Commentationes Mathematicae Universitatis Carolinae

Marián J. Fabián; David Preiss

A generalization of the interior mapping theorem of Clarke and Pourciau

Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 2, 311--324

Persistent URL: http://dml.cz/dmlcz/106543

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

28,2(1987)

A GENERALIZATION OF THE INTERIOR MAPPING THEOREM OF CLARKE AND POURCIAU

M. FABIAN and D. PREISS

Abstract: We prove the following generalization of a result of Clarke and Pourciau. A mapping acting between two (super) reflexive Banach spaces which is locally approximable by convex subsets of linear surjections is locally surjective. The main tool of the proof is a modification of the Caristi's fixed point principle. We also show that this tool can be used for deriving theorems of Cramer and Ray, Džumabaev, and Graves.

Key words: Reflexive Banach space, Clarke's generalized Jacobian, interior mapping theorem.

Classification: 58C15, 47H15

---

1. Introduction. A special case of the well known theorem due to Graves [9], see Corollary 3, asserts that the image of a neighbourhood of $x_0 \in X$ under a mapping F acting between Banach spaces X and Y is a neighbourhood of Fx_0 provided that F is continuously Fréchet differentiable at x_0 and the derivative of F at x_0 is surjective. Clarke [2] for $X=Y=R^n$ and Pourciau [11] for $X=R^n$ and $Y=R^k$, $k \leq n$, have generalized this result for Lipschitz, not necessarily differentiable, mappings by showing

Theorem 1. Let $F:D(F) \subset R^n \longrightarrow R^k$, $k \leq n$, be a Lipschitz mapping and let x_0 be an interior point of the domain D(F) of F. Let $\partial F(x_0)$ denote the set of $n \times k$-matrices obtained as the closed convex hull of all possible limits

$$\lim_{m \to \infty} DF(x_m),$$

where $x_m \longrightarrow x_0$ and the derivatives $DF(x_m)$ exist.

If $\partial F(x_0)$ consists of matrices of maximal rank only, then $F(D(F))$ is a neighbourhood of Fx_0.

Let us suppose that $\partial F(x_0)$ contains matrices of maximal rank only. Then, owing to the finite dimensionality, a compactness argument ensures that there is an $\alpha > 0$ such that. for every $L \in \partial F(x_0)$ and every $y \in R^k$ there exists $x \in R^n$ satisfying

$$Lx=y \text{ and } \|y\| \geqq \alpha \|x\|.$$

Moreover, if $\beta \in (0, \alpha)$ is given, then by using the mean value theorem [11, Theorem 3.1, Proposition 3.2] and the compactness once more, we can find an $r > 0$ such that for any x_1, x_2 in the closed ball $B(x_0.r)$ centred at x_0 and of radius r there exists an $L \in \partial F(x_0)$ such that

$$\|Fx_1-Fx_2-L(x_1-x_2)\| \leqq \beta \|x_1-x_2\|.$$

These observations have led the first named author in [7] to generalize the above theorem to Hilbert spaces. The result obtained there asserts that if the above relations hold when replacing $\partial F(x_0)$ by a convex bounded subset of the space $\mathcal{L}(X,Y)$ of continuous linear mappings from X to Y, then the closure of F(D(F)) is a neighbourhood of Fx_0. Recently Ursescu [13] has shown by a more direct and simpler method that Fx_0 is in fact in the interior of F(D(F)). It should be noted that this can also be derived from the quoted result of [7] by using the Pták´s closed graph theorem [10].

In this paper we go on in generalizing this result:

Theorem 2. Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be two reflexive Banach spaces, $r > 0$, $\varrho > 0$, $\beta \geqq 0$, let $F:D(F) \subset X \longrightarrow Y$ be a mapping and let $x_0 \in D(F)$. Let us suppose that either F is continuous and its domain D(F) is closed or that F has a closed graph and Y is superreflexive. Moreover, let there exist a convex bounded subset $\mathfrak{M}$ of $\mathcal{L}(X,Y)$ such that whenever $x \in B(x_0,r) \cap D(F)$ and $h \in X$, there are $\varepsilon \in (0,1]$ and $L \in \mathfrak{M}$ fulfilling

(1) $$\|F(x-\varepsilon h)-Fx+\varepsilon Lh\| \leqq \varepsilon \beta \|h\|.$$

Finally, let us assume that the mappings from $\mathfrak{M}$ are uniformly open in the sense that, for each $L \in \mathfrak{M}$ and each $y \in Y$, there exists $x \in X$ such that

(2) $$Lx=y \text{ and } \|y\| \geqq (\beta + \varrho)\|x\|.$$

Then the open ball $\mathring{B}(Fx_0, \varrho r)$ of centre Fx_0 and radius ϱr is included in $F(B(x_0,r) \cap D(F))$.

Recall that in [7] it is required that whenever x and $x-\varepsilon h$ belong to $B(x_0,r)$, then (1) holds with some $L \in \mathfrak{M}$. It should be noted that the case $\beta > 0$ can be reduced to that of $\beta=0$; see Remark 5.

The proof consists of three steps. First, from a variant of the Caristi's fixed point principle (Lemma 1), we derive an interior mapping theorem (Corollary 2). Then we prove Lemmas 2 and 3 which show that the hypotheses of Theorem 2 lead to the situation occurred in Corollary 2.

We also show how Lemma 1 may be used to derive the interior mapping theorems of Cramer and Ray [3, Theorem 2.1], Džumabaev [6], and Graves [9, Theorem 1].

2. Caristi's principle and its consequences. We shall use the Caristi's fixed point principle [1],[5] in the following slight reformulation and generalization.

Lemma 1. Let Z be a set and let $d_0, d_1, \ldots, d_k$ be pseudometrics on Z. Suppose further that

$$d(z,\bar{z})=\max\{d_0(z,\bar{z}),\ldots,d_k(z,\bar{z})\},\quad z,\bar{z}\in Z,$$

is a metric in which Z is a complete metric space. Let us consider functions $f_0,f_1,\ldots,f_k:Z\longrightarrow[0,+\infty)$ which are lower semicontinuous with respect to d. Finally fix $z_0\in Z$ and let us assume that, for any $z\in Z$ fulfilling $f_0(z)>0$ and

$$(3)\qquad d_i(z_0,z)\leqq f_i(z_0)-f_i(z),\quad i=0,\ldots,k,$$

there exists $\bar{z}\in Z$, $\bar{z}\neq z$, such that

$$(4)\qquad d_i(z,\bar{z})\leqq f_i(z)-f_i(\bar{z}),\quad i=0,\ldots,k.$$

Then there exists $z\in Z$ such that $f_0(z)=0$ and $d_i(z_0,z)\leqq f_i(z_0)$, $i=0,\ldots,k$.

Proof. A simple induction argument ensures that there exists a sequence $\{z_0,z_1,\ldots\}\subset Z$ such that for all $n=0,1,\ldots$

$$d_i(z_n,z_{n+1})\leqq f_i(z_n)-f_i(z_{n+1}),\quad i=0,\ldots,k,$$

and

$$d_0(z_n,z_{n+1})\geqq \tfrac{1}{2}s_n,$$

where

$$s_n=\sup\{d_0(z_n,z):z\in Z, d_i(z_n,z)\leqq f_i(z_n)-f_i(z),\ i=0,\ldots,k\}.$$

Clearly

$$d_i(z_n,z_{n+m})\leqq d_i(z_n,z_{n+1})+\ldots+d_i(z_{n+m-1},z_{n+m})\leqq$$

$$\leqq f_i(z_n)-f_i(z_{n+1})+\ldots+f_i(z_{n+m-1})-f_i(z_{n+m})=f_i(z_n)-f_i(z_{n+m})$$

for all n, m and

$$f_i(z_0)\geqq f_i(z_1)\geqq \ldots \geqq 0.$$

It follows that $\{z_n\}$ is a Cauchy sequence in each pseudometric d_i, hence in the metric d. As d is complete, $\{z_n\}$ converges in the metric d to some $z\in Z$. Then from the lower semicontinuity of f_i we have for all n

$$d_i(z_n,z)=\lim_{m\to\infty} d_i(z_n,z_{n+m})\leqq f_i(z_n)-\lim_{m\to\infty} f_i(z_{n+m})\leqq f_i(z_n)-f_i(z)$$

and, especially,

$$d_i(z_0,z)\leqq f_i(z_0)-f_i(z)\leqq f_i(z_0),\ i=0,\ldots,k.$$

If $f_0(z)=0$, we are done. Further let us assume that $f_0(z)>0$. Then, by the assumptions, there exists $\bar{z}\in Z$, $\bar{z}\neq z$, such that

$$d_i(z,\bar{z})\leqq f_i(z)-f_i(\bar{z}).$$

For each i, we add this inequality and

$$d_i(z_n,z)\leqq f_i(z_n)-f_i(z),$$

and we obtain

$$d_i(z_n,\bar{z})\leqq d_i(z_n,z)+d_i(z,\bar{z})\leqq$$

$$\leqq f_i(z_n)-f_i(z)+f_i(z)-f_i(\bar{z})=f_i(z_n)-f_i(\bar{z}).$$

The definition of s_n then yields that $s_n\geqq d_0(z_n,\bar{z})$. But $s_n\leqq 2d_0(z_n,z_{n+1})\longrightarrow 0$. Hence

$$d_0(z,\bar{z})=\lim_{n\to\infty} d_0(z_n,\bar{z})\leqq\lim_{n\to\infty} s_n=0,$$

a contradiction with $\bar{z}\neq z$. We have thus shown that the possibility $f_0(z)>0$ cannot occur and so the proof is completed.

Remark 1. a) In applications the existence of $\bar{z}$ is often required for any $z\in Z$ with $f_0(z)>0$, which strengthens a little the assumptions of Lemma 1.

b) It is obvious but useful to realize that the functions f_i can be replaced by $\Phi_i\circ f_i$, where f_i are as in Lemma 1 and $\Phi_i: [0,+\infty)\longrightarrow[0,+\infty)$ are nondecreasing lower semicontinuous with $\Phi_i(s)=0$ if and only if $s=0$.

c) Another useful variant of b) is to replace (3) and (4) by

(3´) $d_i(z_0,z)\leqq A_i(f_i(z_0))(f_i(z_0)-f_i(z))$ and

$$(4')\qquad d_i(z,\bar z)\leqq A_i(f_i(z))(f_i(z)-f_i(\bar z))$$

respectively. If $A_i\geqq 0$, if Φ_i fulfil the same hypotheses as in b), and if

$$(5)\quad A_i(u)(u-v)\leqq \Phi_i(u)-\Phi_i(v) \text{ whenever } 0\leqq v<u,$$

then, according to b), there is $z\in Z$ such that $f_0(z)=0$ and $d_i(z_0,z)\leqq \Phi_i(f_i(z_0))$. Often used situations in which (5) holds are, for example:

$A: (0,+\infty)\longrightarrow(0,+\infty)$, $\int_0^c A(s)ds<+\infty$ for all $c>0$, and

α) A is nonincreasing. Then, if we take $\Phi(u)=\int_0^u A(s)ds$, we get for $0\leqq v<u$

$$\Phi(u)-\Phi(v)=\int_v^u A(s)ds\geqq A(u)(u-v).$$

Hence (5) is satisfied.

β) A is nondecreasing. Then the choice $\Phi(u)=\int_0^{2u}A(s)ds$ yields for $0\leqq v<u$

$$\Phi(u)-\Phi(v)=\int_{2v}^{2u}A(s)ds\geqq\int_u^{2u}A(s)ds\geqq A(u)u>A(u)(u-v)$$

if $2v\leqq u$, and

$$\Phi(u)-\Phi(v)\geqq A(2v)(2u-2v)>A(u)(u-v)$$

if $2v>u$. Thus (5) holds again.

γ) A(s)s is nondecreasing. Then for $\Phi(u)=\int_0^{eu}A(s)ds$ and $0\leqq v<u$ we have

$$\Phi(u)-\Phi(v)=\int_{ev}^{eu}A(s)ds\geqq\int_u^{eu}A(s)s\,\frac{1}{s}\,ds\geqq A(u)u\int_u^{eu}\frac{1}{s}\,ds=$$
$$=A(u)u>A(u)(u-v)$$

if $ev\leqq u$, and

$$\Phi(u)-\Phi(v)\geqq A(ev)ev\int_{ev}^{eu}\frac{1}{s}\,ds>A(u)u\cdot\ln\left(\frac{u}{v}\right)>A(u)(u-v)$$

if $ev>u$.

d) Requiring stronger versions of (3'), (4') we can get better choices for Φ. For example, if $A_i(s)s$ are nondecreasing and if we replace (3'),(4') by the inequalities

$$d_i(z_0,z)\leqq A_i(f_i(z_0))(f_i(z_0)-\max(q_if_i(z_0),f_i(z))),$$
$$d_i(z,\bar z)\leqq A_i(f_i(z))(f_i(z)-\max(q_if_i(z),f_i(\bar z)))$$

respectively, where $q_i\in[0,1)$ are fixed, then we can take $\Phi_i(u)=\int_0^{ue^{1-q_i}}A_i(s)ds$.

Corollary 1. Let (X,d), (Y,d) be complete metric spaces, $r>0$, $\wp>0$, $c>0$, let $F:D(F)\subset X\to Y$ be a mapping and let $x_0\in D(F)$. Let us assume that F is continuous and $D(F)$ is closed (or that F has a closed graph only). Finally, suppose that, for any $x\in B(x_0,r)\cap D(F)$ and any $y\in B(Fx_0,\wp r)$, $y\neq Fx$, there exists $\bar{x}\in D(F)$, $\bar{x}\neq x$, such that

(6) $$\wp d(x,\bar{x})\leqq d(Fx,y)-d(F\bar{x},y)$$

(and moreover, if F has a closed graph only, that

(7) $$cd(Fx,F\bar{x})\leqq d(Fx,y)-d(F\bar{x},y)).$$

Then $B(Fx_0,\wp r)\subset F(B(x_0,r)\cap D(F))$.

Proof. Fix $y\in B(Fx_0,\wp r)$. We are to find an $x\in B(x_0,r)\cap D(F)$ such that $Fx=y$. Denote $Z=D(F)$

$$d_0(x,\bar{x})=\wp d(x,\bar{x}),\ d_1(x,\bar{x})=cd(Fx,F\bar{x}),\ x,\bar{x}\in Z,$$
$$f_0(x)=f_1(x)=d(Fx,y),\ x\in Z.$$

If F is continuous and $D(F)$ is closed, take $k=0$, while in the parenthetic case consider $k=1$. Clearly Z is complete and f_0, f_k are continuous in the metric $\max(d_0,d_k)$. Also, the inequalities (6) and (7) pass exactly to (4). The assumptions of Lemma 1 are thus verified and so there exists an $x\in D(F)$ such that $f_0(x)=0$, i.e., $Fx=y$, and that $d_0(x_0,x)\leqq f_0(x_0)$, which implies that $x\in B(x_0,r)$.

Remark 2. a) For slightly weaker assumptions of the above corollary see the exact formulation of Lemma 1.

b) In the same way as in Remark 1 - b),c),d) one can replace (6),(7) by using the functions A and Φ . In fact, the version of Corollary 1 obtained by the use of d) implies [3, Theorem 2.1].

c) Corollary 1 can be extended to multivalued mappings. Thus, if $F:D(F)\subset X\longrightarrow 2^Y$ is upper semicontinuous closed valued and $D(F)$ is closed, then (6) should be replaced by

$$\wp d(x,\bar{x})\leqq \mathrm{dist}(Fx,y)-\mathrm{dist}(F\bar{x},y),$$

while if F has a closed graph only, then (6) and (7) should read as

$$\max(\wp d(x,\bar{x}),cd(v,\bar{v}))\leqq d(v,y)-d(\bar{v},y),$$

where (x,v) and $(\bar{x},\bar{v})$ lie in the graph of F.

d) We also notice that, if F is continuous, the completeness of Y is not necessary. A similar remark applies also to the consequences of Corollary 1.

Corollary 2. Let X, Y be Banach spaces, $r > 0$, $\rho > 0$, $q \in [0,1)$, let $F: D(F) \subset X \longrightarrow Y$ be a mapping and let x_0 be in $D(F)$. Let us assume that F is continuous and $D(F)$ is closed (or that F has a closed graph only). Finally, suppose that, for any $x \in B(x_0,r) \cap D(F)$ and any $y \in B(Fx_0, \rho r)$, $y \neq Fx$, there exist $0 \neq h \in X$ and $\varepsilon \in (0,1]$ such that

(8) $$\rho\|h\| + \frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx+\varepsilon(Fx-y)\| \leqq \|Fx-y\|$$

(and moreover, if F has a closed graph only, that

(9) $$\frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx+\varepsilon(Fx-y)\| \leqq q\|Fx-y\|).$$

Then $B(Fx_0, \rho r) \subset F(B(x_0,r) \cap D(F))$.

Proof. Take $x \in B(x_0,r) \cap D(F)$, $y \in B(Fx_0, \rho r) \setminus \{Fx\}$ arbitrarily. By the hypotheses find h and ε corresponding to x and y. The triangle inequality then yields

$$\|F(x-\varepsilon h)-y\| \leqq \|F(x-\varepsilon h)-Fx+\varepsilon(Fx-y)\|+(1-\varepsilon)\|Fx-y\|.$$

Thus, by (8),

$$\|F(x-\varepsilon h)-y\| \leqq \|Fx-y\| - \rho\varepsilon\|h\|,$$

and after denoting $\bar{x}=x-\varepsilon h$, we get

$$\rho\|x-\bar{x}\| \leqq \|Fx-y\| - \|F\bar{x}-y\|,$$

which is the inequality (6). If (9) holds, then

$$\|F\bar{x}-y\| \leqq \varepsilon q\|Fx-y\|+(1-\varepsilon)\|Fx-y\| = (1-\varepsilon(1-q))\|Fx-y\|,$$

and

$$\|Fx-F\bar{x}\| \leqq \|F\bar{x}-Fx+\varepsilon(Fx-y)\| + \varepsilon\|Fx-y\| \leqq \varepsilon(1+q)\|Fx-y\| =$$

$$= \frac{1+q}{1-q}(\|Fx-y\|-(1-\varepsilon(1-q))\|Fx-y\|) \leqq \frac{1+q}{1-q}(\|Fx-y\|-\|F\bar{x}-y\|)$$

and so (7) is verified. It means that Corollary 1 can be applied and consequently $B(Fx_0, \rho r) \subset F(B(x_0,r) \cap D(F))$.

In the proof of Theorem 2 we shall need only Corollary 2. But we feel that further consequences of this corollary should also be mentioned.

Corollary 3. Let X, Y be Banach spaces, $r>0$, $\varrho>0$, $\beta\geqq 0$, let $F:D(F)\subset X\longrightarrow Y$ be a mapping with a closed graph and let $x_0\in D(F)$. Let us assume that there exists $L\in\mathcal{L}(X,Y)$ such that, for every $x\in B(x_0,r)\cap D(F)$ and every $h\in X$, there is $\varepsilon\in(0,1]$ fulfilling

(10) $$\|F(x-\varepsilon h)-Fx+\varepsilon Lh\|\leqq\varepsilon\beta\|h\|.$$

Finally, suppose that the L is such that to each $y\in Y$ there is $x\in X$ satisfying $Lx=y$ and $\|y\|\geqq(\beta+\varrho)\|x\|$.

Then $B(Fx_0,\varrho r)\subset F(B(x_0,r)\cap D(F))$.

Proof. Fix $x\in B(x_0,r)\cap D(F)$ and $y\in B(Fx_0,\varrho r)$, $y\neq Fx$. Find $h\in X$ such that $Lh=Fx-y$ and $(\beta+\varrho)\|h\|\leqq\|Fx-y\|$. Let ε correspond to x and h. Then

$$\frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx+\varepsilon(Fx-y)\|=\frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx+\varepsilon Lh\|\leqq$$
$$\leqq\beta\|h\|\leqq\frac{\beta}{\beta+\varrho}\|Fx-y\|$$

and so both (8) and (9) hold. Now apply Corollary 2.

Remark 3. The above corollary is a slight improvement of the result of Graves [9, Theorem 1], where (10) is required to hold whenever x and $x-\varepsilon h$ belong to $\mathring{B}(x_0,r)$. Another proof of the theorem of Graves, by using Nadler´s contraction principle for multivalued mappings is due to Szilágyi [12].

Corollary 4. Let X, Y be Banach spaces, $r>0$, $\alpha>0$, $\Theta\in(0,1)$, let $F:D(F)\subset X\longrightarrow Y$ be a mapping with a closed graph and let $x_0\in D(F)$. Let us assume that, for every x from $B(x_0,r)\cap D(F)$, there are $\alpha_x>\alpha$, $\beta_x\in[0,\Theta\alpha_x)$, $\delta_x>0$ and a mapping $C_x:\mathring{B}(0,\delta_x)\subset X\longrightarrow Y$ such that $\|Fu-Fx-C_x(u-x)\|\leqq\beta_x\|u-x\|$ whenever $u\in\mathring{B}(x_0,r)\cap\mathring{B}(x,\delta_x)$, and that for every $y\in Y$ there is $h\in X$ satisfying $\|y\|\geqq\alpha_x\|h\|$ and $C_x(\varepsilon h)=\varepsilon y$ for all $\varepsilon>0$ sufficiently small.

Then $\mathring{B}(Fx_0,(1-\Theta)\alpha r)\subset F(\mathring{B}(x_0,r)\cap D(F))$.

Proof. Choose a fixed $\hat{r}\in(0,r)$. Take arbitrary x in $B(x_0\hat{r})\cap D(F)$ and y in $B(Fx_0,(1-\Theta)\alpha\hat{r})\setminus\{Fx\}$. Find $h\in X$ and $\varepsilon_0\in(0,1)$ such that $\|Fx-y\|\geqq\alpha_x\|h\|$ and $C_x(-\varepsilon h)=\varepsilon(Fx-y)$ whenever $\varepsilon\in(0,\varepsilon_0)$. Fix an $\varepsilon\in(0,\varepsilon_0)$ so small that

$$\|x-\varepsilon h-x\|<\delta_x \text{ and } \|x-\varepsilon h-x_0\|<r.$$

Then we can estimate

$$\frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx+\varepsilon(Fx-y)\| = \frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx-C_x(-\varepsilon h)\| \leqq \beta_x \|h\| < \leqq \Theta\|Fx-y\|$$

and

$$(1-\Theta)\alpha\|h\| + \frac{1}{\varepsilon}\|F(x-\varepsilon h)-Fx+\varepsilon(Fx-y\| \leqq \frac{(1-\Theta)\alpha}{\alpha_x}\|Fx-y\| + \Theta\|Fx-y\| < \|Fx-y\|.$$

It means that Corollary 2 applies. Hence $B(Fx_0,(1-\Theta)\alpha\hat{r})$ $F(B(x_0,\hat{r})\cap D(F))$ and by letting $\hat{r}$ go to r the result follows.

Remark 4. This corollary is a slight improvement of the result of Džumabaev [6], where the C_x are assumed to have inverses and an additional condition $\sigma_x > tr(1-\Theta)\alpha/\alpha_x$ with a fixed $t \in (0,1)$ is required.

Corollary 5. Let X, Y be Banach spaces, $r > 0$, $\rho > 0$, and $q \in [0,1)$. Let $F: D(F) \subset X \longrightarrow Y$ be a continuous mapping (or a mapping with a closed graph), Gâteaux differentiable on $B(x_0,r) \subset \subset D(F)$. Let for every $x \in B(x_0,r)$ and every $y \in B(Fx_0,\rho r)$, $y \neq Fx$, there exist $0 \neq h \in X$ such that

$$\rho\|h\| + \|Fx-y-DF(x)h\| \leqq \|Fx-y\|$$

(and moreover, if F has a closed graph only, that

$$\|Fx-y-DF(x)h\| \leqq q\|Fx-y\|).$$

Then $\mathring{B}(Fx_0,\rho r) \subset F(\mathring{B}(x_0,r))$.

Proof. Take $\hat{\rho} \in (0,\rho)$, $\hat{q} \in (q,1)$, and $0 < \tilde{r} < \hat{r} < r$. Let $\hat{F}$ be the restriction of F to $B(x_0,\hat{r})$. Then $D(\hat{F})$ is closed and the above inequalities ensure that there exists $\varepsilon \in (0,1]$ such that the assumptions of Corollary 2 hold with r, ρ, q, and F replaced by $\tilde{r}$, $\hat{\rho}$, $\hat{q}$, and $\hat{F}$ respectively. Hence $B(Fx_0,\hat{\rho}\tilde{r}) \subset F(B(x_0,\tilde{r}))$ and we conclude the proof by letting $\hat{\rho}$ converge to ρ and $\tilde{r}$ converge to r.

Corollary 6 ([9, Theorem 3]). Let $L_0 \in \mathcal{L}(X,Y)$ and let there exist $\alpha > 0$ such that to every $y \in Y$ there is $x \in X$ satisfying $L_0x=y$ and $\|y\| \geqq \alpha\|x\|$.

If $L \in \mathcal{L}(X,Y)$ is such that $\|L-L_0\| < \alpha$, then $(\alpha - \|L-L_0\|)B_Y \subset L(B_X)$, where B_X and B_Y denote the closed unit balls in X and Y respectively.

Proof. Let L be as above and take any $x\in X$ and any $y\in Y\setminus\{Lx\}$. Find $h\in X$ such that $L_0h=Lx-y$ and $\alpha\|h\|\leqq\|Lx-y\|$. Then $h\neq 0$ and

$$(\alpha-\|L-L_0\|)\|h\|+\|L(x-h)-Lx+(Lx-y)\|=(\alpha-\|L-L_0\|)\|h\|+\|Lh-L_0h\|\leqq$$
$$\leqq\alpha\|h\|\leqq\|Fx-y\|.$$

Thus (8) holds with $\varepsilon=1$ and so, by Corollary 2

$$(\alpha-\|L-L_0\|)B_Y=B(L(0),(\alpha-\|L-L_0\|))\subset L(B(0,1))=L(B_X).$$

Remark 5. The above corollary enables us to reduce in Theorem 2 the case $\beta>0$ to that of $\beta=0$. Let us show it. Define

$$\mathfrak{M}'=\{L'\in\mathcal{L}(X,Y):\|L'-L\|\leqq\beta \text{ for some } L\in\mathfrak{M}\}.$$

Let $x\in B(x_0,r)\cap D(F)$ and $h\in X$ be given. Clearly, we may suppose that $h\neq 0$. Let $G\subset X$ be a hyperplane such that $\|h+g\|\geqq\|h\|$ for all $g\in G$. Define

$$L'(th+g)=-\frac{t}{\varepsilon}(F(x-\varepsilon h)-Fx)+Lg,\ t\in R,\ g\in G,$$

where $\varepsilon>0$ and $L\in\mathfrak{M}$ correspond to x and h. Then $F(x-\varepsilon h)-Fx+\varepsilon L'h=0$. Further, L' is linear and by (1)

$$\|L'(th+g)-L(th+g)\|=\|\frac{t}{\varepsilon}(F(x-\varepsilon h)-Fx)+L(th)\|\leqq\frac{|t|}{\varepsilon}\varepsilon\beta\|h\|=$$
$$\leqq\frac{|t|}{\varepsilon}\varepsilon\beta\|h\|=|t|\beta\|h\|\leqq\beta\|th+g\|$$

for all $t\in R$ and all $g\in G$. Hence $\|L'-L\|\leqq\beta$, and so $L'\in\mathfrak{M}'$. Now applying Corollary 6 we get that

$$\rho B_Y\subset(\beta+\rho-\|L'-L\|)B_Y\subset L'(B_X).$$

From this inclusion it easily follows that for every $y\in Y$ there is $x\in X$ such that $L'x=y$ and $\|y\|\geqq\rho\|x\|$. Thus we have shown that the assumptions of Theorem 2 are fulfilled with β and $\mathfrak{M}$ replaced by 0 and $\mathfrak{M}'$ respectively.

3. Geometrical lemmas and the proof of Theorem 2. If X is a Banach space, let X^* denote its dual, X^{**} its second dual, $\varkappa:X\longrightarrow X^{**}$ the canonical embedding and $\langle x^*,x\rangle$ the value of $x^*\in X^*$ at $x\in X$. If $L\in\mathcal{L}(X,Y)$, then L^* means the adjoint to L.

Lemma 2. Let X and Y be Banach spaces, $\mathfrak{M}\subset\mathcal{L}(X,Y)$ be a convex set, and $\alpha > 0$. Let us consider the following assertions:

(i) for every $y^* \in Y^*$ there is $0 \neq x \in X$ such that $\langle y^*, Lx\rangle \geqq \alpha \|y^*\| \|x\|$ whenever $L \in \mathfrak{M}$;

(ii) for every $y^* \in Y^*$ and every $L \in \mathfrak{M}$ $\|L^* y^*\| \geqq \alpha \| y^*\|$;

(iii) whenever $y \in Y$ and $L \in \mathfrak{M}$ then there is $x \in X$ such that $Lx=y$ and $\|y\| \geqq \alpha \|x\|$.

Then (i) $\Longrightarrow$ (ii), (iii) $\Longrightarrow$ (ii) and, if X is reflexive, then all the assertions are equivalent.

Proof. (i) $\Longrightarrow$ (ii). Let $y^* \in Y^*$ and $L \in \mathfrak{M}$. By (i) there is $0 \neq x \in X$ such that $\langle y^*, Lx\rangle \geqq \alpha\|y^*\| \|x\|$. Hence

$$\|L^* y^*\| \|x\| \geqq \langle L^* y^*, x\rangle = \langle y^*, Lx\rangle \geqq \alpha \|y^*\| \|x\|, \ \|L^* y^*\| \geqq \alpha \|y^*\|.$$

(iii) $\Longrightarrow$ (ii). For $y^* \in Y^*$ and an arbitrary $\delta > 0$ find $y \in Y$, $\|y\| = 1$, such that $\langle y^*, y\rangle \geqq (1-\delta)\|y^*\|$. Then by (iii), for any $L \in \mathfrak{M}$ there exists $x \in X$ such that $Lx=y$ and $1 = \|y\| \geqq \alpha \|x\|$. Hence

$$\|L^* y^*\| \|x\| \geqq \langle L^* y^*, x\rangle = \langle y^*, Lx\rangle = \langle y^*, y\rangle \geqq (1-\delta)\|y^*\| \geqq$$
$$\geqq (1-\delta)\alpha \|y^*\| \|x\|, \|L^* y^*\| \geqq (1-\delta)\|y^*\|.$$

And since $\delta > 0$ was arbitrary, we get (ii).

Let X be reflexive by the end of the proof. Let us prove (ii) $\Longrightarrow$ (i). We shall proceed as Clarke in the proof of [2, Lemma 3]. Fix $0 \neq y^* \in Y^*$. Let us remark that the set $\{L^* y^* : L \in \mathfrak{M}\}$ is convex and disjoint from $\{x^* \in X^* : \|x^*\| < \alpha \|y^*\|\}$. Hence by the separation theorem and reflexivity there is $x \in X$, $x \neq 0$, such that for any $L \in \mathfrak{M}$

$$\langle y^*, Lx\rangle = \langle L^* y^*, x\rangle \geqq \sup \{\langle x^*, x\rangle : \|x^*\| < \alpha \|y^*\|\} = \alpha \|y^*\| \|x\|.$$

It remains to prove (ii) $\Longrightarrow$ (iii). By (ii) L^* maps Y^* onto the closed subspace $Z = L^*(Y^*)$ of X^* and there exists $S \in \mathcal{L}(Z, Y^*)$ such that $\|S\| \leqq 1/\alpha$ and $S(L^* y^*) = y^*$ for all $y^* \in Y^*$. Then S^* maps Y^{**} into Z^*. Fix now $y \in Y$, $y \neq 0$. Then $S^*(\varkappa(y))$ is in Z^* and hence, by the Hahn Banach theorem, there exists $x^{**} \in X^{**}$ such that $\|x^{**}\| = \|S^* \varkappa(y)\|$ and that $\langle x^{**}, z\rangle = \langle S^* \varkappa(y), z\rangle$ for all $z \in Z$. As X is reflexive, we can write $x^{**} = \varkappa(x)$ with some $x \in X$. Then we have

$$\langle y^*, Lx\rangle = \langle L^* y^*, x\rangle = \langle \varkappa(x), L^* y^*\rangle = \langle S^* \varkappa(y), L^* y^*\rangle =$$
$$= \langle \varkappa(y), SL^*(y^*)\rangle = \langle \varkappa(y), y^*\rangle = \langle y^*, y\rangle$$

for all $y^* \in Y^*$. Hence $Lx=y$. Moreover $\|S^*\| = \|S\| \leqq 1/\alpha$ and so

$$\|x\|=\|x^{**}\| = \|S^*\varkappa(y)\| \leq \frac{1}{\alpha}\|\varkappa(y)\| = \frac{1}{\alpha}\|y\|.$$

Thus (iii) holds.

Lemma 3. Let X, Y be Banach spaces, let $\alpha > 0$, $\gamma \in (0,\alpha)$ be given and let $\mathfrak{M} \subset \mathscr{L}(X,Y)$ be a nonempty convex bounded set such that for every $y^* \in Y^*$ there exists $0 \neq x \in X$ satisfying

$$\langle y^*, Lx\rangle \geqq \alpha\|y^*\|\,\|x\| \text{ whenever } L \in \mathfrak{M}.$$

If the norm of Y is Fréchet differentiable off the origin, then for every $0 \neq y \in Y$ there are $t \in (0,1/\gamma)$ and $g \in X$ such that

(11) $\|g\|=\|y\|$ and $\|y-tLg\| < (1-\gamma t)\|y\|$ whenever $L \in \mathfrak{M}$.

If the norm of Y is uniformly Fréchet differentiable on the unit sphere, then there exists $t \in (0,1/\gamma)$ such that for every $0 \neq y \in Y$ there is $g \in X$ fulfilling (11).

Proof. Let $0 \neq y \in Y$ be given. Let $\|y\|'$ denote the Fréchet derivative of $\|\cdot\|$ at y. By assumptions, to $y^*=\|y\|'$, there exists $g \in X$, $\|g\|=\|y\|$, such that

$$\langle \|y\|', Lg\rangle \geqq \alpha\,\|\,\|y\|'\,\|\,\|g\| = \alpha\|y\|$$

for all $L \in \mathfrak{M}$. Denote $c=\sup\{\|L\| : L \in \mathfrak{M}\}$. As $\mathfrak{M}$ is bounded, c is finite. Since the norm is Fréchet differentiable at y, there is $t > 0$ such that

$$\|y-z\| \leqq \|y\| - \langle \|y\|', z\rangle + \frac{\alpha-\gamma}{2c}\|z\|$$

whenever $z \in Y$ and $\|z\| \leqq tc\|y\|$. We note that if the norm on Y is uniformly Fréchet differentiable on the unit sphere, then t can be chosen independently of the concrete y. As $\|tLg\| \leqq tc\|y\|$ for all $L \in \mathfrak{M}$, we have

$$\|y-tLg\| \leqq \|y\| - \langle \|y\|', tLg\rangle + \frac{\alpha-\gamma}{2c}\|tLg\| \leqq$$

$$\leqq \|y\| - \alpha t\|y\| + \frac{\alpha-\gamma}{2c}tc\|y\| < (1-\gamma t)\|y\|,$$

which was to prove.

Proof of Theorem 2. According to Troyanski [4, p. 164] Y^* admits an equivalent locally uniformly rotund norm. If Y is superreflexive, so is Y^* [4, p. 87] and by Enflo [4, p. 87] there exists an equivalent uniformly rotund norm on Y^*. Further, it is known and easy to check [8] that such norms can be taken arbitrarily close, in the sense of Banach-Mazur distance, to the original norm on Y^*. Hence, by an easy duality argument [4] we get that Y admits an equivalent norm which is Fréchet (or uniformly Fréchet) differentiable on the unit sphere and is arbitrarily close to the original norm on Y. Thus we may assume that the original norm on Y is Fréchct (uniformly Fréchet) differentiable on the unit sphere and that the assumptions of Theorem 2 hold with β and $\wp$ replaced by $\beta+\sigma$ and $\wp-2\sigma$ respectively, where σ is some fixed number from $(0,\wp/3)$.

Take $x\in B(x_0,r)\cap D(F)$ and $y\in B(Fx_0,(\wp-3\sigma)r)$, $y\neq Fx$. From (2), by applying subsequently Lemmas 2 and 3 we can find $t\in(0,1/(\beta+\wp-2\sigma))$ and $g\in X$ such that

$$\|g\|=\|Fx-y\| \text{ and } \|Fx-y-tLg\|<(1-(\beta+\wp-2\sigma)t)\|g\|$$

for all $L\in\mathfrak{M}$. Let us note that in the case of uniform Fréchet differentiability the t does not depend on the choice of x and y. From the hypotheses choose $\varepsilon\in(0,1]$ and $L\in\mathfrak{M}$ such that

$$\|F(x-\varepsilon tg)-Fx+\varepsilon L(tg)\|\leqq\varepsilon(\beta+\sigma)\|tg\|.$$

Then the last two inequalities yield

$$\frac{1}{\varepsilon}\|F(x-\varepsilon tg)-Fx+\varepsilon(Fx-y)\|\leqq\frac{1}{\varepsilon}\|F(x-\varepsilon tg)-Fx+\varepsilon L(tg)\|+$$
$$+\|Fx-y-tLg\|<(\beta+\sigma)\|tg\|+(1-(\beta+\sigma-2\sigma)t)\|g\|=$$
$$=\|Fx-y\|-(\wp-3\sigma)\|tg\|=(1-(\wp-3\sigma)t)\|Fx-y\|,$$

$$(\wp-3\sigma)\|tg\|+\frac{1}{\varepsilon}\|F(x-\varepsilon tg)-Fx+\varepsilon(Fx-y)\|<\|Fx-y\|.$$

It means that (8) and (9) hold with $h=tg$, $q=1-(\wp-3\sigma)t$, and with $\wp$ replaced by $\wp-3\sigma$. Thus by Corollary 2 $B(Fx_0,(\wp-3\sigma)r)\subset F(B(x_0,r)\cap D(F))$. And since $\sigma>0$ could be arbitrarily small, the conclusion of Theorem 2 follows.

Remark 6. From the above proof one can see that the version of Theorem 2 with F continuous and D(F) closed holds under weaker assumptions. Namely, the reflexivity of Y can be replaced by the requirement that the set of equivalent Fréchet differentiable norms on Y is dense in the sense of Banach-Mazur distance. We do not know whether this case occurs if one such norm exists.

Final note. After this paper had been prepared for publication, we learned about the paper of P.H. Dien, Some results on locally Lipschitzian mappings, Acta Math. Vietnamica 6(1981), 97-105. Here a theorem similar to our Theorem 2 is presented under a little stronger assumptions. Its proof is based on the Ekeland's variational principle.

References

[1] J. CARISTI: Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc. 215(1976), 241-251.

[2] F.H. CLARKE: On the inverse function theorem, Pacific J. Math. 64(1976), 97-102.

[3] W.J. CRAMER, Jr. and W.O. RAY: Solvability of nonlinear operator equations, Pacific J. Math. 95(1981), 37-50.

[4] J. DIESTEL: Geometry of Banach spaces - Selected topics, Lect. Notes 485, Springer Verlag, Berlin 1975.

[5] D. DOWNING and W.A. KIRK: A generalization of Caristi's theorem with applications to nonlinear mapping theory, Pacific J. Math. 69(1977), 339-346.

[6] D.S. DŽUMABAEV: On solvability of nonlinear closed operator equations (Russian), Izv. Akad. Nauk Kazach. SSR, Ser. Fyz.-Mat. No. 1, 1984, 31-34.

[7] M. FABIAN: Concerning interior mapping theorem, Comment. Math. Univ. Carolinae 20(1979), 345-356.

[8] M. FABIAN, L. ZAJÍČEK and V. ZIZLER: On residuality of the set of rotund norms on a Banach space, Math. Ann. 258(1982), 349-351.

[9] L.M. GRAVES: Some mapping theorems, Duke Math. J. 17(1950), 111-114.

[10] V. PTÁK: A quantitative refinement of the closed graph theorem, Czech. Math. J. 24(1974), 503-506.

[11] B.H. POURCIAU: Analysis and optimization of Lipschitz continuous mappings, J. Opt. Theory Appl. 22(1977), 311-351.

[12] T. SZILÁGYI: Generalization of the implicit function theorem and of Banach's open mapping theorem, Acta Scient. Math. 38(1977), 391-396.

[13] C. URSESCU: Letter to the first named author, January 1984.

Institute for Computation Technique of the ČVUT, Horská 3, 12800 Prague 2, Czechoslovakia

Faculty of Mathematics and Physics, Charles' University, Sokolovská 83, 18600 Prague 8, Czechoslovakia

(Oblatum 11.2. 1987)