and [K].

5) Put $Pr_1(\langle x_1, x_2 \rangle) = x_1$, put $\overset{+}{=} \; = V \times V$. Pr_1 is a designated

class, a function continuous w.r.t. $\stackrel{+\times*}{=}$ and $\stackrel{*}{=}$. Now use the same theorem from [K] as in the proof of 1). The proof for $X\times V$ is analogous.

6) Use the equality $X\times Y=X\times V\cap V\times Y$, 5),4),3).

7) We firstly prove the following assertion. Let G_1 be a generating system for a totally disconnected indiscernibility equivalence $\stackrel{1}{=}$ on $V\times V$. We define by a normal formula generating systems G_2, G_3 for totally disconnected indiscernibility equivalences $\stackrel{2}{=},\stackrel{3}{=}$ such that $\stackrel{2}{=}$ is finer than $\stackrel{1}{=}$ and $\langle x_1,y_1\rangle \stackrel{2}{=} \langle x_2,y_2\rangle \Rightarrow x_1\stackrel{3}{=}x_2$ (thus Pr_1 is continuous). Define now G_3. If $X_1,\dots,X_k$ is the partition corresponding to $G_1''\{n\}$ then let us put $G_3''\{n\}$ to be the equivalence corresponding to the partition consisting of the Boolean combinations of $Pr_1''X_1,\dots,Pr_1''X_k$. This partition has maximally 2^k elements and thus the compactness is preserved. Let $\stackrel{\times}{=}$ denote the product equivalence of $\stackrel{3}{=}$ and $V\times V$. Now it suffices to put $\stackrel{2}{=}$ = $=\stackrel{1}{=}\cap\stackrel{\times}{=}$. We have proved the assertion. To prove 7) use the given assertion,, 1) and the equality $rng(Y)= Pr_1''(Y)$. The fact that $Pr_1''Y$ is a projective figure in $\stackrel{3}{=}$ now follows from [K].

8) Use the equality $R''Y = rng(R\cap(V\times Y))$.

Remark: As we use the classical definition for the predicate "to be projective" we do not define the projective codes. Note here that if we have a countable amount of projective classes having the projective codes bounded from above then their union is also projective.

Definition 2.4: 1) Let $\doteq$ denote the indiscernibility equivalence on rational numbers (also infinite) given by the formula $x \doteq y \equiv (\forall n\in FN)(x<-n \,\&\, y<-n)\vee(\forall n\in FN)(|x-y|<1/n)\vee$ $\vee(\forall n\in FN)(x>n \,\&\, y>n)$. (The factor space $RN/\doteq$ is the common space of real numbers with $+\infty$ and $-\infty$.)

2) For $\alpha \in N-FN$ we put $a_k^{\alpha} = \{\beta/\alpha ; -k \leq \beta/\alpha < k\}$ (β denotes here a variable for integers). We usually omit the superscript α keeping in mind a fixed infinitely large natural number for α.

<u>Definition 2.5</u>. A semiset $\wp \subseteq a$ is said to be a-measurable iff $(\forall n \in FN)(\exists b,c)(b \subseteq \wp \subseteq c \subseteq a \,\&\, (|c-b|/|a| < 1/n)$.

The reasonability of this definition is justified by the following theorem.

<u>Theorem 2.6</u>. If $\wp \subseteq a_k$ is a figure in $\doteq$ then $\wp$ is a_k-measurable iff $\wp/\doteq$ is Lebesgue measurable. Moreover $m(\wp/\doteq) = \sup(\{st|b|/\alpha\}; b \subset \wp\})$.

Proof: It suffices to prove that the measure, nonstandardly described in the theorem, is σ-additive, translation invariant and such that the measure of intervals is equal to their lengths. We omit here the easy proof of the assertion for intervals. The invariance on translations is proved by the following consideration. Let β/α be such that $x = st(\beta/\alpha)$. Let us put $\wp_x = \{y+\beta/\alpha ; y \in \wp\}$. We have then that $\wp_x/\doteq$ is obtained by a translation of $\wp/\doteq$ and if $b \subseteq \wp$ then $b_x \subseteq \wp_x$ (where $b_x = \{y+\beta/\alpha ; y \in b\}$). σ-additivity we prove in the following theorem generally (not only for figures).

<u>Theorem 2.7</u>: If $\{\wp_i ; i \in FN\}$ is a sequence of disjoint a-measurable semisets then $\wp = \bigcup_{i \in FN} \wp_i$ is an a-measurable semiset.

Proof: Let us fix n. Let $b_i \subseteq \wp_i \subseteq c_i$ be such that

$$(*) \qquad |c_i - b_i|/|a| < (1/n) \cdot 2^{-(i+2)}.$$

As $\wp_i$ are disjoint, b_i are disjoint, too. Let us prolong the sequences $\{b_i ; i \in FN\}$, $\{a_i ; i \in FN\}$ in such a manner that the condition $*$ and the disjointness of b_i hold. We prove now that there are j,γ such that $j>2$ and $|\bigcup\{b_i; j\leq i\leq\gamma\}|/|a|< 1/4n$. Let us put $x_i = st(|\bigcup\{b_j; j\leq i\}|/|a|)$. Then $\{x_i; i\in\omega\}$ is an increasing sequence of real positive numbers ≤ 1. This sequence has a limit, We put $y = \lim_{n\to\infty} x_n$. Now it is easy to prove that there is an infinitely large γ such that $y=st(|\bigcup\{b_i; i\in\gamma\}/|a|)$. For every infinitely large δ such that $\delta<\gamma$ we have $0 \doteq |\bigcup\{b_i; \delta\leq i\leq\gamma\}|/|a|< 1/4n$ and hence we have the inequality also for a suitable finite $j>2$. If we now put $b = \bigcup\{b_i; i<j\}$ and $c = \bigcup\{c_i; i\leq\gamma\}$ then we obtain $b\subseteq\wp\subseteq c\subseteq a \& |c-b|/|a|< 1/n$.

The proof of the assertion that if $\{\wp_i; i\in FN\}$ is a countable sequence of disjointed measurable figures being parts of some a_k then $m(\bigcup\{\wp_i; i\in FN\}/\doteq) = \sum m(\wp_i/\doteq)$, can be obtained by an easy modification of the last theorem (Th. 2.7).

For a-measurable semisets we now define a measure by the following description.

Definition 2.8: If $\wp$ is an a-measurable semiset then we put $m_a(\wp) = \sup(\{st|b|/|a|); b\subseteq\wp\})$.

We shall not prove analogues of classical assertions concerning the measure. Also, the following easy assertion we give without any proof.

Theorem 2.9: Let $a\subset b \& |a|/|b| \not\doteq 0$. If $\wp$ is b-measurable then $\wp\cap a$ is a-measurable and $m_a(\wp\cap a) = st(|b|/|a|)\cdot m_b(\wp\cap a)$.

Now we can solve our basic problem and namely to prove the consistency of the assertion that every projective semiset is a-measurable for every $a\supseteq\wp$.

Theorem 2.10: Suppose that all the projective subsets of real numbers are Lebesguely measurable. If $\wp\subseteq a$ is a projective semiset then $\wp$ is a-measurable.

Proof: Let $\wp$ be a projective figure in a totally disconnected indiscernibility equivalence $\overset{*}{=}$ on a. Using the theorem 1.17 we may suppose that a is a natural number and that the elements of the partitions corresponding to the elements of the generating system are intervals. Hence we shall use α instead of a in the following text. By Th. 1.18 there is a function f such that $f:\alpha \xrightarrow{1-1} \xrightarrow{1-1} \{\beta/\alpha ; 0 \leq \beta \leq 2\alpha\}$ and $f(x) \doteq f(y) \Rightarrow x \overset{*}{=} y$ and Fig(rng(f)/$\doteq$ is a closed set with the Lebesgue measure 1. It suffices to justify the measurability of $(f^{-1})''\wp$ w.r.t. $(f^{-1})''\alpha$ as f is 1-1. Let us denote $\sigma = (f^{-1})''\wp$, $\bar{a} = (f^{-1})''\alpha$ and $b = \{\beta/\alpha ; 0 \leq \beta < 2\alpha\}$. Fig$(f^{-1})/{}^{\bullet}\!\overset{*}{\leq}$ is continuous (Th. 1.14). Hence $\text{Fig}_{\doteq}(\sigma)/\doteq$ is a projective subset of the interval [0,2] by [K] and Lebesguely measurable by the assumption. Thus $\text{Fig}_{\doteq}(\sigma)$ is a_2-measurable by Th. 2.6 and $\sigma = \bar{a} \cap \text{Fig}_{\doteq}(\sigma)$ is a-measurable by Th. 2.9. This completes the proof.

To prove the consistency of the measurability of projective semisets in AST it suffices to justify the validity of the continuum hypothesis in the Solovay's model (see [J] Th. 43) for Lebesgue measurability of projective sets, since the model of AST we obtain by the ultrapower (with ω as the index set). To obtain the consistency we proceed as follows: Let $\mathfrak{M}$ be the initial model (and let us suppose $\mathfrak{M} \models V=L$), let $\varkappa$ be an inaccessible cardinal. Further let B be the $(\aleph_0, \varkappa)$ Levy's algebra and G an $\mathfrak{M}$- generic ultrafilter on B. The consistency is considered in the model $\mathfrak{M}[G]$. Remember that in this model ($\mathfrak{M}[G]$) all the cardinals $< \varkappa$ are countable (in the sense of the model) and $\varkappa = (\aleph_1)^{\mathfrak{M}[G]}$ Remember also that $(2^{\aleph_0})^{\mathfrak{M}[G]} \leq (|B|^{\aleph_0})^{\mathfrak{M}}$. To prove $(2^{\aleph_0} = \aleph_1)^{\mathfrak{M}[G]}$ it suffices now to justify the equality $|B| = \varkappa$. Since B has a dense part of the cardinality $\varkappa$ (the set of conditions) and the $\varkappa$-chain condition for B holds, we have (by Lemma 58 [J]) $|B| \leq \varkappa^{<\varkappa} = \varkappa$.

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