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A RESONANCE PROBLEM FOR NONLINEAR DUFFING EQUATION

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Abstract: The purpose of this paper is to study a semilinear periodic problem for the forced Duffing equation at resonance. Nonlinear perturbation is allowed to grow linearly. Also some nonuniform nonresonance conditions concerning the nonlinear perturbation are discussed.

Key words and phrases: Duffing equation, periodic solutions, nonlinear perturbations with linear growth, Leray-Schauder degree.

Classification: 34C25, 34B15, 34C15

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1. Introduction. We consider the periodic boundary value problem BVP for an ordinary differential equation of Duffing type

(1.1) $u''+cu'+f(t,u)=e(t)$ a.e. on $[0,T]$,

(1.2) $u(T)=u(0),\ u'(T)=u'(0)$.

The right hand side e is an element of $L^1(0,T;\mathbf{R})$, $c\in\mathbf{R}$ is the damping and f is a nonlinear Carathéodory function.

Our aim is to give sufficient conditions for the existence of periodic solutions of (1.1) - (1.2). We can formulate either nonresonance or resonance conditions according to the lower and upper limits of $s^{-1}f(t,s)$ as $s\longrightarrow\pm\infty$. As for the nonresonance case our result is related to Drábek, Invernizzi [7]. The resonance conditions are related to the papers of Landesman, Lazer [8], Ward [9], Ahmad [1,2] and Drábek [3].

To prove our results we use essentially the structure of the set of all couples $(a,b)\in\mathbf{R}^2$ for which the nonlinear Dirichlet BVP

$u''+au^+-bu^-=0$ on $[0,T]$,
$u(0)=u(T)=0$

has a nontrivial solution. The reader is referred to the Fučík's monograph [6] in order to see the important role played by the set of such couples $(a,b)\in\mathbf{R}^2$.

When resonance occurs, we obtain Landesman-Lazer type sufficient conditions for the solvability of (1.1) - (1.2) (see (3.6), (3.7) below). Similarly as in [1],[2] and [3], some upper bounds for $s^{-1}f(t,s)$ are necessary if $s \to \pm\infty$. In case of the Duffing equation we have an explicit dependence of these bounds on the damping. Roughly speaking, the stronger the damping $|c|$ is, the weaker assumptions laid on $s^{-1}f(t,s)$ (as $s \to \pm\infty$) are necessary.

The paper is organized as follows. In Section 2 we formulate some auxiliary assertions. The main result of this paper is formulated in Section 3. The proof of the main result is contained in Section 4. Some remarks in Section 5 conclude the paper.

2. Preliminary assertions. In this section we shall summarize some results on BVP's for second order ODE's. Let $T>0$ be given number, and let $\alpha:=\pi/T$. Denote $u^{\pm}:=(|u|\pm u)/2$.

Lemma 2.1. <u>Let</u> $(a,b)\in R^2$. <u>The nonlinear BVP</u>

(2.1) $u''+a\,u^{+}-b\,u^{-}=0$ on $[0,T]$,

(2.2) $u(0)=u(T)=0$

<u>has a nontrivial solution if and only if</u>

$$(a,b)\in C_0^{*}\cup\left[\bigcup_{k=1}^{\infty}(C_k\cup C_k^{*})\right],$$

<u>where</u>

$C_0^{*}:=\{(a,b)\in R^2;(a-\alpha^2)(b-\alpha^2)=0\}$,

$C_k:=\{(a,b)\in R^2;a>k^2\alpha^2,\ b>0,\ b^{1/2}=k\alpha a^{1/2}/(a^{1/2}-k\alpha)\}$,

$C_k^{*}:=\{(a,b)\in R^2;a>k^2\alpha^2,\ b>0,\ b^{1/2}=(k+1)\alpha a^{1/2}/(a^{1/2}-k\alpha)\}\ \cup$
$\cup\{(a,b)\in R^2;a>(k+1)^2\alpha^2,\ b>0,\ b^{1/2}=k\alpha a^{1/2}/(a^{1/2}-(k+1)\alpha)\}$.

Remark 2.1. The proof of this lemma can be found in Fučík [6, Lemma 42.2]. Let $C_0:=\{(a,b)\in R^2;\ ab=0\}$.
Assuming $\alpha=1/2$ and plotting $(a^{1/2},b^{1/2})$ for $(a,b)\in C_k$ or C_k^{*}, $k\geq 0$, $a\geq 0$, $b\geq 0$, it is possible to get a picture of the sets C_k and C_k^{*} which can be found in Drábek, Invernizzi [4, p. 645].

The following assertion can be proved similarly as Lemma 2.1.

Lemma 2.2. <u>Let</u> $(a,b)\in R^2$. <u>The nonlinear periodic BVP</u>

(2.1) $u''+a\,u^{+}-b\,u^{-}=0$ on $[0,T]$,

(2.3) $u(T)=u(0),\ u'(T)=u'(0)$

has a nontrivial solution if and only if

$$(a,b) \in \bigcup_{k=0}^{\infty} C_k.$$

The set of nontrivial solutions of nonlinear periodic BVP is reduced only to constants if we consider a nonzero damping term ($c \neq 0$) in the equation (2.1).

Lemma 2.3. Let $(a,b) \in \mathbf{R}^2$, $c \in \mathbf{R}$, $c \neq 0$. The nonlinear periodic BVP

(2.4) $u'' + c\,u' + a\,u^+ - b\,u^- = 0$ on $[0,T]$,

(2.3) $u(T)=u(0)$, $u'(T)=u'(0)$

has a nontrivial solution if and only if

$$(a,b) \in C_0.$$

Proof. Multiplying both parts of (2.4) by u' and integrating on $[0,T]$ we get (with respect to (2.3)) that $u \equiv$ const. Let us suppose that $(a,b) \notin C_0$. Then the continuous function $\mathbf{R} \to \mathbf{R}$ defined by $s \mapsto a\,s^+ - b\,s^-$ vanishes only at $s=0$. In this case $u \equiv 0$ on $[0,T]$. On the other hand, $u \equiv 1$ and $u \equiv -1$ is a nontrivial solution of (2.4) - (2.3) if $a=0$ and $b=0$, respectively.

Substituting a, b by t-dependent mappings, we obtain the following result.

Lemma 2.4. Let $g_{\pm}$ be two mappings in $L^{\infty}(0,T;\mathbf{R})$.
Let us assume one of the following hypotheses is valid:

(H1) there exist an integer $i \geq 1$, two points $(a_i,b_i) \in C_i$, $(a'_{i+1},n_{i+1}) \in C_{i+1}$ such that

$a_i \le g_+(t) \le a_{i+1}$, $b_i \le g_-(t) \le b_{i+1}$

holds a.e. in $[0,T]$, with strict inequality signs on the set of positive measure in $[0,T]$;

(H2) there is $(a_1,b_1) \in C_1$ such that

$g_+(t) \le a_1$, $g_-(t) \le b_1$

holds a.e. in $[0,T]$, with strict inequality signs on the set of positive measure in $[0,T]$.

Then the nonlinear Dirichlet BVP

(2.5) $u'' + g_+(t)\,u^+ - g_-(t)\,u^- = 0$ on $[0,T]$,

(2.2) $u(0)=u(T)=0$

has no nontrivial solutions verifying also

(2.6) $\quad \operatorname{sign} u'(0)=\operatorname{sign} u'(T).$

Remark 2.2. The proof of Lemma 2.4 can be found in Invernizzi [7, Lemma 1.2]. Under more restrictive assumptions than (H1), (H2) laid on $g_{\pm}(t)$, the assertion of Lemma 2.4 is proved in Drábek, Invernizzi [4, Lemma 2.2].

3. Main result. In this section we shall consider the solvability of the nonlinear periodic BVP

(3.1) $\quad u''+c\,u'+f(t,u)=e(t)$ a.e. on $[0,T]$,

(3.2) $\quad u(T)=u(0),\ u'(T)=u'(0).$

We shall suppose that e is a fixed element in the Banach space $X:=L^1(0,T;\mathbf{R})$, with usual norm $\|\cdot\|$, $c\in\mathbf{R}$, and f is a Carathéodory function (i.e. $f(\cdot,s)$ is measurable for all s, and $f(t,\cdot)$ is continuous for a.e. $t\in[0,T]$), satisfying the growth restriction

(3.3) $\quad |f(t,s)|=p_1(t)+p_2|s|$

for a.e. $t\in[0,T]$ and for all $s\in\mathbf{R}$, with $p_1\in X$, $p_2\in\mathbf{R}$, $p_2\geq 0$.

Suppose that there are numbers $r_{\pm}\leq s_{\pm}$ such that

(3.4) $\quad r_{\pm}\leq \liminf\limits_{s\to\pm\infty} s^{-1}f(t,s),$

(3.5) $\quad \limsup\limits_{s\to\pm\infty} s^{-1}f(t,s)\leq s_{\pm}$

a.e. in $[0,T]$.

The solution u of (3.1) - (3.2) is a continuously differentiable mapping $u:[0,T]\to\mathbf{R}$, such that u' is absolutely continuous and (3.1) - (3.2) hold.

Let us denote by Δ_d the d-th closed quadrant of $\mathbf{R}^2$.

Theorem 3.1 (Nonuniform nonresonance case). Let $[r_+,s_+]\times[r_-,s_-]\subset\Delta_1\cup\Delta_3$ and let either $(r_+-c^2/4,r_--c^2/4)\in C_i$, $(s_+-c^2/4,s_--c^2/4)\in C_{i+1}$ for some fixed $i\geq 1$, or $(s_+-c^2/4,s_--c^2/4)\in C_1$, or $s_+-c^2/4\leq 0, s_--c^2/4\leq 0$. Moreover, let us suppose that (3.4) and (3.5) hold with strict inequality signs for t in a subset of positive measure in $[0,T]$.

Then the BVP (3.1) - (3.2) has a solution for arbitrary $e\in X$.

In the resonance case we shall suppose that the following hypothesis is fulfilled:

(f) $f^{-\infty}(t) = \limsup_{s\to-\infty} f(t,s)$ and $f_{+\infty}(t) = \liminf_{s\to+\infty} f(t,s)$

are bounded from above and from below, respectively, for a.e. $t \in [0,T]$.

Remark 3.1. Note that hypothesis (f) implies that $r_{\pm}$ from (3.4) satisfy $r_{\pm} \gtrless 0$.

Theorem 3.2 (Resonance case). Let us suppose (f), $(s_+ - c^2/4, s_- - c^2/4) \in C_1$ and (3.5) hold with strict inequality signs for t in a subset of positive measure in $[0,T]$.

Then the BVP (3.1) - (3.2) has a solution provided that

$$(3.6) \qquad \int_0^T f^{-\infty}(t)dt < \int_0^T e(t)dt < \int_0^T f_{+\infty}(t)dt.$$

Consider, instead of (f), the following hypothesis:

(f*) $f^{+\infty}(t) = \limsup_{s\to+\infty} f(t,s)$ and $f_{-\infty}(t) = \liminf_{s\to-\infty} f(t,s)$

are bounded from above and from below, respectively, for a.e. $t \in [0,T]$.

Remark 3.2. The hypothesis (f′) implies that $s_{\pm}$ from (3.5) satisfy $s_{\pm} \lessgtr 0$.

Theorem 3.3 (Resonance case). Let us suppose (f′) is valid. Then the BVP (3.1) - (3.2) has a solution provided that

$$(3.7) \qquad \int_0^T f^{+\infty}(t)dt < \int_0^T e(t)dt < \int_0^T f_{-\infty}(t)dt.$$

4. Proof of the main result. The proofs of Theorems 3.1 - 3.3 will be performed in several steps. The main tool we shall use is the homotopy invariance property of the Leray-Schauder degree.

Fix $\lambda \in]0,(2\pi/T)^2[$. Consider the linear operator $K:X \to X$ defined by $Kw :=$ the unique solution u of the linear periodic BVP: $u'' + cu' + \lambda u = w$, $u(T) = u(0)$, $u'(T) = u'(0)$. A straightforward argument shows that K is completely continuous. Moreover, the standard regularity argument for ODE′s proves that K maps bounded sets in $L^1(0,T;\mathbb{R})$ into relatively compact sets in $C([0,T])$. The Nemytskii operator induced by f and the mappings $u \mapsto u^{\pm}$ are all continuous $X \to X$ and map bounded sets into bounded sets.

Step 1. Let $e \in K$ be given and $c \neq 0$. Let us consider the continuous path

$$(4.1) \qquad \sigma \in [0,1] \mapsto (\mu_\sigma, \nu_\sigma) \in (\Delta_1 \cup \Delta_3) \setminus C_0$$

with endpoints (μ_0, ν_0) and (μ_1, ν_1), with $\mu_1 = \nu_1$. Then there is $R > 0$ sufficiently large such that the mapping $X \to X$ defined by

(4.2) $u \mapsto u - K(-\mu_\sigma u^+ + \nu_\sigma u^- + \lambda u + (1-\sigma)e)$

has the Leray-Schauder degree at 0 relative to the ball $\{x \in X;\ \|x\| < R\}$ equal to a fixed odd number.

Proof. The mapping (4.2) is odd for $\sigma = 1$ (we have $\mu_1 = \nu_1$). Hence by the Leray-Schauder continuation theorem it is sufficient to show the existence of such $R > 0$ that

$$\tilde{H}(\sigma, u) := u - K(-\mu_\sigma u^+ + \nu_\sigma u^- + \lambda u + (1-\sigma)e) = 0 ,$$

for $(\sigma, u) \in [0,1] \times X$, implies $\|u\| < R$. Let us suppose the contrary. Then there are $\sigma_n \in [0,1]$ and $u_n \in X$, $\|u_n\| \to \infty$ and

(4.3) $$\tilde{H}(\sigma_n, u_n) = 0,$$

for all $n \in \mathbb{N}$. The equation (4.3) is equivalent to the periodic BVP

(4.4) $u_n'' + c\, u_n' + \mu_{\sigma_n} u_n^+ - \nu_{\sigma_n} u_n^- = (1-\sigma_n)e,$

(4.5) $u_n(T) = u_n(0),\ u_n'(T) = u_n'(0).$

Put $v_n = u_n / \|u_n\|$. Then passing to a subsequence, if necessary, we obtain from (4.4) - (4.5) $v_n \to v$ in X, $\sigma_n \to \sigma_0 \in [0,1]$ and

$$v'' + c\, v' + \mu_{\sigma_0} v^+ - \nu_{\sigma_0} v^- = 0,$$
$$v(T) = v(0),\ v'(T) = v'(0).$$

This, with respect to the assumption (4.1), contradicts the assertion of Lemma 2.3.

Step 2. Let $e \in X$ be given and $c = 0$. Let us consider the continuous path

(4.6) $\sigma \in [0,1] \mapsto (\mu_\sigma, \nu_\sigma) \in (\Delta_1 \cup \Delta_3) \setminus \bigcup_{k=0}^{\infty} C_k$

with endpoints (μ_0, ν_0) and (μ_1, ν_1), with $\mu_1 = \nu_1$. Then the assertion from Step 1 remains true.

Proof. By the same way as in the proof of Step 1 we get $v \in X$, $\|v\| = 1$, $\sigma_0 \in [0,1]$ and

$$v'' + \mu_{\sigma_0} v^+ - \nu_{\sigma_0} v^- = 0,$$

$$v(T) = v(0),\ v'(T) = v'(0).$$

This, with respect to the assumption (4.6), contradicts the assertion of

Lemma 2.2.

Step 3. Let us suppose that the hypotheses of Theorems 3.1 - 3.3 are fulfilled. Let either

$(\mu_0, \nu_0) \in [r_+, s_+] \times [r_-, s_-] \setminus C_0$, for $c \neq 0$, or

$(\mu_0, \nu_0) \in [r_+, s_+] \times [r_-, s_-] \setminus \bigcup_{k=0}^{\infty} C_k$, for $c=0$.

Consider the homotopy $H: [0,1] \times X \to X$ defined by

$H(\tau, u) := u - K(-\tau f(\cdot, u) - (1-\tau)[\mu_0 u^+ - \nu_0 u^-] + \lambda u + e)$.

Then there exists $R > 0$ such that

$$H(\tau, u) = 0,$$

for $(\tau, u) \in [0,1] \times X$ implies $\|u\| < R$.

Proof. Let us suppose the contrary. Then there is a sequence $(\tau_n, u_n) \in [0,1] \times X$ such that

(4.7) $u_n = K(-\tau_n f(\cdot, u_n) - (1-\tau_n)[\mu_0 u_n^+ - \nu_0 u_n^-] + \lambda u_n + e)$,

and $\|u_n\| \to \infty$. The normalized sequence $v_n := u_n / \|u_n\|$ verifies

(4.8) $v_n = K(-\tau_n \|u_n\|^{-1} f(\cdot, u_n) - (1-\tau_n)[\mu_0 v_n^+ - \nu_0 v_n^-] + \lambda v_n + \|u_n\|^{-1} e)$.

According to (3.3) the sequence $f_n := \|u_n\|^{-1} f(\cdot, u_n)$ is bounded in X. Therefore, passing if necessary to subsequence, we can assume that $v_n \to v$ uniformly on $[0,T]$. In this case, (3.3) implies

$|f_n(t)| \leq p_1(t) \|u_n\|^{-1} + p_2 |v_n(t)| \leq p(t)$,

for all $n \in \mathbf{N}$, with some $p \in X$. Hence

$$\int_{t_1}^{t_2} |f_n(t)| dt \to 0, \text{ for } |t_1 - t_2| \to 0,$$

uniformly with respect to $n \in \mathbf{N}$. Therefore the sequence $\{f_n\}_{n=1}^{\infty} \subset X$ is weakly sequentially compact (see Dunford, Schwartz [5, Corollary 8.11]), i.e. there is $g \in X$ such that some subsequence of $\{f_n\}_{n=1}^{\infty}$ converges weakly to g in X. We can suppose $\tau_n \to \tau \in [0,1]$, too. Since any bounded linear mapping $X \to X$ is both continuous and weakly continuous, we can pass to the weak limit in (4.8) and we get

(4.9) $v = K(-\tau g + (1-\tau)[\mu_0 v^+ - \nu_0 v^-] + \lambda v)$.

It is a direct consequence of Lebesgue's theorem, Fatou's lemma, (3.3), (3.4) and (3.5) that

(4.10) $g(t)=h_+(t)v^+(t)-h_-(t)v^-(t)$ a.e. on $[0,T]$,

(4.11) $r_\pm \leq h\ (t) \leq s_\pm$ a.e. on $[0,T]$

The mappings $h_\pm : [0,T] \to R$ are defined as follows:

$h_\pm(t) := g(t)/v(t)$, for $t \in V_\pm$, $h_+(t) := \mu_0, h_-(t) := \nu_0$,

for $t \in V_0$, where $V_0 := \{t \in [0,T];\ v(t)=0\}$, $V_\pm := \{t \in [0,T];\ v(t) \gtrless 0\}$. The reader is referred to Drábek, Invernizzi [4, p. 648] for details.

Note that (4.9) - (4.11) imply that v, $\|v\|=1$, is a solution of the periodic BVP

(4.12) $v''+c\,v'+\gamma_+(t)v^+ - \gamma_-(t)v^- =0$ a.e. on $[0,T]$,

(4.13) $v(T)=v(0),\ v'(T)=v'(0)$,

where the coefficients $\gamma_+(t) := \tau h_+(t)+(1-\tau)\mu_0$ and $\gamma_-(t) := \tau h_-(t)+(1-\tau)\nu_0$ verify

(4.14) $r_\pm \leq \gamma_\pm(t) \leq s_\pm$ a.e. on $[0,T]$.

Our aim is to show that (4.12) - (4.14) yield to the contradiction under the assumptions of Theorems 3.1 - 3.3.

Let us suppose that the assumptions of Theorem 3.1 are satisfied. Then either $\gamma_\pm(t) \geqq 0$ or $\gamma_\pm(t) \leqq 0$ with strict inequality signs on the set of positive measure in $[0,T]$. Then integrating (4.12) on $[0,T]$, we obtain (with respect to (4.13)) that there is neither $v>0$ nor $v<0$ on $[0,T]$. Hence v has at least two zero points in $[0,T]$, i.e. we can find $t^* \in [0,T]$ such that $v(t^*)=0$ and $C := 1/v'(t^*) > 0$. Let us extend v and $\gamma_\pm$ by T-periodicity on the whole real line and define $\tilde{v}, \tilde{\gamma}_\pm : [0,T] \to R$ by the relations $\tilde{v}(t) := Cv(t+t^*)$, $\tilde{\gamma}_\pm(t) := \gamma_\pm(t+t^*)$. We obtain

$$\tilde{v}''+c\,\tilde{v}'+\tilde{\gamma}_+(t)\tilde{v}^+ - \tilde{\gamma}_-(t)\tilde{v}^- =0 \text{ a.e. on } [0,T],$$

$$v(T)=v(0)=0,\ v'(T)=v'(0)=1.$$

Introducing $z : [0,T] \to R$, $z(t) := \exp((c/2)t)\tilde{v}(t)$, a simple computation shows that z is a solution of the nonlinear Dirichlet BVP

(4.15) $z''+(\tilde{\gamma}_+(t)-c^2/4)z^+ -(\tilde{\gamma}_-(t)-c^2/4)z^- =0$,

(4.16) $z(0)=z(T)=0$,

which verifies

(4.17) $\operatorname{sign} z'(T)=\operatorname{sign} z'(0)$.

The inequalities (4.14) and the hypotheses of Theorem 3.1 imply that the functions $g_{\pm}(t):=\tilde{\gamma}_{\pm}(t)-c^2/4$ fulfil the assumptions of Lemma 2.4. Then (4.15) - (4.17) yield to $z(t)\equiv 0$ which contradicts $\|v\|=1$.

Let the assumptions of Theorem 3.2 be satisfied. Then $\gamma_{\pm}(t)=0$ a.e. in $[0,T]$. If these inequalities hold with the strict inequality signs on the subset of positive measure in $[0,T]$ (the resonance does not occur), we can reach the contradiction by the same way as in the previous case. With respect to the assumptions of Theorem 3.2, we have that either $v>0$, or $v<0$ on $[0,T]$ (see Lemma 2.4, (H2)). If $\gamma_+(t)=0$ a.e. on $[0,T]$ and $\gamma_-(t)\neq 0$ in a subset of positive measure in $[0,T]$ then integrating (4.12) on $[0,T]$, we obtain $v>0$. Then it follows directly from (4.12) - (4.13) that $v\equiv 1/T$ on $[0,T]$. Similarly, we have $v\equiv -1/T$ on $[0,T]$ if $\gamma_-(t)=0$ a.e. on $[0,T]$ and $\gamma_+(t)\neq 0$ in a subset of positive measure in $[0,T]$. Finally, it is either $v\equiv 1/T$, or $v\equiv -1/T$ on $[0,T]$ if $\gamma_{\pm}(t)=0$ a.e. on $[0,T]$. Let us suppose that $v\equiv 1/T$ on $[0,T]$ (the other case $v\equiv -1/T$ on $[0,T]$ can be treated similarly). Then with respect to the uniform convergence $v_n\to 1/T$ on $[0,T]$, we have $u_n(x)\to+\infty$ uniformly on $[0,T]$. Hence the operator equation (4.7) is, for n sufficiently large, equivalent to the periodic BVP

(4.18) $u_n''+c\,u_n'+\tau_n f(t,u_n)+(1-\tau_n)\mu_0 u_n^+=e,$

(4.19) $u_n(T)=u_n(0),\ u_n'(T)=u_n'(0).$

Integrating (4.18) on $[0,T]$, we obtain by using Fatou's lemma and (4.19):

(4.20) $$\int_0^T \liminf_{n\to\infty}[\tau_n f(t,u_n)+(1-\tau_n)\mu_0 u_n^+]dt\le\liminf_{n\to\infty}\int_0^T[\tau_n f(t,u_n)+(1-\tau_n)\mu_0 u_n^+]dt=\int_0^T e(t)dt.$$

On the other hand, the assumption (3.6) yields

$$\int_0^T e(t)dt<\int_0^T \liminf_{n\to\infty}[\tau_n f(t,u_n)+(1-\tau_n)\mu_0 u_n^+]dt,$$

which contradicts (4.20).

Let us suppose that the assumptions of Theorem 3.3 are fulfilled. Then $\gamma_{\pm}(t)\le 0$ a.e. in $[0,T]$. If these inequalities hold with the strict inequality signs on the subset of positive measure in $[0,T]$ (resonance does not occur), we can proceed again as in the case of Theorem 3.1. In the opposite case we can derive $v\equiv 1/T$, or $v\equiv -1/T$ on $[0,T]$, and the proof can be performed in a similar way as in the case of Theorem 3.2 but by using the assumption (3.7) instead of (3.6) (note that it is $\mu_0<0$, $\nu_0<0$ under the assumptions of Theorem 3.3).

Step 4. Let us suppose that the hypotheses of Theorems 3.1 - 3.3 are fulfilled. Then the nonlinear periodic BVP (3.1) - (3.2) has a solution.

Proof. The nonlinear periodic BVP (3.1) - (3.2) is equivalent to the operator equation

(4.21) $u-K(-f(\cdot,u)+\lambda u+e)=0.$

By Step 3 the Leray-Schauder degree of the operator standing on the left hand side of (4.21), relative to $\{x \in X;\ \|x\| < R\}$, is equal to the degree of the operator

(4.22) $u \longmapsto u-K(-\mu_0 u^+ + \nu_0 u^- + \lambda u+e),$

provided that $R > 0$ is large enough. But according to Steps 1,2 the degree of (4.22) is different from zero. Hence the operator equation (4.21) has at least one solution which is simultaneously the solution of the nonlinear periodic BVP (3.1) - (3.2).

5. Concluding remarks

Remark 5.1. The assertion of Theorem 3.1 is proved under more restrictive assumptions in Drábek, Invernizzi [4]. More precisely, in [4, Theorem 3.1], we impose uniform nonresonance conditions on nonlinearity f, while the nonresonance conditions in Theorem 3.1 are nonuniform ones.

Remark 5.2. The proof of Theorem 3.1 can be found in Invernizzi [7] if $c=0$.

Remark 5.3. To prove our result, we construct a slightly different homotopy than in [4, 7]. The homotopy of compact perturbations of the identity used here allows us to treat also the resonance case. This is the topic of Theorems 3.2, 3.3. The relation of our result to the paper of Ahmad [2] was already pointed out in the introduction. The author of [2] studies nonselfadjoint resonance problems for PDE's with unbounded perturbations. Note that in case of Duffing equation we received an explicit bound for the ratio $s^{-1}f(t,s)$ (if $|s|$ is large enough) by means of the curve C_1 and the damping c.

Remark 5.4. Our result also generalizes the result contained in Ward [9, Theorem 1] for the case of the Duffing equation.

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