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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
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ON DIMENSIONS OF SEMIMETRIZED MEASURE SPACES

Miroslav KATĚTOV

Abstract: We introduce and examine various kinds of dimensions and dimensional densities defined for semimetric spaces equipped with a finite measure.

Key words: Extended Shannon semientropy, Shannon functional, regularized upper (lower) Rényi dimension, monotone dimension.

Classification: 94A17

In a previous article [4] by the author, there have been introduced, for the class of all semimetrized spaces equipped with a finite measure, dimension functionals which generalize the dimensions defined for vector-valued random variables in [1] and in subsequent papers of A. Rényi. In the present article, we introduce dimension functionals of another kind; in some respects, they behave similarly as dimensions of topological (or uniform, as the case may be) spaces. We also introduce various kinds of dimensional densities generalizing a closely related concept examined in [4]. Among other things, theorems are proved analogous to the sum theorem for the topological dimension and to the theorem on the dimension of the cartesian product of topological spaces.

Section 1 contains preliminaries. In Section 2, functionals of the form φ-udim and some related notions are examined. In Section 3, we investigate dimension functionals for which there is a theorem analogous to Sum Theorem of the topological dimension theory. In Section 4, dimensional densities are considered.

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1.1. The terminology and notation is that of [3] and [4] with two exceptions stated below (1.3 and 1.19). Nevertheless, we will re-state some definitions and conventions.

1.2. The symbols N, R, $\overline{R}$, R_+, $\overline{R}_+$ have their usual meaning. We put 0/0=0, and, for any $b \in \overline{R}$, $0.b=0$; log means $\log_2$; we put $L(0)=0$, $L(t)= -t \log t$ if

$0<t<\infty$. For $t\in\overline{R}$, we put $\operatorname{sgn}(0)=0$, $\operatorname{sgn}(t)=1$ if $t>0$, $\operatorname{sgn}(t)=-1$ if $t<0$. If $f:X\to\overline{R}$ is a function, then sgn f denotes the function $x\mapsto \operatorname{sgn}(f(x))$.

1.3. If $Q\neq\emptyset$ is a set and $\mathcal{A}$ is a σ-algebra of subsets of Q, then, in accordance with the current terminology, a σ-additive function $\mu:\mathcal{A}\to\overline{R}_+$ satisfying $\mu(\emptyset)=0$ will be called a measure on Q (in [2], the term "$\overline{R}$-measure" was used), whereas a μ such that, in addition, $\mu(Q)<\infty$ will be called a finite measure (in [2], [3] and [4], such μ were called "measures").

1.4. If a set A is given, then, for any $X\subset A$, i_X is the indicator of X, i.e., $i_X(x)=1$ if $x\in X$, $i_X(x)=0$ if $x\in A\setminus X$.

1.5. A) If $Q\neq\emptyset$ is a set, then $\mathcal{F}(Q)$ and $\mathcal{M}(Q)$ will denote, respectively, the set of all $f:Q\to\overline{R}$ and that of all measures on Q. - B) The completion of a $\mu\in\mathcal{M}(Q)$ is denoted by $\overline{\mu}$ or $[\mu]$. If $\mu,\nu\in\mathcal{M}(Q)$, we put $\nu\leq\mu$ if $\operatorname{dom}\nu=\operatorname{dom}\mu$ and $\nu(X)\leq\mu(X)$ for all $X\in\operatorname{dom}\mu$. If $\mu\in\mathcal{M}(Q)$, $f,g\in\mathcal{F}(Q)$ and $\overline{\mu}\{x\in Q:f(x)\neq g(x)\}=0$, we write $f=g(\operatorname{mod}\mu)$. - C) Let $\mu\in\mathcal{M}(Q)$. If $f\in\mathcal{F}(Q)$ is $\overline{\mu}$-measurable, we put $[f]_\mu=\{g\in\mathcal{F}(Q):g=f(\operatorname{mod}\mu)\}$ and call $[f]_\mu$ a function (mod μ). We put $\mathcal{F}[\mu]=\{[f]_\mu:f\in\mathcal{F}(Q)$, f is $\overline{\mu}$-measurable$\}$. - D) If $F,G\in\mathcal{F}[\mu]$, then we put $F\leq G$ (respectively, $F<G$) iff there are $f\in F$ and $g\in G$ such that $f(x)\leq g(x)$ (respectively, $f(x)<g(x)$) for all $x\in Q$. - E) If $\mu\in\mathcal{M}(Q)$, $f\in\mathcal{F}(Q)$, then $\sup\,[f]_\mu$ denotes the least $b\in\overline{R}$ such that $[f]_\mu\leq b$, and similarly for $\inf\,[f]_\mu$.

1.6. If $\mu\in\mathcal{M}(Q)$, $f\in\mathcal{F}(Q)$ is $\overline{\mu}$-measurable and $F=[f]_\mu\geq 0$, then the measure $X\mapsto\int_X f\,d\mu$, defined on $\operatorname{dom}\mu$, is denoted by $f.\mu$ or $F.\mu$. - Clearly, $f.\mu\leq\mu$ iff $[f]_\mu\leq 1$, $f.\mu=g.\mu$ iff $f=g$ (mod μ).

1.7. If $K\neq\emptyset$ is countable, $\xi=(x_k:k\in K)$, $x_k\in R_+$, $\sum x_k<\infty$, we put $H(\xi)=H(x_k:k\in K)=\sum(L(x_k):k\in K)-L(\sum(x_k:k\in K))$. If Q is countable, $\mu\in\mathcal{M}(Q)$ is finite and $\operatorname{dom}\mu=\exp Q$, we put $H(\mu)=H(\mu\{q\}:q\in Q)$.

1.8. If M is a (partially) ordered set and x_a, $a\in A$, x, y are in M, we often write $\bigvee(x_a:a\in A)$, $\bigwedge(x_a:a\in A)$, $x\vee y$, etc. instead of $\sup(x_a:a\in A)$, $\inf(x_a:a\in A)$, $\sup\{x,y\}$, etc. In particular, if $x,y\in\overline{R}$, then $x\vee y=\max(x,y)$, $x\wedge y=\min(x,y)$.

1.9. Recall that $P=\langle Q,\rho,\mu\rangle$ is called semimetrized measure space or W-space (or also a semimetric space endowed with a measure) if $\mu\in\mathcal{M}(Q)$ is finite and ρ is a $[\mu\times\mu]$-measurable semimetric. The class of all W-spaces is denoted by $\mathcal{W}$. If $P=\langle Q,\rho,\mu\rangle\in\mathcal{W}$, we put $wP=\mu(Q)$; if $wP=0$, P is called a null space; if Q is finite and $\operatorname{dom}\mu=\exp Q$, we call P an FW-space. The class of all FW-spaces is denoted by $\mathcal{W}_F$. - See, e.g., [3], 1.5.

1.10. Let $P=\langle Q,\rho,\mu\rangle\in\mathcal{W}$. If $f\in\mathcal{F}(Q)$ is $\overline{\mu}$-measurable, $[f]_\mu\geq 0$

and $f.\mu$ is finite, we put $f.P=\langle Q,\varrho,f.\mu\rangle$; if $X\in\operatorname{dom}\bar{\mu}$, we put $X.P=i_X.P$ (see 1.4). If $S\in\mathfrak{M}$, $S=\langle Q,\varrho,\nu\rangle$ and $\nu\leq\mu$, we write $S\leq P$ and call S a subspace of P (a pure subspace if $S=X.P$, $X\in\operatorname{dom}\bar{\mu}$). Clearly, $S\leq P$ iff $S=f.P$ for some $\bar{\mu}$-measurable $f:Q\to\bar{R}_+$. - Cf. [3], 1.6, 1.7.

1.11. If $P\in\mathfrak{M}$, we put $\exp P=\{S:S\leq P\}$. We put $\mathfrak{A}=\bigcup(\exp P\times\exp P:P\in\mathfrak{M})$.

1.12. If $P=\langle Q,\varrho,\mu\rangle\in\mathfrak{M}$, $P_k=\langle Q,\varrho,\mu_k\rangle\in\mathfrak{M}$ for $k\in K$, where $K\neq\emptyset$ is countable, and $\mu=\Sigma(\mu_k:k\in K)$, we put $P=\Sigma(P_k:k\in K)$ and call $(P_k:k\in K)$ an ω-partition of P (merely "partition" if K is finite). - See [3], 1.6.

1.13. **Lemma.** If $P\in\mathfrak{M}$, $P=\Sigma(P_n:n\in N)$, $S\leq P$, then there are $S_n\leq P_n$ such that $\Sigma(S_n:n\in N)=S$.

Proof. Let $S=s.P$, $P_n=f_n.P$ (see 1.10). Put $g_n=sf_n$, $S_n=g_n.P\leq P_n$. Clearly, $\Sigma S_n=S$.

1.14. Let $\mathcal{U}=(U_k:k\in K)$ and $\mathcal{V}=(V_m:m\in M)$ be ω-partitions of $P\in\mathfrak{M}$. If there are pairwise disjoint M_k such that $U_k=\Sigma(V_m:m\in M_k)$, $\bigcup M_k=M$, then $\mathcal{V}$ is said to refine $\mathcal{U}$. - See [3], 1.6.

1.15. If $P=\langle Q,\varrho,\mu\rangle\in\mathfrak{M}$, we put $d(P)=\sup[\varrho]_{\mu\times\mu}$. If $(P_1,P_2)\in\mathfrak{A}$, $P_i=\langle Q,\varrho,\mu_i\rangle$, we put $E(P_1,P_2)=d(P_1+P_2)$, $r(P_1,P_2)=\int\varrho\, d(\mu_1\times\mu_2)/wP_1.wP_2$ if $wP_1.wP_2>0$, $r(P_1,P_2)=0$ if $wP_1.wP_2=0$. - Cf. [3],1.19,

1.16. Let $P=\langle Q,\varrho,\mu\rangle\in\mathfrak{M}$, $\varepsilon>0$. Then $\mathfrak{X}=(X_k:k\in K)$, where $K\neq\emptyset$ is countable, $X_k\in\operatorname{dom}\bar{\mu}$, will be called an ε-covering of P if $\operatorname{diam}X_k\leq\varepsilon$ for all k and $\bar{\mu}(Q\setminus\bigcup X_k)=0$. If, in addition, $X_i\cap X_j=\emptyset$ for $i\neq j$, then $\mathfrak{X}$ will be called an ε-partition of P. - Cf. [3], 1.19.

1.17. If $P=\langle Q,\varrho,\mu\rangle\in\mathfrak{M}$, then we put $\varepsilon * P=\langle Q,\varepsilon*\varrho,\mu\rangle$, where $(\varepsilon*\varrho)(x,y)=0$ if $\varrho(x,y)\leq\varepsilon$, $(\varepsilon*\varrho)(x,y)=1$ if $\varrho(x,y)>\varepsilon$. - See [3], 1.17.

1.18. If $P_i=\langle Q_i,\varrho_i,\mu_i\rangle\in\mathfrak{M}$, i=1,2, then we put $P_1\times P_2=\langle Q,\varrho,\mu\rangle$, where $Q=Q_1\times Q_2$, $\mu=\mu_1\times\mu_2$ and $\varrho((x_1,x_2),(y_1,y_2))=\varrho_1(x_1,y_1)\vee\varrho_2(x_2,y_2)$.

1.19. Let $\varphi:\mathfrak{M}\to\bar{R}_+$ satisfy the following conditions: (1) if $\langle Q,\varrho,\mu\rangle\in\mathfrak{M}$, $a,b\in R_+$, then $\varphi\langle Q,a\varrho,b\mu\rangle=ab\,\varphi\langle Q,\varrho,\mu\rangle$; (2) if $P_i=\langle Q,\varrho_i,\mu\rangle\in\mathfrak{M}$, i=1,2, and $\varrho_1\geq\varrho_2$, then $\varphi P_1\geq\varphi P_2$; (3) if $P=\langle Q,1,\mu\rangle\in\mathfrak{M}_F$, then $\varphi P=H(\mu)$; (4) if $P_i=\langle Q_i,\varrho_i,\mu_i\rangle\in\mathfrak{M}$, i=1,2, and there is an $f:Q_1\to Q_2$ such that (a) $\varrho_2(fx,fy)=\varrho_1(x,y)$ if $x,y\in Q_1$, $\mu_1\{x\}>0$, $\mu_1\{y\}>0$, (b) $\mu_1(f^{-1}\{q\})=\mu_2\{q\}$ for all $q\in Q_2$, then $\varphi P_1=\varphi P_2$; (5a) if $P=\langle Q,\varrho,\mu\rangle\in\mathfrak{M}_F$, $P_n=\langle Q,\varrho_n,\mu\rangle\in\mathfrak{M}_F$ and $\varrho_n\to\varrho$, then $\varphi P_n\to\varphi P$; (5b) if $P=\langle Q,\varrho,\mu\rangle\in\mathfrak{M}_F$, $\langle Q,\varrho,\mu_n\rangle\in\mathfrak{M}_F$, $\mu\{q\}>0$ for all $q\in Q$ and $\mu_n\to\mu$, then $\varphi P_n\to\varphi P$. Then φ will be called an extended Shannon semientropy (in the broad sense), which is the expression introduced in [2] and used in [3] and [4], or a Shannon functional (in the broad sense), which is the expression we use in this article.

1.20. Convention. The letter φ will always stand for a Shannon functional (in the broad sense).

1.21. For the definition of normal gauge functionals (NGF) and of C_τ and C^*_τ, where τ is an NGF, we refer to [2] and [3], since we need only (1) the fact that r and E are NGF's, (2) the fact that C_r and C_E are Shannon functionals (b.s.), and (3) some propositions on C_E, see 1.24 - 1.26 below. It is also useful to note that there are E-projective (see 1.23) φ's distinct from C_E, for instance C_r.

1.22. Convention. The functional C_E will ne often denoted by E, provided there is no danger of confusion with the E introduced in 1.15.

1.23. **Definition.** A functional $\psi : \mathfrak{M} \rightarrow \overline{R}_+$ will be called E-projective if, for any $P \in \mathfrak{M}$ and any partition (S,T) of P, $\psi(P) \leq \psi(S)+\psi(T)+$ $+E(S,T)H(wS,wT)$. - Cf. [2], 3.10.

1.24. Fact. The functional $E: \mathfrak{M} \rightarrow R_+$ is E-projective. - See [2], Theorem II.

1.25. **Proposition.** If $S \leq P \in \mathfrak{M}$, then $E(S) \leq E(P)$. - See [3], 2.3.

1.26. **Proposition.** If $P \in \mathfrak{M}$, then, for all sufficiently small $\varepsilon > 0$, $E(\varepsilon * P)$ is equal to the infimum of all $H(\bar{\mu} X_n : n \in N)$, where $(X_n : n \in N)$ is an ε-partition of P. - See [3], 2.18, 1.19.

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2.1. **Definition** (cf. [4], 2.1). For any φ and any $P \in \mathfrak{M}$, φ-uw(P) (respectively, φ-ℓw(P)) will denote the upper (lower) limit of $\varphi(\varepsilon * P)/\,|\log \varepsilon|$ for $\varepsilon \rightarrow 0$. We put φ-ud(P)=φ-uw(P)/wP, φ-ℓd(P)=φ-ℓw(P)/wP, φ-udim(P)= =sup{φ-ud(S):S$\leq$P}, φ-ℓdim(P)=sup{φ-ℓd(S):S$\leq$P}. If φ-uw(P)=φ-ℓw(P), we put φ-Rw(P)=φ-uw(P), φ-Rd(P)=φ-ud(P). We call φ-udim(P) the monotone φ-dimension of P. For φ-uw(P), etc., the terminology introduced in [4], 2.1, will be used. - If φ=E, we often omit the prefix "φ". - Remark. *In the present note, the functionals φ-ℓdim will not be considered.*

2.2. Fact. For any E-projective φ and any $P \in \mathfrak{M}$, (1) *if P=S+T, then* φ-uw(P)$\leq\varphi$-uw(S)+φ-uw(T), φ-ud(P)$\leq\varphi$-ud(S)$\vee\varphi$-ud(T), (2) *if* φ-udim(P)$<$ $<\infty$ and $P=\Sigma(P_k:k \in N)$, then φ-uw(P)$\leq\Sigma(\varphi$-uw$(P_k):k\in N)$, φ-ud(P) $\leq$ $\leq\vee(\varphi$-ud$(P_k):k\in N)$.

Proof. Since φ is E-projective, we have $\varphi(\varepsilon * S)+\varphi(\varepsilon * T)+H(wS,wT) \geq$ $\geq \varphi(\varepsilon * P)$. This proves the inequalities (1). - If φ-udim(P)=$b < \infty$, put $S_n =$ $= \Sigma(P_k : k > n)$. Then, for each $n \in N$, φ-uw(P) $\leq \Sigma(\varphi$-uw$(P_k) : k \leq n) + \varphi$-uw$(S_n)$. Since $wS_n \to 0$ and φ-uw$(S_n) \leq b.wS_n$, this proves the inequalities (2).

2.3. **Proposition.** For any E-projective φ and any $P \in \mathfrak{M}$, (1) if P=S+T or $P=S \vee T$, then φ-udim(P)=φ-udim(S)$\vee \varphi$-udim(R), (2) if φ-udim(P)$< \infty$ and either $P= \Sigma(P_n : n \in N)$ or $P= \bigvee(P_n : n \in N)$, then φ-udim(P)= $\bigvee(\varphi$-udim$(P_n) : n \in N)$.

Proof. Let P=S+T. Then, for any $V \leq P$, there are, by 1.13, $V_1 \leq S$, $V_2 \leq T$ such that $V_1+V_2=V$. By 2.2, we have φ-ud(V)$\leq \varphi$-ud$(V_1) \vee \varphi$-ud$(V_2) \leq \varphi$-udim(S)$\vee$ $\vee \varphi$-udim(T). This proves (1), since $S \vee T \leq S+T$. The case $P= \Sigma(P_n : n \in N)$ is analogous to that of P=S+T. - Let $P= \bigvee(P_n : n \in N)$. Put $T_0=P_0$, $T_{n+1}=T_n \vee P_{n+1}$. Then $P=T_0+ \Sigma(T_{n+1}-T_n : n \in N)$. Since, clearly, $U \vee V=U+V-U \wedge V$ for any $U \leq P$, $V \leq P$, it is easy to show that φ-udim$(T_n) \leq \bigvee(\varphi$-udim$(P_k) : k \leq n)$. Hence, due to φ-udim(P)$< \infty$, we get φ-udim(P)$\leq \bigvee(\varphi$-udim$(T_n) : n \in N) \leq \bigvee(\varphi$-udim$(P_n)$: $: n \in N)$.

2.4. **Example.** Choose $a_n > 0$, $b_n > 0$, $n \in N$, such that $\Sigma(b_n : n \in N)=1$, $\Sigma(L(b_n) : n \in N)= \infty$; $a_n \to 0$, $|\log a_{n+1}|=(n\, \Sigma(L(b_i) : i \leq n))^{-1}$ for $n \geq 1$. Put $P= \langle N, \rho, \mu \rangle$, where $\rho(i,j)=a_i+a_j$, $\mu\{i\} =b_i$. It is easy to see that ud(P)= $= \ell$d(P)$= \infty$, udim(P)$= \infty$. On the other hand, evidently, udim($\{k\}$.P)=0 for all $k \in N$. This shows that, in 2.3, (2), the assumption φ-udim(P)$< \infty$ cannot be omitted. - For an example connected with the assertion (1) in 2.3, see 2.10,E.

2.5. **Lemma.** For any E-projective φ and any $P \in \mathfrak{M}$, φ-udim(P)= =sup $\{\varphi$-ud(S)$: S \leq P$, S pure$\}$.

Proof. Assume wP=1. Write ud instead of φ-ud, uw instead of φ-uw. Put b=sup $\{$ud(S)$: S \leq P$, S pure$\}$. Let $T \leq P$, T=f.P, $0 \leq f(x) \leq 1$ for all $x \in Q$. Let $m \in$ $\in N$, $m > 1$. Define g as follows: g(x)=k/m if $(k-1)/m < f(x) \leq k/m$; g(x)=1/m if f(x)=0. Clearly, $g-1/m \leq f \leq g$, hence $\int (g-f)d\mu \leq 1/m$. Put U=g.P, $X_k = \{x \in Q$: $: g(x)=k/m\}$. Since X_k.P are pure, we have ud$(X_k.P) \leq b$, hence ud$((k/m).X_k.P) \leq b$ and therefore, by 2.2, ud(U)$\leq b$. Since $f.P \leq g.P$, we get uw(T)$\leq$uw(U)$\leq b$. . $\int g d\mu$, ud(T)$\leq b(\int g d\mu / \int f d\mu) \leq b+b \int f d\mu/m$. Since $m \in N$ has been arbitrary, we get ud(T)$\leq b$.

2.6. **Lemma.** Let J and K be countable non-void sets. Let x_{jk}, where $j \in J$, $k \in K$, be non-negative reals, $\Sigma(x_{jk} : j \in J, k \in K) < \infty$. For $j \in J$, $k \in K$, put $a_j =$ $= \Sigma(x_{jk} : k \in K)$, $b_k = \Sigma(x_{jk} : j \in J)$. Then $H(x_{jk} : j \in J, k \in K) \leq H(a_j : j \in J)+H(b_k : k \in K)$.

This follows easily from the well-known special case with both J and K finite and $\Sigma x_{jk}=1$.

2.7. Fact. If P is a W-space, P=S+T, then $uw(S)\vee uw(T)\leq uw(P)\leq uw(S)+$ $+uw(T)$.

Proof. The first inequality follows from 1.25; for the latter, see 2.2.

2.8. **Proposition.** For any non-null W-spaces P_1 and P_2, $ud(P_1)\vee ud(P_2)\leq$ $\leq ud(P_1\times P_2)\leq ud(P_1)+ud(P_2)$. - See [4], 4.5.

2.9. **Theorem.** For any non-null W-spaces P_1 and P_2, $udim(P_1)\vee udim(P_2)\leq$ $\leq udim(P_1\times P_2)\leq udim(P_1)+udim(P_2)$.

Proof. The first inequality follows at once from [4], 2.8. Let $P_i=$ $=\langle Q_i,\varphi_i,\mu_i\rangle$, i=1,2, $P=P_1\times P_2$, $P=\langle Q,\varphi,\mu\rangle$, $udim(P_i)=b_i<\infty$. Put $b=b_1+$ $+b_2$. We can assume that $wP_1=wP_2=1$. By 2.5, it is sufficient to show that $ud(S)\leq b$ for any pure $S\leq P$. Clearly, there exist sets $A_n\in \dim\mu_1$, $B_n\in\dim\mu_2$ such that $\mu_1A_n>0$, $\mu_2B_n>0$ and $S=X.P$, where $X=\bigcup(A_n\times B_n)$. Put $X_1=\bigcup A_n$, $X_2=$ $=\bigcup B_n$, $S_i=X_i.P_i$. - Let $\sigma>0$. We are going to show that, for every sufficiently small $\varepsilon>0$, (1) there exists an ε-covering $(Y_n:n\in N)$ of S_1 such that, with $U_n=X\cap(Y_n\times Q_2)$, we have $H(\bar\mu U_n:n\in N)\leq(b_1.wS+\sigma)|\log\varepsilon|$, (2) there exists an ε-covering $(Z_n:n\in N)$ of S_2 such that, with $V_n=X\cap(Q_1\times Z_n)$, we have $H(\bar\mu V_n:n\in N)<(b_2.wS+\sigma)|\log\varepsilon|$. For any $x\in Q_1$, put $f_1(x)=\mu_2(\bigcup(B_n:n\in N,x\in$ $\in A_n))$. Clearly, f_1 is μ_1-measurable and $X_1=\{x:f_1x>0\}$. Put $S_1'=f_1.P$. We have $S_1'\leq P_1$, hence $ud(S_1')\leq b_1$ and therefore $\overline{\lim}(E(\varepsilon * S_1')/|\log\varepsilon|)\leq b_1.wS_1'=b_1.wS$. Hence, for every sufficiently small $\varepsilon>0$, there exists, by 1.26, an ε-covering $(Y_n:n\in N)$ of S_1' such that $H(w(Y_n.S_1'):n\in N)<(b_1.wS+\sigma)|\log\varepsilon|$. Clearly, $(Y_n:n\in N)$ is an ε-covering of S_1 as well. Put $U_n=X\cap(Y_n\times Q_2)$. It is easy to see that $\bar\mu U_n=w(Y_n.S_1')$, hence $H(\bar\mu U_n:n\in N)<(b_1.wS+\sigma)|\log\varepsilon|$: This proves the assertion (1). The proof of (2) is analogous.

Put $T_{mn}=U_m\cap V_n$. Then $(T_{mn}:m\in N,n\in N)$ is an ε-covering of S. By 2.6, we obtain $H(\bar\mu T_{mn}:m\in N,n\in N)\leq H(\bar\mu U_m:m\in N)+H(\bar\mu V_n:n\in N)<(b.wS+2\sigma)|\log\varepsilon|$, hence $E(\varepsilon * S)<(b.wS+2\sigma)|\log\varepsilon|$. Since this inequality holds for all sufficiently small $\varepsilon>0$, we get $uw(S)\leq b.wS+2\sigma$. This proves $ud(S)\leq b$, for $\sigma>0$ has been arbitrary.

2.10. **Example.** A) For $n\in N$, let $P_n=\langle Q_n,\varphi_n,\mu_n\rangle\in\mathfrak{M}$, $wP_n=1$, diam $P_n<$ $<\infty$. Let a_n be positive reals, and let a_n diam $P_n\longrightarrow 0$. Then $\prod_\alpha(P_n:n\in N)$, where $\alpha=(a_n:n\in N)$, will denote the W-space $\langle Q,\varphi,\mu\rangle$, where $\langle Q,\mu\rangle=$ $=\prod(\langle Q_n,\mu_n\rangle:n\in N)$, $\varphi((x_n),(y_n))=\sup(a_n\varphi_n(x_n,y_n):n\in N)$. If $p=(p_n:n\in N)$, $p_n\in N$, $p_n\geq 1$, then S(p) will denote the W-space $\prod_\alpha(P_n:n\in N)$, where $\alpha=$ $=(2^{-n}:n\in N)$, $P_n=\langle Q_n,1,\nu_n\rangle$, card $Q_n=p_n$, $\nu_n\{q\}=1/p_n$ for $q\in Q_n$. - B) It is easy to show that $E(\varepsilon * S(p)) = \Sigma(\log p_k : k \leq n)$ for $2^{-n} \geq \varepsilon > 2^{-n-1}$, and therefore $ud(S(p)) = \overline{\lim}(\Sigma(\log p_k : k \leq n)/n)$, $\ell d(S(p)) = \underline{\lim}(\Sigma(\log p_k : k \leq n)/n)$. - C) Let $r(0)=2$, $r(k+1)=2^{r(k)}$ for $k \in N$; put $A = \{n \in N : r(2k) \leq n < r(2k+1)$ for some $k \in \in N\}$. Put $u_n = 2$ if $n \in A$, $u_n = 4$ if $n \in N \setminus A$, put $v_n = 8/u_n$ for all $n \in N$. Put $u = (u_n : n \in N)$, $v = (v_n : n \in N)$, $U = S(u)$, $V = S(v)$. It is easy to show (cf. [4], 3.10) that if X is a non-null subspace of U or of V, then $\ell d(X) = 1$, $ud(X) = 2$; hence $udim(U) = udim(V) = 2$. - D) Put $T = U \times V$. It can be easily proved that, for any non-null subspace $Y \leq T$, we have $ud(Y) = \ell d(Y) = 3$. This shows that, in 1.8 and 2.9, no $\leq$ can be replaced by $=$. - E) Let M be a "free sum" of U and V and let U' and V' denote the subspaces of M corresponding to U and V, respectively. Then $M = U' + V'$, and it is easy to show that $uw(M) = 2$, hence $ud(M) = 1$ and therefore $uw(M) < uw(U') + uw(V')$, $ud(M) < ud(U') \wedge ud(V')$. Thus, $\leq$ cannot be replaced by $=$ in 2.2, (1), and φ-udim cannot be replaced by φ-ud in 2.3, (1).

3

3.1. **Definition.** For any φ and any $P \in \mathfrak{M}$, (1) φ-UW(P) (respectively, φ-LW(P)) will denote the infimum of all $b \in \overline{R}_+$ for which there is an ω-partition $\mathcal{U}$ of P such that, for any $(V_k : k \in K)$ refining $\mathcal{U}$, $\Sigma(\varphi\text{-}uw(V_k) : k \in K) \leq b$ (respectively, $\Sigma(\varphi\text{-}\ell w(V_k) : k \in K) \leq b$). We put φ-UD(P) = φ-UW(P)/wP, φ-LD(P) = φ-LW(P)/wP, φ-UDim(P) = $\sup\{\varphi\text{-UD}(S) : S \leq P\}$, φ-LDim(P) = $\sup\{\varphi\text{-LD}(S) : S \leq P\}$. We will call φ-UDim(P) and φ-LDim(P) the regularized upper (lower) monotone φ-dimension of P. For φ-UW(P), etc., we will use the names introduced in [4] for the values of the corresponding functionals (i.e., for φ-uw(P), etc.), with the additional qualification "regularized"; thus, e.g., φ-UW(P) will be called the regularized Rényi φ-weight of P. - If $\varphi = E$, the prefix "φ " will be, as a rule, omitted.

3.2. **Theorem.** For any φ and any $P = \langle Q, \rho, \mu \rangle \in \mathfrak{M}$, (1) if $P = \Sigma(P_k : k \in \in N)$, then $\varphi\text{-UW}(P) = \Sigma(\varphi\text{-UW}(P_k) : k \in N)$, $\varphi\text{-LW}(P) = \Sigma(\varphi\text{-LW}(P_k) : k \in N)$, (2) the functions $X \mapsto \varphi\text{-UW}(X.P)$, $X \mapsto \varphi\text{-LW}(X.P)$, defined on dom $\bar{\mu}$, are measures.

Proof. The assertion (2) is an immediate consequence of (1). We prove (1) for φ-UW; for φ-LW, the proof is analogous. If $S \leq P$, put $\psi(S) = \varphi\text{-uw}(S)$, $\Phi(S) = \varphi\text{-UW}(S)$. Let $P = \Sigma(P_n : n \in N)$. - I. We are going to show that $\Phi(P) \leq \Sigma\Phi(P_n)$. We can assume that all $\Phi(P_n)$ are finite. Let $b_n \in R_+$, $b_n > \Phi(P_n)$ for all n. For any $n \in N$, there is an ω-partition $\mathcal{U}_n = (U_{nk} : k \in K_n)$ of P_n such that $\Sigma(\psi(V_j) : j \in J) \leq b_n$ for any $(V_j \in j \in J)$ refining $\mathcal{U}_n$. Put $\mathcal{U} = (U_{nk} : n \in N, k \in K_n)$. Let $(V_m : m \in M)$ be an arbitrary ω-partition of P refining $\mathcal{U}$. Let $(M_{nk} : n \in N, k \in K_n)$ be an ω-partition of the set M such that $\Sigma(V_m : m \in M_{nk}) = U_{nk}$

for all $n\in N$, $k\in K_n$. Put $M_n=\bigcup(M_{nk}:k\in K_n)$. Then $(V_m:m\in M_n)$ refines $\mathcal{U}_n$ and therefore $\sum(\psi(V_m):m\in M_n)\le b_n$, hence $\sum(\psi(V_m):m\in M)\le\sum b_n$. We have shown that $\Phi(P)\le\sum b_n$. Since $b_n>\Phi(P_n)$ have been arbitrary, we get $\Phi(P)\le$ $\le\sum\Phi(P_n)$. - II. Suppose that $\Phi(P)<\sum\Phi(P_n)$. Choose reals $a_n<\Phi(P_n)$ such that $\sum a_n>\Phi(P)$. Then there is an ω-partition $\mathcal{U}=(U_m:m\in M)$ of P such that (1) $\sum(\psi(V_k):k\in K)<\sum a_n$ whenever $(V_k:k\in K)$ refines $\mathcal{U}$. Let $U_m=u_m.P$; for $m\in N$, $n\in N$, put $U_{mn}=u_m.P$. Put $\mathcal{U}'=(U_{mn}:m\in M,n\in N)$, Then $\mathcal{U}'$ refines $\mathcal{U}$ and, for any $n\in N$, $(U_{mn}:m\in M)$ is an ω-partition of P_n. For each $n\in N$, there exists, due to $a_n<\Phi(P_n)$, an ω-partition $(V_{nj}:j\in J_n)$ of P_n refining $(U_{mn}:$ $:n\in N)$ and satisfying (2) $\sum(\psi(V_{nj}):j\in J_n)>a_n$. Clearly, $(V_{nj}:n\in N,j\in J_n)$ refines $\mathcal{U}'$, hence $\mathcal{U}$, and therefore, by (1), $\sum(\psi(V_{nj}):n\in N,j\in J_n)<\sum a_n$, which contradicts (2). We have shown that $\Phi(P)=\sum\Phi(P_n)$.

3.3. Fact. For any φ and any $P\in\mathcal{M}$, φ-LD(P)$\le\varphi$-UD(P)$\le\varphi$-UDim(P)$\le$ $\le\varphi$-udim(P).

Proof. If φ-udim(P)=$b<\infty$ and $P=\sum(P_n:n\in N)$, then $\sum(\varphi\text{-uw}(P_n):n\in N)\le$ $\le\sum(b.wP_n:n\in N)=b.wP$. This proves the last inequality; the remaining ones are evident.

3.4. **Proposition.** For any φ and any $P\in\mathcal{M}$, if $P=\sum(P_n:n\in N)$, then φ-LD(P)$\le\bigvee(\varphi$-LD$(P_n):n\in N)$, φ-UD(P)$\le\bigvee(\varphi$-UD$(p_n):n\in N)$.

This follows at once from 3.2.

3.5. **Theorem.** For any φ and any $P\in\mathcal{M}$, if $P=\sum(P_n:n\in N)$ or $P=\bigvee(P_n:$ $:n\in N)$, then φ-LDim(P)$=\bigvee(\varphi$-LDim$(P_n):n\in N)$, φ-UDim(P)$=\bigvee(\varphi$-UDim$(P_n):n\in$ $\in N)$.

Proof. Let $P=\sum P_n$. Put $b_n=\varphi$-UDim(P_n), $b=\varphi$-UDim(P). Clearly, $b\ge b_n$ for all $n\in N$. Let $S\le P$. Then, by 1.13, there are $S_n\le P_n$ such that $S=\sum S_n$. We have φ-UD$(S_n)\le b_n$ and hence, by 3.4, φ-UD(S)$\le\bigvee(b_n:n\in N)$. This proves $b\le\bigvee(b_n:$ $:n\in N)$. - If $P=\bigvee P_n:n\in N)$, then the proof is similar to the corresponding part of the proof of 2.3.

Remark. The theorem shows that, in some respects, the behavior of φ-Udim and φ-LDim is similar to that of various kinds of dimension of topological spaces (for instance, for normal spaces, dim $P=\bigvee(\dim P_n:n\in N)$ whenever $P=\bigcup P_n$, P_n are closed). On the other hand, the behavior of φ-udim (where φ is E-projective) is different from that of the topological dimension and rather resembles the behavior of the dimension δd of uniform spaces (the equality $\delta d(S\cup T)=\delta d(S)\vee\delta d(T)$ does hold whereas $\delta d(\bigcup(P_n:n\in N))=$ $=\bigvee(\delta d(P_n):n\in N)$ does not, in general).

3.6. **Lemma.** Let $\mathcal{X} \subset \mathcal{W}$ and assume that $\mathcal{X}$ contains all null spaces. Then, for any $P \in \mathcal{W}$, there is an $S \leq P$ such that (1) S has an ω-partition consisting of spaces in $\mathcal{X}$, (2) if $T \leq P-S$, $T \in \mathcal{X}$, then $wT=0$.

Proof. It is easy to show by transfinite induction that there is a countable ordinal $\alpha \geq 0$ and an indexed collection $(X_\beta : \beta < \alpha)$ such that (a) for all $\beta < \alpha$, $X_\beta \in \mathcal{X}$, $wX_\beta > 0$, (b) $\Sigma(X_\beta : \beta < \alpha) \leq P$, (c) if $Y \leq P - \Sigma(X_\beta : : \beta < \alpha)$, $Y \in \mathcal{X}$, then $wY=0$. Put $S = \Sigma(X_\beta : \beta < \alpha)$. Clearly, S satisfies (1) and (2).

3.7. **Lemma.** For any φ and any $P \in \mathcal{W}$, if $wP > 0$, $b \in \overline{R}_+$ and $\varphi\text{-udim}(S) \geq \geq b$ whenever $S \leq P$, $wS > 0$, then $\varphi\text{-UD}(P) \geq b$.

Proof. Let $a < b$. Let $\mathcal{U} = (U_n : n \in N)$ be an ω-partition of P. Put $M = \{n : : wU_n > 0\}$. If $n \in M$, then, by 3.6, there are $S_{nk} \leq U_n$, $k \in N$, such that $\Sigma(S_{nk} : k \in \in N) \leq U_n$, $\varphi\text{-uw}(S_{nk}) \geq a.wS_{nk}$ and $\varphi\text{-ud}(T) \geq a$ for no $T \leq V_n = P - \Sigma(S_{nk} : k \in N)$, hence $\varphi\text{-udim}(V_n) \leq a$. This implies $wV_n = 0$, $U_n = \Sigma(S_{nk} : k \in N)$. Hence $(S_{nk} : n \in M, k \in N)$ is an ω-partition of P refining $\mathcal{U}$. Clearly, $\Sigma(\varphi\text{-uw}(S_{nk}) : n \in M, k \in N) > > a.wP$. Since $\mathcal{U}$ has been arbitrary, this proves $\varphi\text{-UW}(P) \geq a.wP$.

3.8. **Proposition.** For any φ and any $P \in \mathcal{W}$, $\varphi\text{-UDim}(P)$ is equal to the infimum of all $b \in \overline{R}_+$ for which there exist $P_n \leq P$ such that $\Sigma P_n = P$, $\varphi\text{-udim}(P_n) \leq b$ for all $n \in N$.

Proof. Put $s = \varphi\text{-UDim}(P)$; let t be the infimum in question. If $b \in \overline{R}_+$ and there are P_n with properties stated above, then, by 3.3 and 3.4, $s \leq b$. This proves $s \leq t$. - Let $s' > s$. By 3.6, there are $S_n \leq P$, $n \in N$, such that $\varphi\text{-udim}(S_n) \leq s'$, $\Sigma(S_n : n \in N) \leq P$ and $\varphi\text{-udim}(T) \leq s'$ for no non-null $T \leq V = P - - \Sigma S_n$. By 3.7, $wV > 0$ would imply $\varphi\text{-UD}(V) \geq s'$, hence $\varphi\text{-UDim}(P) \geq s'$. Hence $wV=0$, $\Sigma S_n = P$ and therefore $t \leq s'$.

3.9. **Proposition.** If φ is E-projective, $P \in \mathcal{W}$ and $\varphi\text{-udim}(P) < \infty$, then $\varphi\text{-UDim}(P) = \varphi\text{-udim}(P)$.

Proof. If $S \leq P$ and $S = \Sigma(S_n : n \in N)$, then, by 2.3, $\varphi\text{-uw}(S) \leq \Sigma(\varphi\text{-uw}(S_n) : : n \in N)$. This implies $\varphi\text{-uw}(T) \leq \varphi\text{-UW}(T)$ for all $T \leq P$. Hence, $\varphi\text{-udim}(P) \leq \leq \varphi\text{-UDim}(P)$. By 3.3, this proves the proposition.

3.10. **Theorem.** Let P_1 and P_2 be W-spaces. Then $\mathrm{UDim}(P_1 \times P_2) \leq \mathrm{UDim}(P_1) + + \mathrm{UDim}(P_2)$.

Proof. Put $b_i = \mathrm{UDim}(P_i)$, $b = b_1 + b_2$. We can assume that $b < \infty$. Let $\varepsilon > 0$. For $i=1,2$, there exists, by 3.8, an ω-partition $(P_{in} : n \in N)$ of P_i such that $\mathrm{udim}(P_{in}) < b_i + \varepsilon/2$ for all $n \in N$. Put $T_{mn} = P_{1m} \times P_{2n}$. By 2.8, $\mathrm{udim}(T_{mn}) \leq b + \varepsilon$

for all $m,n\in N$, hence, by 3.5 and 3.3, $\mathrm{UDim}(P_1\times P_2)\leq b+\varepsilon$. Since $\varepsilon>0$ has been arbitrary, the theorem is proved.

Remark. Let U and V be as in 2.10. Put $T=U\times V$. It is easy to prove $\mathrm{UDim}(U)=\mathrm{UDim}(V)=2$, $\mathrm{UDim}(T)=3$. This shows that $\leq$ cannot be replaced by = in 3.10.

4

4.1. **Proposition and definition.** For any φ and any $P=\langle Q,\rho,\mu\rangle\in\mathfrak{M}$, there is exactly one function (mod μ) f (respectively, g) such that φ-$\mathrm{UW}(X.P)=\int_X f\,d\mu$ (respectively, φ-$\mathrm{LW}(X.P)=\int_X g\,d\mu$) for all $X\in\mathrm{dom}\,\bar{\mu}$. - We denote f and g by φ-$\nabla^U(P)$ (or $\nabla^U_\varphi(P)$) and φ-$\nabla^L(P)$ (or $\nabla^L_\varphi(P)$), respectively; $\nabla^U_\varphi(P)$ (respectively, $\nabla^L_\varphi(P)$) will be called the upper (lower) φ-dimensional density of P. If $\varphi=E$, we often omit the prefix "φ".

Proof. The proposition follows from 3.2 and the Radon-Nikodým theorem.

4.2. Conventions. To express the subsequent propositions 4.3, 4.4, 4.6 and 4.16 in a concise and exact manner, we introduce some ad hoc conventions. - A) If $\mu\in\mathcal{M}(Q)$, f and g are $\bar{\mu}$-measurable, $F=[f]_\mu$, $G=[g]_\mu$, we put $fG=FG=[fg]_\nu$, where $\nu=f.\mu$. Observe that, under this convention, FG=GF does not hold in general. - B) Let $\mu,\nu\in\mathcal{M}(Q)$, let μ be finite, let $\nu\leq\mu$ and let $f\in\mathcal{F}(Q)$ be $\bar{\mu}$-measurable. Then $\int[f]_\nu\,d\mu$ is defined as follows: let X be a support of ν with respect to μ (i.e., (1) $\nu\leq X.\mu$, (2) if $\nu\leq Y.\mu$, then $\bar{\mu}(X\setminus Y)=0$); we put $\int[f]_\nu\,d\mu=\int_X f\,d\mu$. - C) If $\mu\in\mathcal{M}(Q)$ is finite and, for $n\in N$, $\mu_n\leq\mu$, $\mu=\bigvee(\mu_n:n\in N)$, $F_n\in\mathcal{F}[\mu_n]$ and $F_n\geq 0$, then we put $\bigvee(F_n:n\in N)=[\bigvee(f_n i_{X(n)}:n\in N)]_\mu$, where, for each $n\in N, f_n\in F_n$ and X(n) is a support of μ_n with respect to μ. - D) If $\mu_i\in\mathcal{M}(Q_i)$, $F_i\in\mathcal{F}[\mu_i]$, i=1,2, then we put $F_1+F_2=[f]_\mu$, where $\mu=\mu_1\times\mu_2$ and, for some $f_i\in F_i$, f is the function $(x,y)\mapsto f_1(x)+f_2(y)$.

4.3. **Proposition.** For any φ and any $P=\langle Q,\rho,\mu\rangle\in\mathfrak{M}$, if $S=s.P\leq P$, then φ-$\mathrm{UW}(S)=\int s\,\nabla^U_\varphi(P)d\mu$, φ-$\mathrm{LW}(S)=\int s\,\nabla^L_\varphi(P)d\mu$.

Proof. It is easy to see that there are sets $X(n)\in\mathrm{dom}\,\bar{\mu}$ and reals a_n such that $\sum(a_n i_{X(n)}:n\in N)=s$ (mod μ). Then φ-$\mathrm{UW}(S)=\sum(a_n\varphi$-$\mathrm{UW}(X(n).P):n\in N)=\sum a_n\int_{X(n)}\nabla^U_\varphi(P)d\mu=\int(\sum a_n i_{X(n)})\nabla^U_\varphi(P)d\mu=\int s\,\nabla^U_\varphi(P)d\mu$. For φ-LW, the proof is analogous.

4.4. **Proposition.** For any φ and any $P=\langle Q,\rho,\mu\rangle\in\mathfrak{M}$, if $S=s.P\leq P$, then $\nabla^U_\varphi(S)=(\mathrm{sgn}\,s).\nabla^U_\varphi(P)$, $\nabla^L_\varphi(S)=(\mathrm{sgn}\,s).\nabla^L_\varphi(P)$.

Proof. Put $\nu = s.\mu$, $t = \operatorname{sgn} s$. Let $f \in \nabla_\varphi^U(P)$. If $X \in \operatorname{dom} \bar{\mu}$, then $\int_X tfd\nu = \int_X tfsd\mu = \int_X sfd\mu$, hence, by 4.3, $\int_X tfd\nu = \varphi\text{-UW}(X.s.P) = \varphi\text{-UW}(X.S)$. This proves that $tf \in \nabla_\varphi^U(S)$, and therefore (see 4.2, A) $\nabla_\varphi^U(S) = t\,\nabla_\varphi^U(P)$. The proof for ∇_φ^L is analogous.

4.5. **Theorem.** For any φ and any $P = \langle Q, \rho, \mu \rangle \in \mathfrak{M}$, $\varphi\text{-UDim}(P) = \sup \nabla_\varphi^U(P)$, $\varphi\text{-LDim}(P) = \sup \nabla_\varphi^L(P)$.

Proof. Put $a = \varphi\text{-UDim}(P)$, $b = \sup \nabla_\varphi^U(P)$. For any $S = s.P \leq P$, we have $\varphi\text{-UD}(S) = \int s\,\nabla_\varphi^U(P)d\mu/wS$, hence $\varphi\text{-UD}(S) \leq b$. This proves $a \leq b$. - Let $c < b$; let $f \in \nabla_\varphi^U(P)$. Then there is an $X \in \operatorname{dom} \bar{\mu}$ such that $\bar{\mu}X > 0$, $f(x) \geq c$ if $x \in X$. Clearly, $\varphi\text{-UD}(X.P) = \int_X fd\mu/\bar{\mu}X \geq c$. This proves $a \geq b$. - The proof for φ-LDim is analogous.

Remark. There are examples (not quite simple) of W-spaces P satisfying $\nabla^L(P) = \nabla^U(P)$ and such that UDim(S), where $S \leq P$, assumes all values from a certain interval.

4.6. **Theorem.** For any φ and any $P \in \mathfrak{M}$, if $P = \sum(P_n : n \in N)$ or $P = \bigvee(P_n : n \in N)$, then $\nabla_\varphi^U(P) = \bigvee(\nabla_\varphi^U(P_n) : n \in N)$, $\nabla_\varphi^L(P) = \bigvee(\nabla_\varphi^L(P_n) : n \in N)$.

Proof. We only prove the first equality. Clearly, it is sufficient to show that the equality holds if $P = \bigvee P_n$. Let $P_n = f_n.P$. Put $g_n = \operatorname{sgn} f_n$. Then, by 4.4, $\nabla_\varphi^U(P_n) = g_n.\,\nabla_\varphi^U(P)$. Since, clearly, $\mu = \bigvee(g_n.\mu : n \in N)$, $\bigvee g_n = 1 \pmod{\mu}$, we get $\bigvee(\nabla_\varphi^U(P_n) : n \in N) = \nabla_\varphi^U(P)$.

4.7. **Definition.** For any φ, a W-space P will be called φ-dimension-bounded (or merely " φ-bounded") if $\varphi\text{-udim}\, P < \infty$. It will be called fully φ-exact if $\varphi\text{-ud}(S) = \varphi\text{-}\ell d(S)$ for all $S \leq P$. If $\varphi = E$, we often omit the prefix " φ " in " φ -dimension-bounded" and "fully φ-exact".

4.8. Remark. It is easy to prove that, for any φ and any $P \in \mathfrak{M}$, there is exactly one partition (P_1, P_2, P_3, P_4) such that $\nabla_\varphi^L(P_1) = \nabla_\varphi^U(P_1) < \infty$, $\nabla_\varphi^L(P_2) < \nabla_\varphi^U(P_2) < \infty$, $\nabla_\varphi^L(P_3) = \nabla_\varphi^U(P_3) = \infty$, $\nabla_\varphi^L(P_4) < \nabla_\varphi^U(P_4) = \infty$. The spaces $P_1, \dots, P_4$ can be characterized as follows: (1) P_1 has an ω-partition consisting of φ-bounded fully φ-exact subspaces, (2) P_2 has an ω-partition consisting of φ-bounded subspaces and contains no fully φ-exact subspace, (3) every non-null subspace $S \leq P_3$ contains subspaces T with $\varphi\text{-}\ell d(T)$ arbitrarily large, (4) if $S \leq P_4$ is non-null, then it is neither φ-bounded nor fully φ-exact.

4.9. **Fact and definition.** For any φ and any $P = \langle Q, \rho, \mu \rangle \in \mathfrak{M}$, if there exists a function (mod μ) F such that ($*$) $\int_X F\, d\mu = \varphi$-uw(X.P)= $=\varphi$- ℓw(X.P) for all $X \in \operatorname{dom} \bar{\mu}$, then this F is unique. It will be denoted by φ- $\nabla^R(P)$ or $\nabla_\varphi^R(P)$ and called the exact φ-dimensional density for P. If there is no F satisfying ($*$), we will say that φ - $\nabla^R(P)$ does not exist. - If φ=E, we often omit the prefix " φ ". - Remark. If f is an Rw-density function for P in the sense of [4], 3.12, then $\nabla^R(P) = [f]_\mu$; conversely, if $\nabla^R(P)$ exists, then every $f \in \dot{\nabla}^R(P)$ is an Rw-density function for P.

4.10. **Proposition.** For any φ and any $P \in \mathfrak{M}$, if φ - $\nabla^R(P)$ exists, then P is fully φ-exact and $\nabla_\varphi^U(P) = \nabla_\varphi^L(P) = \nabla_\varphi^R(P)$.

Proof. If φ- $\nabla^R(P)$ exists, then, for any $S \preceq P$, φ-uw(S)=φ-ℓw(S) and if $S = \sum(S_n : n \in N)$, then φ-uw(S)= $\sum$(φ-uw(S_n)). This implies that P is fully φ-exact and φ-UW(S)=φ-uw(S)=φ-ℓw(S)=φ-LW(S) for each $S \preceq P$.

4.11. **Proposition.** For any φ and any $P \in \mathfrak{M}$, if there are fully φ-exact P_n such that $P = \sum(P_n : n \in N)$, then $\nabla_\varphi^U(P) = \nabla_\varphi^L(P)$.

Proof. If P is fully φ-exact, then φ-uw(T)=φ-ℓw(T) for all $T \preceq P$, hence φ-UW(S)= φ-LW(S) for all $S \preceq P$ and therefore $\nabla_\varphi^U(P) = \nabla_\varphi^L(P)$. If P= $= \sum(P_n : n \in N)$ and P_n are fully φ-exact, apply 4.6.

4.12. Remark. Let $P = \langle R^n, \rho, f.\lambda \rangle$, where ρ is any usual metric on R^n, λ is the Lebesgue measure and $\mu = f.\lambda$ is a finite measure. Then (1) P is fully exact, (2) for any non-null $S \preceq P$, UDim(S)=LDim(S)=n, (3) $\nabla^U(P)$= $= \nabla^L(P) = n.\,[\operatorname{sgn} f]_\mu$; this follows from [4], 2.9. However, if e.g. n=1, f(x)= $= |x|^{-1} |\log x|^{-3-2}$, then Rd(P)= ∞, whereas Rd(X.P)=1 whenever $X \in \operatorname{dom} \bar{\mu}$ is bounded and $\bar{\mu} X > 0$; thus $\nabla^R(P)$ does not exist.

4.13. Fact. For any $P \in \mathfrak{M}$ and any $P_n \preceq P$ satisfying $\sum(P_n : n \in N) = P$, (1) $\sum(\ell w(P_n) : n \in N) \leq \ell w(P)$, (2) if P is dimension-bounded, then uw(P) $\preceq$ $\preceq \sum(uw(P_n) : n \in N)$.

Proof. The assertion (1) follows at once from [4], 3.1. For (2), see [4], 3.4.

4.14. Fact. For any $P \in \mathfrak{M}$, (1) LW(P) $\leq$ ℓw(P), (2) if P is dimension-bounded, then uw(P) $\leq$ UW(P).

This is an immediate consequence of 4.13.

4.15. **Proposition.** Let $P \in \mathfrak{M}$ be dimension-bounded. Then the following conditions are equivalent: (1) P is fully exact, (2) $\nabla^R(P)$ exists, (3) $\nabla^L(P) = \nabla^U(P)$.

Proof. I. If (1) holds, then uw(T)=ℓw(T) for all $T \preceq P$. Hence, by 4.13, if $S \preceq P$, $S = \sum(S_n : n \in N)$, then $\sum(Rw(S_n) : n \in N) \preceq Rw(S) \preceq \sum(Rw(S_n) : n \in N)$. This proves that $X \mapsto Rw(X.P)$ is a measure, hence $\nabla^R(P)$ does exist. - II. By 4.10, (2) implies (3). - III. If $\nabla^L(P) = \nabla^U(P)$, then, for any $S \leq P$, $UW(S) = LW(S)$ and hence, by 4.14, $uw(S) = \ell w(S)$.

4.16. **Theorem.** For any W-spaces P_1 and P_2, $\nabla^U(P_1 \times P_2) \leq \nabla^U(P_1) + \nabla^U(P_2)$.

Proof. Let $P_i = \langle Q_i, \rho_i, \mu_i \rangle$, $P = P_1 \times P_2 = \langle Q, \rho, \mu \rangle$. Let $A \in \mathrm{dom}\,\bar{\mu}$, $B \in \mathrm{dom}\,\bar{\mu}$; put $C = A \times B$. Then, by 3.9, $UD(C.P) \leq UD(A.P_1) + UD(B.P_2)$, hence $UW(C.P) \leq UW(A.P_1) \cdot \mu_2 B + UW(B.P_2) \cdot \mu_1 A$. Clearly, $UW(C.P) = \int_C \nabla^U(P)\,d\mu$, $UW(A.P_1) \cdot \mu_2 B = \int_B \int_A \nabla^U(P_1)\,d\mu_1\,d\mu_2$, $UW(B.P_2) \cdot \mu_1 A = \int_A \int_B \nabla^U(P_2)\,d\mu_2\,d\mu_1$. This proves that $\int_{A\times B} \nabla^U(P)\,d\mu \leq \int (\nabla^U(P_1) + \nabla^U(P_2))\,d\mu$ for all $A \in \mathrm{dom}\,\bar{\mu}_1$, $B \in \mathrm{dom}\,\bar{\mu}_1$, and therefore $\nabla^U(P) \leq \nabla^U(P_1) + \nabla^U(P_2)$.

Remark. The equality $\nabla^U(P_1 \times P_2) = \nabla^U(P_1) + \nabla^U(P_2)$ does not hold, in general. For instance, for U and V from 2.10, we have $\nabla^U(U \times V) < \nabla^U(U) + \nabla^U(V)$.

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