Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 27 (1986), No. 2, 307--323

Persistent URL: `http://dml.cz/dmlcz/106454`

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27,2 (1986)

ON A CLASS OF LOCALLY COMPLETELY DECOMPOSABLE ABELIAN GROUPS

Ladislav BICAN, Jaroslav HORA

Abstract: This paper deals with the clas $\mathfrak{M}$ of all torsionfree abelian groups G for which there is a partition $\pi = \pi_1 \cup \pi_2 \cup \cup \dots \cup \pi_n$ of the set π of all primes such that for each $j \in \in \{1,2,\dots,n\}$ the group $G \otimes Z_{\pi_j}$ is completely decomposable with the ordered type set $T(G \otimes Z_{\pi_j})$. The subclasses of $\mathfrak{M}$ consisting of the groups having all the pure (regular) subgroups in $\mathfrak{M}$ are characterized.

Key words: Completely decomposable group, pure subgroup, regular subgroup, type set.

Classification: 20K20

In the papers [2] and [3] an almost complete description of all completely decomposable torsionfree abelian groups any pure (regular) subgroup of which is completely decomposable was presented. The results obtained have been recently completed by A.A. Kravčenko in [6]. In the past ten years the class of Butler groups (torsionfree homomorphic images of completely decomposable torsionfree groups of finite rank) was studied very intensively by several authors. Among other results, the first author in [4] showed that G is a Butler group if and only if there is a partition $\pi = \pi_1 \cup \pi_2 \cup \dots \cup \pi_n$ of the set π of all primes such that $G \otimes Z_{\pi_j}$ is completely decomposable with the ordered type set for each $j \in \{1,2,\dots,n\}$. So, it is natural to study the properties of "locally completely decomposable groups" G in the sense that there is a partition $\pi = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n$ of π such that $G \otimes Z_{\pi_j}$ is completely decomposable with the ordered type set for each $j \in \{1,2,\ldots,n\}$. The purpose of this note is to characterize the classes of such groups which are closed under pure (regular) subgroups.

By the word "group" we shall always mean an additively written abelian group. The symbols $\underline{N}, \underline{N}_0$ and π are used for the set of all positive integers, non-negative integers and the set of all primes, respectively. If π' is a subset of π then $Z_{\pi'}$ will denote the group of rationals with denominators prime to every $p \in \pi'$. If J is a rank one torsionfree group of the type $\hat{\tau}$ and π' is a set of primes, then the type of $J \otimes Z_{\pi'}$ will be simply denoted by $\hat{\tau} \otimes Z_{\pi'}$. If G is a completely decomposable group, $G = \bigoplus_{i \in I} J_i$, then the set of types $\hat{\tau}(J_i)$, $i \in I$, is denoted by $T(G)$. Other notations and terminology are essentially the same as in [5].

Lemma 1: Let $\hat{\tau}$ and $\hat{\sigma}$ be the types and π' be a set of primes such that $\hat{\tau} \otimes Z_{\pi'} < \hat{\sigma} \otimes Z_{\pi'}$. If $\pi = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n$ is a partition of π then $\hat{\tau} \otimes Z_{\pi' \cap \pi_j} < \hat{\sigma} \otimes Z_{\pi' \cap \pi_j}$ for some $j \in \{1, 2,\ldots,n\}$.

Proof: If τ and σ are characteristics belonging to the types $\hat{\tau}$ and $\hat{\sigma}$, respectively, and $\tau(p) < \sigma(p) = \infty$ for some $p \in \pi'$, then $p \in \pi_j$ for some $j \in \{1,2,\ldots,n\}$ and we are through. In the opposite case, there is an infinite subset $\pi'' \subseteq \pi'$ such that $\tau(p) < \sigma(p)$ for each $p \in \pi''$. Then, for some $j \in \{1,2,\ldots$ $\ldots,n\}$, the intersection $\pi'' \cap \pi_j$ is infinite and the assertion follows.

Lemma 2: Let $\hat{\tau}_1, \hat{\tau}_2, \ldots$ be the types such that

$\hat{\tau}_1 \otimes Z_{\pi'} < \hat{\tau}_2 \otimes Z_{\pi'} < \ldots$ for some subset $\pi' \subseteq \pi$. If $\pi = \pi_1 \cup \cup \pi_2 \cup \ldots \cup \pi_n$ is a partition of π and $\pi'_j = \pi' \cap \pi_j$, then for some $j \in \{1,2,\ldots,n\}$ the sequence $\hat{\tau}_1 \otimes Z_{\pi'_j} \leq \hat{\tau}_2 \otimes Z_{\pi'_j} \leq \ldots$ contains infinitely many different terms.

Proof: It follows easily from Lemma 1.

Notation. A sequence $\{\hat{\tau}_1, \hat{\tau}_2, \ldots\}$ of types will be simply denoted by $\{\hat{\tau}_i\}$. If $\{\hat{\tau}_i\}$ and $\{\hat{\sigma}_i\}$ are two sequences of types then the symbol $\{\hat{\tau}_i\} < \{\hat{\sigma}_i\}$ means that $\hat{\tau}_i < \hat{\sigma}_j$ for all $i,j \in \underline{N}$.

Lemma 3: Let $\{\hat{\tau}_i\}$ and $\{\hat{\sigma}_i\}$ be two sequences of types and π' be a set of primes such that $\{\hat{\tau}_i \otimes Z_{\pi'}\}$ and $\{\hat{\sigma}_i \otimes Z_{\pi'}\}$ are increasing sequences of types and $\{\hat{\tau}_i \otimes Z_{\pi'}\} < \{\hat{\sigma}_i \otimes Z_{\pi'}\}$. If $\pi = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n$ is a partition of π then, for some $j \in \{1,2, \ldots,n\}$, $\{\hat{\tau}_i \otimes Z_{\pi' \cap \pi_j}\} < \{\hat{\sigma}_i \otimes Z_{\pi' \cap \pi_j}\}$ and $\{\hat{\tau}_i \otimes Z_{\pi' \cap \pi_j}\}$ contains an infinite increasing sequence of types.

Proof: By Lemma 2 there is $j \in \{1,2,\ldots,n\}$ such that the sequence $\hat{\tau}_1 \otimes Z_{\pi' \cap \pi_j} \leq \tau_2 \otimes Z_{\pi' \cap \pi_j} \leq \ldots$ contains infinitely many different terms. The assertion now follows from the obvious fact that $\hat{\tau}_i \otimes Z_{\pi' \cap \pi_j} \leq \hat{\sigma}_1 \otimes Z_{\pi' \cap \pi_j} \leq \hat{\sigma}_2 \otimes Z_{\pi' \cap \pi_j} \leq \ldots$ for each $i \in \underline{N}$.

Lemma 4: Let $\{\hat{\tau}_i^1\}, \{\hat{\tau}_i^2\}, \ldots, \{\hat{\tau}_i^l\}$ be sequences of types and π' be a set of primes such that $\{\hat{\tau}_i^k \otimes Z_{\pi'}\}$ is an infinite increasing sequence of types for each $k \in \{1,2,\ldots,l\}$ and $\{\hat{\tau}_i^1 \otimes Z_{\pi'}\} < \{\hat{\tau}_i^2 \otimes Z_{\pi'}\} < \ldots < \{\hat{\tau}_i^l \otimes Z_{\pi'}\}$. Let $\pi = \pi_1 \cup \pi_2 \ldots \cup \pi_n$ be a partition of the set π and $\pi'_j = \pi' \cap \pi_j$, $j \in \{1,2,\ldots,n\}$. If l_j, $j \in \{1,2,\ldots,n\}$ is the umber of pairs $\{\hat{\tau}_i^k \otimes Z_{\pi'_j}\} < \{\hat{\tau}_i^{k+1} \otimes Z_{\pi'_j}\}$, $k \in \{1,2,\ldots,l-1\}$, where $\hat{\tau}_i^k \otimes Z_{\pi'_j}\}$ contains an infinite increasing sequence of types, then

$+ l_2 + \ldots + l_n \geq l - 1.$

Proof: For any $k \in \{1,2,\ldots,l-1\}$ Lemma 3 yields the existence of $j_k \in \{1,2,\ldots,n\}$ such that $\{\hat{\tau}_i^k \otimes Z_{\pi'_{j_k}}\} < \{\hat{\tau}_i^{k+1} \otimes Z_{\pi'_{j_k}}\}$ and $\{\hat{\tau}_i^k \otimes Z_{\pi'_{j_k}}\}$ contains an infinite sequence of types.

Lemma 5: Let M be an ordered set of types having chains of increasing sequences of arbitrary lengths. Then M contains an infinite chain of increasing sequences.

Proof: On the set $\mathcal{K}$ of all increasing sequences of types from M we define the equivalence relation $\equiv$ in the following way: $\{\hat{\tau}_i\} \equiv \{\hat{\sigma}_i\}$ if and only if there is an increasing sequence $\{\hat{\varrho}_i\}$ of types from M such that all $\hat{\tau}_i$'s and all $\hat{\sigma}_i$'s are equal to some element of $\{\hat{\varrho}_i\}$. Further, we define the ordering $\trianglelefteq$ on $\mathcal{K}$ in such a way that $\{\hat{\tau}_i\} \trianglelefteq \{\hat{\sigma}_i\}$ if and only if either $\{\hat{\tau}_i\} \equiv \{\hat{\sigma}_i\}$ or there is $m \in \underline{N}$ such that $\hat{\sigma}_m > \hat{\tau}_i$ for all $i \in \underline{N}$. Since the relation $\trianglelefteq$ is obviously a total ordering on $\mathcal{K}$, the assertion follows now easily.

Lemma 6: Let M be an ordered set of types satisfying the following condition:

($*$) If π' is a subset of π such that the set $M \otimes Z_{\pi'} = \{\hat{\tau} \otimes Z_{\pi'} \mid \hat{\tau} \in M\}$ contains an increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \ldots$ then there is a prime $p \in \pi'$ such that $\tau(p) = \infty$ for each type $\hat{\tau} \in M \otimes Z_{\pi'}$ with $\hat{\tau}_1 < \hat{\tau}_2 < \ldots < \hat{\tau}$.

If M contains a chain $\{\hat{\tau}_i^1\} < \{\hat{\tau}_i^2\} < \ldots < \{\hat{\tau}_i^l\}$ of l increasing sequences of types then there is a partition $\pi = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_l$ of π such that $\{\hat{\tau}_i^j \otimes Z_{\pi_k}\}$ contains an infinite increasing sequence for $j \neq k$, $j,k \in \{1,2,\ldots,l\}$, and it is a finite set of types otherwise.

Proof: Setting $\pi_1 = \{p \in \pi \mid \tau_1^1(p) < \infty\}$ we obviously get that $\{\hat{\tau}_i^1 \otimes Z_{\pi_1}\}$ is an infinite increasing sequence. Moreover, the assumption that $\{\hat{\tau}_i^j \otimes Z_{\pi_1}\}$ contains for some $j \in \{1,2,\ldots,l-1\}$ an infinite increasing sequence of types leads to a contradiction with the condition ($*$).

Assume that for some $r \in \{1,2,\ldots,l-1\}$ we have constructed the subsets $\pi_{r+1},\ldots,\pi_l$ of π such that $\{\hat{\tau}_i^j \otimes Z_{\pi_j}\}$ contains an infinite sequence of types for each $j \in \{r+1,\ldots,l\}$ and the set $\{\hat{\tau}_i^j \otimes Z_{\pi_k}\}$ is finite whenever $j \neq k$ and $j \in \{1,2,\ldots,l\}$, $k \in$ $\in \{r+1,\ldots,l\}$. Denoting $\pi'_r = \pi \setminus (\pi_{r+1} \cup \ldots \cup \pi_l)$ we easily get from Lemma 2 that the set $\{\hat{\tau}_i^r \otimes Z_{\pi'_r}\}$ contains an infinite increasing sequence of types. Setting $\pi_r = \{p \in \pi'_r \mid \tau_1^r(p) < \infty\}$ we see that the set $\{\hat{\tau}_i^r \otimes Z_{\pi_r}\}$ contains an infinite increasing sequence of types, too. As above, the assumption that the set $\{\hat{\tau}_i^j \otimes Z_{\pi_r}\}$, $j \in \{1,2,\ldots,r-1\}$, contains an infinite increasing sequence of types leads to a contradiction with the condition ($*$). Moreover, the choice of π'_r gives that $\tau_1^{r+1}(p) = \infty$ for each $p \in \pi'_r$ and so the set $\{\hat{\tau}_i^j \otimes Z_{\pi_r}\}$ is finite for each $j \in$ $\in \{r+1,\ldots,l\}$. Finally, we set $\pi_1 = \pi \dot{-} (\pi_2 \cup \ldots \cup \pi_l)$ and the proof is finished.

<u>Lemma 7</u>: Let M be an ordered set of types satisfying the condition ($*$) from the preceding Lemma. If M contains no two increasing sequences $\{\hat{\tau}_i\}$ and $\{\hat{\sigma}_i\}$ with $\{\hat{\tau}_i\} < \{\hat{\sigma}_i\}$ then there is a partition $\pi = \pi_1 \cup \pi_2$ of π such that $M \otimes Z_{\pi_1}$ is inversely well-ordered and either $\pi_2 = \emptyset$ or $M \otimes Z_{\pi_2}$ contains an infinite increasing sequence $\hat{\sigma}_1 < \hat{\sigma}_2 < \ldots$ such that for each $n \in \underline{N}$ the set $\{\hat{\tau} \in M \otimes Z_{\pi_2} \mid \hat{\tau} \leq \hat{\sigma}_n\}$ is inversely well-ordered and for each $\hat{\tau} \in M \otimes Z_{\pi_2}$ it is either $\hat{\tau} = \hat{R}$ (the type of the additive group of all rationals) or $\hat{\tau} \leq \hat{\sigma}_n$ for some $n \in \underline{N}$.

<u>Proof</u>: If M contains no infinite increasing sequence, then

it is inversely well-ordered and it suffices to put $\pi_1 = \pi$, $\pi_2 = \emptyset$.

Assume that M contains an infinite increasing sequence $\{\hat{\tau}_i\}$. Set $T = \{\hat{\tau} \in M \mid \hat{\tau} > \hat{\tau}_i$ for all $i \in \underline{N}\}$, $\pi_1 = \{p \in \pi \mid \tau(p) < \infty$ for some $\hat{\tau} \in T\}$ and $\pi_2 = \pi \setminus \pi_1$. Since M satisfies the condition $(*)$, the set $\{\hat{\tau}_i \otimes Z_{\pi_1}\}$ is finite and consequently the set $M \otimes Z_{\pi_1}$ is inversely well-ordered. By Lemma 2, the sequence $\{\hat{\tau}_i \otimes Z_{\pi_2}\}$ contains infinitely many different terms and the assertion follows easily (by the choice of π_1).

<u>Definition 1</u>: For a positive integer n let $\mathfrak{M}(n)$ be the class of all torsionfree groups G having the property that there is a partition $\pi = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n$ of the set π of all primes such that the group $G \otimes Z_{\pi_j}$ is completely decomposable with the ordered type set $T(G \otimes Z_{\pi_j})$ for every $j \in \{1,2,\ldots,n\}$. For completeness set $\mathfrak{M}(0) = \emptyset$ and let $\mathfrak{M}$ be the union of all $\mathfrak{M}(n)$'s, $\mathfrak{M} = \bigcup_{n=1}^{\infty} \mathfrak{M}(n)$.

<u>Lemma 8</u>: Let G be a completely decomposable group of the form $G = J \oplus \bigoplus_{i=1}^{\infty} J_i$, where J and J_i are of rank one and of the types $\hat{\tau}$ and $\hat{\tau}_i$, $i \in \underline{N}$, respectively. If $\pi' = \{p_1, p_2, \ldots\}$ is a set of primes such that $\hat{\tau}_1 \otimes Z_{\pi'} < \hat{\tau}_2 \otimes Z_{\pi'} < \ldots < \hat{\tau} \otimes Z_{\pi'}$ and $\tau(p) < \infty$ for each $p \in \pi'$, then G contains a pure subgroup S not belonging to the class $\mathfrak{M}$.

<u>Proof</u>: In each J_i, $i \in \underline{N}$, select an element u_i with $h^G_{p_j}(u_i) = 0$ for all $j \in \{1,2,\ldots,i\}$ and let $0 \neq u \in J$ be arbitrary. If $h^G_{p_i}(u) = e_i$, we choose the elements $v_1, v_2, \ldots$ in J such that $p_1^{e_1} v_1 = u$ and $p_i^{e_i} v_i = v_{i-1}$ for all $i \in \{2,3,\ldots\}$.

Considering the pure subgroup

$$S = \langle v_1 + p_1 u_1, v_2 + p_1 p_2 u_2, \ldots, v_i + p_1 p_2 \ldots p_i u_i, \ldots \rangle^G_*$$

of G, we are going to show that $S \notin \mathfrak{M}$.

Proving indirectly, suppose that there is a partition $\pi = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n$ of π such that $S \otimes Z_{\pi_j}$ is completely decomposable with the ordered type set $T(S \otimes Z_{\pi_j})$ for each $j \in \{1,2,\ldots,n\}$. By Lemma 2 there is $j \in \{1,2,\ldots,n\}$ such that for $\pi'_j = \pi' \cap \pi_j$ the sequence $\{\hat{\tau}_i \otimes Z_{\pi'_j}\}$ contains infinitely many different terms. Obviously, $S \otimes Z_{\pi'_j}$ is completely decomposable and $\hat{\tau} \otimes Z_{\pi'_j} > \hat{\tau}_i \otimes Z_{\pi'_j}$ for all $i \in \underline{N}$.

The group $S \otimes Z_{\pi'_j}$ can be written in the form $S \otimes Z_{\pi'_j} = \bigoplus_{k=1}^{\infty} S_k$, where S_k is a homogeneous completely decomposable group of the type $\hat{\sigma}_k$ and $\hat{\sigma}_k < \hat{\sigma}_{k+1}$, $k \in \underline{N}$. If $p_s \in \pi'_j$ is any prime, then $v_s + p_1 p_2 \ldots p_s u_s \in H_1 = S_1 \oplus S_2 \oplus \ldots \oplus S_r$. By hypothesis, there exists an element $v_t + p_1 p_2 \ldots p_t u_t \in H_2 = \bigoplus_{k=r+1}^{\infty} S_k$. Further, $p_1 p_2 \ldots p_s(u_s - p_{s+1}^{e_{s+1}+1} \ldots p_t^{e_t+1} u_t) = v_s + p_1 p_2 \ldots p_s u_s - p_{s+1}^{e_{s+1}} \ldots p_t^{e_t} (v_t + p_1 p_2 \ldots p_t u_t) \in S$. Hence $u_s - p_{s+1}^{e_{s+1}+1} \ldots p_t^{e_t+1} u_t \in S$, S being pure in G. Therefore, $u_s - p_{s+1}^{e_{s+1}+1} \ldots p_t^{e_t+1} u_t = h_1 + h_2$, $h_1 \in H_1$, $h_2 \in H_2$, and from the form of $S \otimes Z_{\pi'_j}$ it follows $p_1 p_2 \ldots p_s h_1 = v_s + p_1 p_2 \ldots p_s u_s$ which contradicts $h^G_{p_s}(v_s) = 0$.

Lemma 9: Let G be a completely decomposable group of the form $G = \bigoplus_{i=1}^{\infty} J_i + \bigoplus_{i=1}^{\infty} J'_i$, where J_i, J'_i are of rank one and of the types $\hat{\tau}_i$, $\hat{\tau}'_i$, $i \in \underline{N}$, respectively. If $\pi' = \{p_1, p_2, \ldots\}$ is a set of primes such that $\hat{\tau}_1 \otimes Z_{\pi'} < \hat{\tau}_2 \otimes Z_{\pi'} < \ldots < \hat{\tau}_i \otimes Z_{\pi'} < \ldots < \hat{\tau}'_i \otimes Z_{\pi'} < \ldots < \hat{\tau}'_2 \otimes Z_{\pi'} < \hat{\tau}'_1 \otimes Z_{\pi'}$ and $\tau'_i(p_i) < \infty$ for each $i \in \underline{N}$, then G contains a pure subgroup S not belonging to the class $\mathfrak{M}$.

Proof: In each J'_i, $i \in \underline{N}$, select an element u'_i with $h^G_{p_i}(u'_i) = 0$

and for all $i,j \in \underline{N}$, select the elements $u_{ij} \in J_{p_i^j}$ with $h^G_{p_i}(u_{ij}) = 0$, $j \in \underline{N}$.

Considering the pure subgroup

$$S = \langle u_i' + p_i u_{ij} | i,\ j \in \underline{N} \rangle^G_*$$

of G, we are going to show that $S \notin \mathfrak{M}$.

Proving indirectly, suppose that there is a partition $\pi = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n$ of π such that $S \otimes Z_{\pi_j}$ is completely decomposable with the ordered type set $T(S \otimes Z_{\pi_j})$ for each $j \in \{1,2,\ldots,n\}$. By Lemma 2 there is $j \in \{1,2,\ldots,n\}$ such that for $\pi'_j = \pi' \cap \pi_j$ the sequence $\{\hat{\tau}_i \otimes Z_{\pi'_j}\}$ contains infinitely many different terms. Obviously, $S \otimes Z_{\pi'_j}$ is completely decomposable and $\hat{\tau}'_k \otimes Z_{\pi'_j} > \hat{\tau}_i \otimes Z_{\pi'_j}$ for all $i,\ k \in \underline{N}$.

The group $S \otimes Z_{\pi'_j}$ can be written in the form $S \otimes Z_{\pi'_j} = \bigoplus_{k=1}^{\infty} S_k$, where S_k is a homogeneous completely decomposable group of the type $\hat{\sigma}_k$ and $\hat{\sigma}_k < \hat{\sigma}_{k+1}$, $k \in \underline{N}$. If $p_s \in \pi'_j$ is any prime, then $u_s' + p_s u_{s1} \in H_1 = S_1 \oplus S_2 \oplus \ldots \oplus S_r$. By hypothesis, there exists an element $u_s' + p_s u_{st}$, the type of which is greater than $\hat{\sigma}_r$, so that $u_s' + p_s u_{st} \in H_2 = \bigoplus_{k=r+1}^{\infty} S_k$. Further, $p_s(u_{s1} - u_{st}) = (u_s' + p_s u_{s1}) - (u_s' + p_s u_{st}) \in S$. Hence $u_{s1} - u_{st} \in S$, S being pure in G. Therefore $u_{s1} - u_{st} = h_1 + h_2$, $h_1 \in H_1$, $h_2 \in H_2$, and from the form of $S \otimes Z_{\pi'_j}$ it follows $p_s h_1 = u_s' + p_s u_{s1}$ which contradicts $h^G_{p_s}(u_s') = 0$

<u>Lemma 10</u>: Let G be a completely decomposable group of the form $G = \bigoplus_{i,j=1}^{\infty} J_i^j$, where J_i^j are of rank one and of the types $\hat{\tau}_i^j$, $i,j \in N$. If the sequences $\{\hat{\tau}_i^1\}, \{\hat{\tau}_i^2\}, \ldots$ form an infinite chain of increasing sequences of types, then G contains a pure subgroup S not belonging to the class $\mathfrak{M}$.

<u>Proof</u>: Let $\pi = \{p_1, p_2, \ldots\ldots\ldots\}$ be the set of all primes and p be a fixed prime. For each pair of sequences with $\{\hat{\tau}_i^s\} >$

$> \{\hat{\tau}_i^r\}$ and $\tau_i^s(p) \neq \infty$ for all $i \in N$ select in $\{\hat{\tau}_i^s\}$ the type with index p^{p_r} and in the sequence $\{\hat{\tau}_i^r\}$ select the types with indices $p^{p_s^l}$, $l \in N$. In the corresponding groups choose the elements u^s, u_1^r, $u_2^r, \ldots\ldots$ with zero p-heights. Finally, denote $\,{}_r^s M(p) = \{u^s + pu_1^r, u^s + pu_2^r, \ldots\ldots\}$.

Now, let us use the construction described above for all pairs $r,s \in N$ with $\{\hat{\tau}_i^r\} < \{\hat{\tau}_i^s\}$ and $\tau_i^s(p) \neq \infty$ for all $i \in N$ and all $p \in \pi$.

We set $S = \langle \bigcup_{p \in \pi} {}_r^s M(p),\ s,r \in N \rangle_*^G$. Suppose that $S \in \mathfrak{M}$ and let $\mathfrak{X} = \pi'_1 \cup \pi'_2 \cup \ldots \cup \pi'_l$ be the corresponding partition of the set π. From Lemma 3 it easily follows that for any π'_j, $j \in \{1,2,\ldots,l\}$ there are $u,v,w \in N$ such that

$$\{\hat{\tau}_i^u \otimes Z\pi'_j\} < \{\hat{\tau}_i^v \otimes Z\pi'_j\} < \{\hat{\tau}_i^w \otimes Z\pi'_j\},$$

and each of these sequences contains infinitely many different terms. Clearly, for some $p \in \pi'_j$ $\tau_i^v(p) \otimes Z\pi'_j \neq \infty$ for any $i \in N$. However, for these $p \in \pi$ and $u,v \in N$ we have already constructed the set ${}_v^u M(p)$ and for the elements of this set we get contradiction by the same methods as in the preceding proof.

<u>Lemma 11</u>: Let $G = D \oplus H$ be a completely decomposable group with the ordered type set, where D is divisible and H reduced. If T(H) is either inversely well-ordered or it contains an infinite increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \ldots$ such that for every $\hat{\tau} \in T(H)$ it is $\hat{\tau} < \hat{\tau}_n$ for some $n \in \underline{N}$ and the set $\{\hat{\tau} \in T(H) \mid \hat{\tau} \leq \hat{\tau}_n\}$ is inversely well-ordered for every $n \in \underline{N}$, then any pure subgroup of G is completely decomposable.

<u>Proof</u>: See [2, Theorem 2] and [6, Theorem 1].

<u>Definition 2</u>: We shall say that a torsionfree group G satis-

fies the condition (P) if for any subset $\pi' \subseteq \pi$ such that $G \otimes Z_{\pi'}$ is completely decomposable with the ordered type set $T(G \otimes Z_{\pi'})$ containing an increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \ldots$ there is a prime $p \in \pi'$ such that $\tau(p) = \infty$ for each type $\hat{\tau} \in T(G \otimes Z_{\pi'})$ with $\hat{\tau}_1 < \hat{\tau}_2 < \ldots < \hat{\tau}$.

<u>Proposition 1</u>: If G is a torsionfree group which does not satisfy the condition (P), then G contains a pure subgroup not belonging to the class $\mathfrak{M}$.

<u>Proof</u>: By hypothesis, there is a subset $\pi' \subseteq \pi$ such that $G \otimes Z_{\pi'}$ is completely decomposable with the ordered type set $T(G \otimes Z_{\pi'})$ containing an increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \ldots$ such that for every prime $p \in \pi'$ there is a type $\hat{\tau} \in T(G \otimes Z_{\pi'})$ with $\hat{\tau}_1 < \hat{\tau}_2 < \ldots < \hat{\tau}$ and $\tau(p) < \infty$.

Let $\pi' = \{p_1, p_2, \ldots\}$ be any ordering on the set π'. By hypothesis, there is a type $\hat{\tau}'_1 \in T(G \otimes Z_{\pi'})$ with $\hat{\tau}_1 < \hat{\tau}_2 < \ldots < \hat{\tau}'_1$ and $\tau'_1(p_1) < \infty$. Suppose that we have found the types $\hat{\tau}'_1, \hat{\tau}'_2, \ldots, \hat{\tau}'_k$ in $T(G \otimes Z_{\pi'})$ in such a way that $\hat{\tau}_1 < \hat{\tau}_2 < \ldots < \hat{\tau}'_k \leq \ldots \ldots \leq \hat{\tau}'_2 \leq \hat{\tau}'_1$ and $\tau'_i(p_i) < \infty$ for each $i \in \{1, 2, \ldots, k\}$. If $\tau'_k(p_{k+1}) < \infty$, then we set $\hat{\tau}'_{k+1} = \hat{\tau}'_k$. If $\tau'_k(p_{k+1}) = \infty$ then, by hypothesis, there is $\hat{\tau}'_{k+1} \in T(G \otimes Z_{\pi'})$ such that $\hat{\tau}_1 < \hat{\tau}_2 < \ldots \ldots < \hat{\tau}'_{k+1} \leq \hat{\tau}'_k$ and $\tau'_{k+1}(p_{k+1}) < \infty$. Thus, by the induction, we have constructed the types $\hat{\tau}'_1, \hat{\tau}'_2, \ldots$ in $T(G \otimes Z_{\pi'})$ in such a way that $\hat{\tau}_1 < \hat{\tau}_2 < \ldots < \hat{\tau}_n < \ldots < \hat{\tau}'_n \leq \ldots \leq \hat{\tau}'_2 \leq \hat{\tau}'_1$ and $\tau'_i(p_i) < \infty$ for all $i \in \underline{N}$. An application of Lemma 8 or 9 gives the existence of a pure subgroup S of $G \otimes Z_{\pi'}$ which does not belong to the class $\mathfrak{M}$. This finishes the proof owing to the simple facts that $S \cap G$ is pure in G and $(S \cap G) \otimes Z_{\pi'} = S$.

<u>Definition 3</u>: We shall sav that a torsionfree group G satis-

fies the condition (R) if there is a non-negative integer l such that for any subset $\pi' \subseteq \pi$ for which $G \otimes Z_{\pi'}$ is completely decomposable with the ordered type set $T(G \otimes Z_{\pi'})$ any chain of increasing sequences $\{\hat{\tau}_i^1\} < \{\hat{\tau}_i^2\} < \ldots$ of elements from $T(G \otimes Z_{\pi'})$ contains at most l terms.

Proposition 2: If G is a torsionfree group which does not satisfy the condition (R), then G contains a pure subgroup not belonging to the class $\mathfrak{M}$.

Proof: We can suppose that $G \in \mathfrak{M}$ and so there is a partition $\pi = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n$ of π such that $G \otimes Z_{\pi_j}$ is a completely decomposable group with the ordered type set $T(G \otimes Z_{\pi_j})$ for each $j \in \{1,2,\ldots, n\}$. By hypothesis, for every $l \in \underline{N}$ there is a subset $\pi'_l \subseteq \pi$ such that $G \otimes Z_{\pi'_l}$ is completely decomposable with the ordered type set $T(G \otimes Z_{\pi'_l})$ containing at least l increasing sequences $\{\hat{\tau}_i^1\} < \{\hat{\tau}_i^2\} < \ldots < \{\hat{\tau}_i^l\}$ of elements. It follows from Lemma 4 easily that for some $j \in \{1,2,\ldots,n\}$ the type set $T(G \otimes Z_{\pi_j})$ contains the increasing chains of increasing sequences of arbitrary lengths. Consequently, Lemma 5 yields that $T(G \otimes T_{\pi_j})$ contains an infinite chain of increasing sequences. Now it suffices to apply Lemma 10.

The following example shows that the condition (P) is not sufficient for a group $G \in \mathfrak{M}$ to have all pure subgroups in the class $\mathfrak{M}$.

Example: Let $\pi = \bigcup_{k=1}^{\infty} \pi_k$ be a disjoint decomposition of π into infinite subsets, $\pi_k = \{p_{k1}, p_{k2}, \ldots\}$ for each $k \in \underline{N}$. For each pair $i,j \in \underline{N}$ we define the characteristic τ_i^j such that for each $l \in \underline{N}$ we set $\tau_i^j(p_{kl}) = \infty$ for $k < j$, $\tau_i^j(p_{kl}) = i$ for $k = j$

and $\alpha_i^j(p_{kl}) = 0$ for $k > j$. Then the corresponding types form an infinite increasing chain of increasing sequences $\{\hat{\tau}_i^1\} < \{\hat{\tau}_i^2 <$ $< \dots$. If J_i^j is a rank one group of the type $\hat{\tau}_i^j$, then the group $G = \bigoplus_{i,j=1}^{\infty} J_i^j$ obviously satisfies the condition (P), but does not satisfy the condition (R).

Definition 4: We shall say that a torsionfree group G is of the type (n, l) if $G \in \mathfrak{M}(n) \setminus \mathfrak{M}(n-1)$ and G satisfies the condition (R) with l the smallest possible.

Proposition 3: If G is a torsionfree group of the type (n,l) satisfying the condition (P), then every pure subgroup of G belongs to some class $\mathfrak{M}(m)$, where $m \leq 2nl$.

Proof: By Lemmas 6 and 7 there is a partition of the set π into at most 2nl parts, $\pi = \pi_1 \cup \pi_2 \cup \dots \cup \pi_k$, such that for each $j \in \{1,2,\dots,k\}$ the group $G \otimes Z_{\pi_j}$ is completely decomposable of the form $D \oplus H$, where D is divisible, H reduced and T(H) is either inversely well-ordered or it contains an infinite increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \dots$ such that for every $\hat{\tau} \in T(H)$ it is $\hat{\tau} < \hat{\tau}_i$ for some $i \in \underline{N}$ and the set $\{\hat{\tau} \in T(H) \mid \hat{\tau} \leq \hat{\tau}_i\}$ is inversely well-ordered for every $i \in \underline{N}$. An application of Lemma 11 now finishes the proof.

Remark: With respect to the proofs of Lemmas 6 and 7 it is not too hard to show that to any $n \in \underline{N}$, $l \in \underline{N}_0$ and $m \in \underline{N}$ with $m \leq 2nl$ there exists a torsionfree group G of the type (n,l) containing a pure subgroup S belonging to the class $\mathfrak{M}(m) \setminus \mathfrak{M}(m-1)$.

Theorem 1: Any pure subgroup of a torsionfree group G belongs to $\mathfrak{M}$ if and only if G satisfies conditions (P) and (R).

Proof: By Propositions 1, 2 and 3.

Definition 5: We shall say that a torsionfree group G satisfies the condition (Rg) if for any subset $\pi' \subseteq \pi$ such that $G \otimes Z_{\pi'}$ is completely decomposable with the ordered type set $T(G \otimes Z_{\pi'})$ containing an increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \ldots$ there is a prime $p \in \pi'$ such that $\tau_k(p) = \infty$ for some $k \in \underline{N}$.

Lemma 12: If a torsionfree group G satisfies the condition (Rg) then it satisfies the condition (P).

Proof: Obvious.

Lemma 13: Let M be an ordered set of types satisfying the following condition:

(**) If π' is a subset of π such that the set $M \otimes Z_{\pi'} = \{\tau' \otimes Z_{\pi'} \mid \hat{\tau} \in M\}$ contains an increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \ldots$ then there is a prime $p \in \pi'$ such that $\tau_k(p) = \infty$ for some $k \in \underline{N}$.

If M contains no two increasing sequences $\{\hat{\tau}_i\}$ and $\{\hat{\sigma}_i\}$ with $\{\hat{\tau}_i\} < \{\hat{\sigma}_i\}$ then there is a partition $\pi = \pi_1 \cup \pi_2$ of π such that $M \otimes Z_{\pi_1}$ is inversely well-ordered and either $\pi_2 = \emptyset$ or $M \otimes Z_{\pi_2}$ contains an infinite increasing sequence $\hat{\sigma}_1 < \hat{\sigma}_2 < \ldots$ such that for each $n \in \underline{N}$ the set $\{\hat{\tau} \in M \otimes Z_{\pi_2} \mid \hat{\tau} \leq \hat{\sigma}_n\}$ is inversely well-ordered, for each $\hat{\tau} \in M \otimes Z_{\pi_2}$ it is either $\hat{\tau} = \hat{R}$ or $\hat{\tau} < \hat{\sigma}_n$ for some $n \in \underline{N}$ and for every prime p it is $\sigma_k(p) = \infty$ for some $k \in \underline{N}$.

Proof: If M contains no infinite increasing sequence, then it is inversely well-ordered and it suffices to put $\pi_1 = \pi$, $\pi_2 = \emptyset$.

Assume that M contains an infinite increasing sequence $\{\hat{\tau}_i\}$. Set $\pi_1 = \{p \in \pi \mid \tau_i(p) < \infty$ for all $i \in \underline{N}\}$ and $\pi_2 = \pi \setminus \pi_1$. Since M satisfies the condition (**), the set $\{\hat{\tau}_i \otimes Z_{\pi_1}\}$ is finite and consequently the set $M \otimes Z_{\pi_1}$ is inversely well-ordered. By Lemma 2, the sequence $\{\hat{\tau}_i \otimes Z_{\pi_2}\}$ contains infinitely many different terms and the assertion follows easily.

Lemma 14: Let G be a completely decomposable group of the form $G = \bigoplus_{i=1}^{\infty} J_i$, where J_i are of rank one and of the types $\hat{\tau}_i$, $i \in \underline{N}$. If $\pi' = \{p_1, p_2, \dots\}$ is a set of primes such that $\hat{\tau}_1 \otimes Z_{\pi'} < \hat{\tau}_2 \otimes Z_{\pi'} < \dots$ and $\tau_i(p) < \infty$ for all $i \in \underline{N}$ and $p \in \pi'$, then G contains a regular subgroup H not belonging to the class $\mathfrak{M}$.

Proof: For each $i \in \underline{N}$ set $U_i = \bigoplus_{k=1}^{\infty} J_{p_i^k}$ and decompose G into $G = \bigoplus_{i=1}^{\infty} U_i \oplus V$. In each $J_{p_i^k}$ select an element u_{ik} with zero p_i-height in G and consider the subgroup

$$H = \langle V, p_i U_i, u_{ik} - u_{i,k+1} \mid i,k \in \underline{N} \rangle$$

of G. It is easy to see that H is a regular subgroup of G.

First, we shall show that $u_{i1} \notin H$ for all $i \in \underline{N}$. Proving indirectly, suppose that $u_{i1} \in H$ for some $i \in \underline{N}$. In view of the form of G we then have $u_{i1} = p_i u_i + \sum_{k=1}^{r-1} \lambda_k (u_{ik} - u_{i,k+1})$. Since $u_i \in U_i$, there are integers $m, \mu_1, \mu_2, \dots, \mu_r$ such that $m u_i = \sum_{k=1}^{r} \mu_k u_{ik}$ and owing to $h_{p_i}^G(u_{ik}) = 0$ we can suppose that $(m, p_i) = 1$. Thus we have $m\, u_{i1} = p_i \sum_{k=1}^{r} \mu_k u_{ik} + m \sum_{k=1}^{r-1} \lambda_k (u_{ik} - u_{i,k+1})$ and consequently

$$\begin{aligned}
p_i \mu_1 \qquad\quad + m\lambda_1 &= m,\\
p_i \mu_2 - m\lambda_1 + m\lambda_2 &= 0,\\
\dots\dots\dots\dots\dots\dots\dots\dots &\\
p_i \mu_{r-1} - m\lambda_{r-2} + m\lambda_{r-1} &= 0,\\
p_i \mu_r - m\lambda_{r-1} \qquad\quad &= 0.
\end{aligned}$$

Adding all these equalities we get $p_i \sum_{k=1}^{r} \mu_k = m$, which contradicts the hypothesis $(m,p_i) = 1$.

Suppose now that $H \in \mathfrak{M}$. Then there is a partition $\pi = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n$ of π such that $H \otimes Z_{\pi_j}$ is completely decomposable with the ordered type set $T(H \otimes Z_{\pi_j})$ for each $j \in \{1,2,\ldots,n\}$. By Lemma 2 there is $j \in \{1,2,\ldots,n\}$ such that the sequence $\hat{\tau}_1 \otimes Z_{\pi'_j} \leq \hat{\tau}_2 \otimes Z_{\pi'_j} \leq \ldots$, where $\pi'_j = \pi' \cap \pi_j$, contains infinitely many different terms.

The group $H \otimes Z_{\pi'_j}$ is obviously completely decomposable, $H \otimes Z_{\pi'_j} = \bigoplus_{\alpha \in A} I_\alpha$. Let p_i be any prime from π'_j. Because $p_i u_{i1} \in H \subseteq H \otimes Z_{\pi'_j}$, the element $p_i u_{i1}$ has a non-zero component in finitely many I_α's. Let H_1 be the direct sum of those direct summands I_α of $H \otimes Z_{\pi'_j}$, in which $p_i u_{i1}$ has a non-zero component, and H_2 be the direct sum of all other direct summands I_α of $H \otimes Z_{\pi'_j}$. From the finiteness of $T(H_1)$ and from the preceding part the existence follows of $\hat{\tau}_s \otimes Z_{\pi'_j}$ with $\hat{\tau}_s \otimes Z_{\pi'_j} > \hat{\tau}$ for all $\hat{\tau} \in T(H_1)$ and so $p_i u_{is} \in H_2$.

Further, $u_{i1} - u_{is} = (u_{i1} - u_{i2}) + (u_{i2} - u_{i3}) + \ldots + (u_{i,s-1} - u_{is}) \in H \otimes Z_{\pi'_j}$ and hence $u_{i1} - u_{is} = h_1 + h_2$, $h_1 \in H_1$, $h_2 \in H_2$. Multiplying by p_i we get $p_i u_{i1} = p_i h_1$ and so $u_{i1} = h_1 \in H$. This contradiction completes the proof.

Lemma 15: Let $G = D \oplus H$ be a completely decomposable group with the ordered type set, where D is divisible and H reduced. If $T(H)$ either is inversely well-ordered or it contains an infinite increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \ldots$ such that for every $\hat{\tau} \in T(H)$ it is $\hat{\tau} < \hat{\tau}_n$ for some $n \in \underline{N}$, the set $\{\hat{\tau} \in T(H) \mid \hat{\tau} \leq \hat{\tau}_n\}$ is inversely well-ordered for every $n \in \underline{N}$ and for every prime p it is $\tau_k(p) = \infty$ for some $k \in \underline{N}$, then every regular subgroup of G is completely decomposable.

Proof: See [3, Theorem 2] and [6, Theorem 2].

Proposition 4: If G is a torsionfree group which does not satisfy the condition (Rg), then G contains a regular subgroup not belonging to the class $\mathfrak{M}$.

Proof: By hypothesis, there is a subset $\pi' \subseteq \pi$ such that $G \otimes Z_{\pi'}$ is completely decomposable with the ordered type set $T(G \otimes Z_{\pi'})$ containing an increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \dots$ such that $\tau_i(p) < \infty$ for all $i \in \underline{N}$ and $p \in \pi'$. An application of Lemma 14 now finishes the proof.

Proposition 5: If G is a torsionfree group of the type (n,l) satisfying the condition (Rg) then every regular subgroup of G belongs to some class $\mathfrak{M}(m)$ where $m \leq 2nl$.

Proof: Using Lemma 12 we see that by Lemmas 6 and 13 there is a partition of the set π into at most 2nl parts, $\pi = \pi_1 \cup \pi_2 \cup \cup \dots \cup \pi_k$, such that for each $j \in \{1,2,\dots,k\}$ the group $G \otimes Z_{\pi_j}$ is completely decomposable of the form $D \oplus H$ where D is divisible, H reduced and T(H) is either inversely well-ordered or it contains an in finite increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \dots$ such that for every $\hat{\tau} \in T(H)$ it is $\hat{\tau} < \hat{\tau}_i$ for some $i \in \underline{N}$, the set $\{\hat{\tau} \in T(H) \mid \hat{\tau} \leq \hat{\tau}_i\}$ is inversely well-ordered for every $i \in \underline{N}$ and for every prime p it is $\tau_r(p) = \infty$ for some $r \in \underline{N}$. Now it suffices to apply Lemma 15.

Theorem 2: Any regular subgroup of a torsionfree group $G \in \mathfrak{M}$ belongs to $\mathfrak{M}$ if and only if G satisfies conditions (Rg) and (R).

Proof: By Propositions 2, 4 and 5.

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(Oblatum 11.9. 1985)