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BOOLEAN GRAPHS

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Abstract: The concept of Boolean graphs is introduced and Boolean graphs are characterized by means of prime convexes having special properties.

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Let L be a lattice and $\mathcal{P}(L)$ the poset of its prime ideals not containing the trivial prime ideals L and $\emptyset$. The poset $\mathcal{P}(L)$ is ordered by the set theoretical inclusion $\subseteq$. Nachbin has proved (see [1, Theorem II.1.22]) that a distributive lattice L with $1 \neq 0$ is a Boolean lattice if and only if the poset $\mathcal{P}(L)$ is unordered. As well known, a lattice is distributive if and only if its every ideal is an intersection of all prime ideals containing it. By combining these two results, we obtain a theorem, the generalization of which we will prove for graphs.

Theorem 1. *A lattice L with $1 \neq 0$ is a Boolean lattice if and only if the following two conditions hold:*

(i) *every ideal of L is an intersection of all prime ideals containing it;*

(ii) *the poset $\mathcal{P}(L)$ is unordered.*

The graphs satisfying the analogies of the conditions (i) and (ii) above constitute the class of Boolean graphs, as it will be shown in Theorem 2. We introduce first some graph theoretic concepts.

The graphs $G = (V, X)$ considered here are finite, undirected and connected without loops and multiple lines. The set V is the set of points of G and X its set of lines. The graph theoretic concept corresponding to convex sublattices (and thus to ideals, too), is the convex of a graph. A pointset $A \subset V$ is a *convex* of G, if A contains all points on every shortest $a-b$ path ($a-b$ geodesic) for any pair $a, b \in A$. The least convex containing a pointset $A \subset V$ is denoted by $[A] = \bigcap\{C \mid C$ is a convex of G and $A \subset C\}$; $[a, b]$ is the brief substitute for the complete expression $[\{a, b\}]$. The sets V and $\emptyset$ are trivial convexes. A convex B is *prime* if also the set $V \backslash B$ is a convex of G; the convexes V and $\emptyset$ are trivial prime convexes of G. We denote by $\mathcal{P}(G)$ the poset of nontrivial prime convexes of a graph G; this poset is ordered by the set theoretical inclusion. A graph G is a *prime convex intersection graph* (or has the prime convex intersection property), if its every convex is the intersection of all prime convexes containing it.

A graph $G = (V, X)$ is a *Boolean graph* if the following conditions (A), (B) and (C) hold:

(A) Let b be an arbitrary point of V. An order relation $\preceq$ can be defined on the set V such that b is the least element with respect to $\preceq$ and the Hasse diagram graph H of the ordered set $(V, \preceq)$ is isomorphic to G under the mapping f : $f(v) = v$ for every $v \in V$.

(B) The ordered set $(V, \preceq)$ of (A) is a lattice H satisfying the Jordan–Hölder chain condition (i.e. a graded lattice).

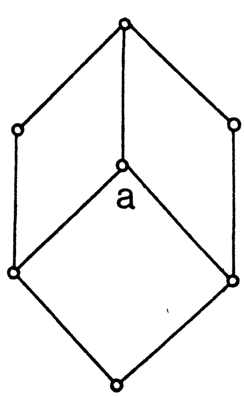
(C) Any two lattices H_1 and H_2 derived from G in (A) are order isomorphic.

In [2] the graphs satisfying (A) and (B) were called highly symmetric covering graphs of finite lattices (or briefly, highly symmetric graphs). Such graphs are single cycles of even length, the covering graphs of finite Boolean lattices, and the products of these two kinds of graphs. The main result of this note is the following theorem which, by Theorem 1, motivates the name Boolean graphs.

Theorem 2. *A graph G is a Boolean graph if and only if the following two conditions hold:*

(i) *G is a prime convex intersection graph;*

(ii) *if $P \neq V, \emptyset$ is a prime convex of G, the graphs $G(P)$ and $G(V \backslash P)$ induced by P and $V \backslash P$, respectively, are isomorphic under the mapping $\varphi : G(P) \to G(V \backslash P)$ such that if $x \in P$ and $y \in V \backslash P$ are adjacent in G, then $\varphi(x) = y$.*



Let G be a graph satisfying the conditions (i) and (ii) of Theorem 2, and consider the condition (ii). If P and Q are prime convexes of G such that $P \subset Q$ and $P, Q \neq V, \emptyset$, then $| P | \leq | Q | = | V \backslash Q | \leq V \backslash P |$, and because $| P | = | V \backslash P |$, we obtain the equality $| P | = | Q |$, which implies that $P = Q$. Hence the poset $\mathcal{P}(G)$ is unordered, and thus the analogy of the condition (ii) of Theorem 1 is contained in (ii) of Theorem 2. In fact, the condition (ii) of Theorem 2 holds for Boolean lattices, but it is not necessary to present it in so strong form as above. On the other hand, one cannot substitute (ii) in Theorem 2 by the condition "$\mathcal{P}(G)$ is unordered". A counterexample G is given in the figure: this graph has the prime convex intersection property and its $\mathcal{P}(G)$ is unordered, but by putting the point a the least element, one cannot obtain a lattice. Note that in this graph $| P | \neq | V \backslash P |$ for all prime convexes P.

One can easily see that a graph J is a prime convex intersection graph if and only if there is for any pair C, x, C is a nonempty convex of J and $x \notin C$ a point of J, a prime convex P of J separating C and x i.e., $C \subset P$ and $x \notin P$. In the proof of theorem 2 we need a lemma, which we present first.

Lemma 1. *The convex $[a, b]$ of a prime convex intersection graph G consists of points on a–b geodesics for every pair $a, b \in V$.*

Proof. Let a and b be a pair of points such that the convex $[a, b]$ contains at least one point v which is not on any $a-b$ geodesic. This implies the existence of two points x and z, x is on an $a-b$ geodesic and z is on another $a-b$ geodesic, such that no point $x_1, ..., x_m$ of an $x-z$ geodesic $x = x_0, x_1, ..., x_m, x_{m+1} = z$ is on any $a-b$ geodesic. Clearly a and b can be chosen such that every convex $[u, w]$ with $d(u, w) < d(a, b)$ is the set of all points on $u-w$ geodesics. We may assume further that $d(a, b) \geq d(x, b)$, $d(z, b) \geq d(x, b)$, and that x and z are as near to b as possible. Let us consider the point x_1. Because $d(a, x) < d(a, b)$, the convex $[a, x]$ consists of points on $a-x$ geodesics, and thus $x_1 \notin [a, x]$. The prime convex intersection property of G implies now the existence of a prime convex P separating $[a, x]$ and x_1 : $[a, x] \subset P$ and $x_1 \in V \backslash P$. Because $x_1 \in [a, b]$, we have $x_1, b \in P$. Let $x = b_0, b_1 b_2, ..., b_{k-1}, b_k = b$ be the points of an $x-b$ geodesic. Because x and z are as near to b as possible, $d(z, b) \geq d(x, b)$ and $d(x_1, z) \geq d(x_1, x) = 1$, then a $b_i - x_1$ geodesic goes over x, $i = 1, ..., k$. This implies that there is no prime convex separating $[a, x]$ and x_1, which is a contradiction. Thus the assumption is false and the convex $[a, b]$ consists of points on $a-b$ geodesics for every pair $a, b \in V$, and the lemma follows.

Proof of Theorem 2. Assume that G is a Boolean graph. We should show that G has the properties (i) and (ii) of the theorem.

(i) Let $C \neq V, \emptyset$ be a convex of G. There exists a line xy with $x \in C$ and $y \in V \backslash C$. Because G is a Boolean graph, we can make up a lattice L_y with y as the least and 1_y as the greatest element. Because 1_y is the unique point for y such that $[y, 1_y] = V$ [2, Thm. 6], the point y is not on any $x - 1_y$ geodesic, and thus $y \notin [x, 1_y]$ and $[x, 1_y] \neq V$. Consider an arbitrary point c of C. Because $V = [y, 1_y]$, the point c is on some $y - 1_y$ geodesic, and because $c, x \in C$ and $y \notin C$, the point y does not locate on any $x - c$ geodesic. Assume that c is not on any $1_y - x$ geodesic. This implies that no $1_y - c - y$ geodesic contains the point x. Thus $d(c, y) < d(c, x) + 1$. On the other hand, all points of any $c - x$ geodesic belong to C and cannot contain the point y, whence $d(c, x) < d(c, y) + 1$. These inequalities imply that $d(c, x) \leq d(c, y)$ and $d(c, x) \leq d(c, y)$, whence $d(c, x) = d(c, y)$. Because $d(x, y) = 1$, the points of $c - y$ and $c - x$ geodesics and the line xy contain now an odd cycle, which is absurd by [2]. Hence the assumption is false, and the point c is on some $1_y - x$ geodesic. The point $c \in C$ was an arbitrary point of C, and thus the convex C is contained in the convex $[1_y, x] \neq V, \emptyset$.

Let $C \neq V, \emptyset$ be a a convex, q a point, $q \notin C$, and xy $(x \in C, y \in V \backslash C)$ a line on a $q - c$ path giving the minimum value to the expression $\{d(q, c) \mid c \in C\}$. Consider now the convex $[x, 1_y]$ constructed above. We saw that $C \subset [x, 1_y]$ and $y \notin [x, 1_y]$. If $q \in [x, 1_y]$, then also $y \in [x, 1_y]$, because y is on a $q - x$ geodesic; a contradiction. Hence $q \notin [x, 1_y]$, and thus $[x, 1_y]$ is a convex separating C and q. The prime convex intersection property of G follows if

$[x, 1_y]$ is a prime convex.

Because G is a Boolean graph, there is, by [2], a unique point u such that $[u, x] = V$. Then the point 1_y is on some $u - x$ geodesic, and because $[x, 1_y] \neq V$, we have $u \notin [x, 1_y]$. Because, by [2], every point of the convex $[x, 1_y]$ is on some $x - 1_y$ geodesic, there is a point z of a $u - x$ geodesic such that $z \notin [1_y, x]$ and z and 1_y are adjacent. Obviously, $u = z$, and because y is on $u - x$ geodesic and $d(x, y) = 1$, every point $p \neq x$ adjacent to y is on a $u - y$ geodesic. By the symmetry, we see that every point $r \neq u$ adjacent to 1_y is on a $1_y - x$ geodesic. But then $[1_y, x] \cap [u, y] = \emptyset$ and $[1_y, x] \cup [u, y] = V$, whence $[1_y, x]$ is prime, and the prime convex itersection property follows.

Let $P \neq V, \emptyset$ be a prime convex of G and $V = [v, u]$ with $v \in P$. Because $P \neq V$, then $u \notin P$, and by [2], the point u is unique. Thus there is for any point $p \in P$ a unique point $q \in V \backslash P$ such that $[p, q] = V$, whence $| P | \leq | V \backslash P |$. The set $V \backslash P \neq V, \emptyset$ is also a prime convex, and by applying the proof above to $V \backslash P$ we obtain $| V \backslash P | \leq | P |$. Accordingly, $| P | = | V \backslash P |$, for every prime convex $P \neq V, \emptyset$ of G, and thus $\mathcal{P}(G)$ is unordered. As in the proof of (i) above, the prime convex $P \neq V, \emptyset$ is contained in another prime convex $[x, 1_y]$. Because $| V | = 2 | P | = 2 | [x, 1_y] |$, we see that $P = [x, 1_y]$, and thus every prime convex $P \neq V, \emptyset$ has an expression $P = [a, b]$ for some $a, b \in P$.

(ii) Let $P \neq V, \emptyset$ be a prime convex. As in the proof of (i), we see that $P = [x, 1_y]$ and $V \backslash P = [y, k]$, where $k1_y$ and yx are two lines of G. On the other hand, $G = [y, 1_y] = [x, k]$, and by the condition (C), the lattices H_1 (where y is the least and 1_y the greatest element) and H_2 (where x is the least and k the greatest element) are order isomorphic. This order isomorphism implies now that the sublattices of $[y, k]$ and $[x, 1_y]$ are also order isomorphic, and because, by [2], the Hasse diagram graph of H_1 contains exactly the lines of G, this latter isomorphism implies the isomorphism of the graphs $G([x, 1_y]) = G(P)$ and $G([y, k]) = G(V \backslash P)$. By the symmetry (every two lattices H_1 and H_2 are order isomorphic), one can always construct an isomorphism satisfying the demands of the theorem, and the first part of the theorem follows.

Conversely, let G be a graph satisfying the conditions (i) and (ii) of the theorem. By [2, Thm. 6], a graph G is a symmetric covering graph of a finite meetsemilattice if

(1) the relation $u \in [x, y]$ implies that u is on an $x - y$ geodesic in G;

(2) every cycle of G is even;

(3) there is for any three points $p, x, y \in V$ a point $v \in V$ such that the equation $[p, x] \cap [p, y] = [p, z]$ holds.

By [2, Thm. 6], G is a graph satisfying the conditions (A) and (B) of Boolean graphs if G satisfies the conditions (1)–(4), where

(4) every meetsemilattice derived from G is a lattice.

We will show that the conditions (1)–(4) and (C) hold for G, which proves the converse part of the theorem. The condition (1) holds by Lemma 1, and so we concentrate on (2)–(4) and (C).

(2) Assume that G contains an odd cycle. This implies that G also contains a minimal odd cycle Q : if x and y are two arbitrary points of Q, then one $x - y$ geodesic goes along the arc of Q. Trivially, Q contains at least three points x, y and z with $xy,\ yz \in X$. The points x and y constitute a convex not containing the point z, and thus there is a prime convex P

separating $\{x, y\}$ and $z : \{x, y\} \subset P$ and $z \in V \backslash P$. Because Q is a cycle and $zy \in X$, there is a line vu in Q such that the $z-u$ geodesic of Q is contained in $V \backslash P$ and the $y-v$ geodesic of Q is contained in P. Because Q is a minimal odd cycle, $d(y, v) \neq d(u, z)$. This is absurd because, by the isomorphism φ of (ii), $\varphi(y) = z$ and $\varphi(v) = u$. Hence G cannot contain odd cycles and thus (2) holds for G.

(3) Choose an arbitrary point p of G. If (3) holds for all pairs $x, y \in V$, we are done, and hence we assume that there exists at least one pair x, y such that at least two points z_1 and z_2 are needed to generate the convex $[p, x] \cap [p, y]$ (i.e. $[p, z_1, z_2] \subset [p, x] \cap [p, y]$ and there is no point w such that $[p, z_i] \subset [p, w] \subset [p, x] \cap [p, y]$, $i = 1, 2$). If there are several triples p, x, y, we choose that one for which the sum $d(x, p) + d(y, p)$ is the least. Let $a \neq y$ be a point of a $y - z_1 - p$ geodesic closest to y. By the choice of the pair x, y, z_2 is not on any $a - p$ geodesic, and thus $z_2 \notin [a, p]$. Because G is a prime convex intersection graph, there is a prime convex P containing $[a, p]$ but not z_2. If $y \in P$, then z_2 as a point of a $y-p$ geodesic belongs to P, which is absurd. Similarly we see that $x \notin P$. Accordingly, $x, y \in V \backslash P$, and because we can substitute z_2 by z_1, we see that no $x-y$ geodesic contains either z_1 or z_2.

Let k_i be a point on a $y - z_i - p$ geodesic adjacent to p, $i = 1, 2$. The relation $z_1 \notin [y, k_2]$ holds, because otherwise p can be substituted by k_2 and we obtain a triple k_2, x, y with $d(k_2, x) + d(k_2, y) < d(p, x) + d(p, y)$, which is a contradiction. The prime convex intersection property of G implies the existence of a prime convex P_2 containing $[y, k_2]$ and not z_1. If $p \in P_2$, then z_1 as apoint of a $y-p$ geodesic also belongs to P_2, which is absurd, and thus $p \in V \backslash P_2$. If $x \in V \backslash P_2$, then z_2 as a point of an $x-p$ geodsic belongs to $V \backslash P_2$, which is a contradiction. Hence $x \in P_2$. Accordingly, $[x, y, k_2] \subset P_2$ and $z_1, p \in V \backslash P_2$. Let a_i be a point of a $y - z_i - p$ geodesic adjacent to y and b_i a point of an $x - z_i - p$ geodesic adjacent to x, $i = 1, 2$. If $b_1 \in P_2$, then either $d(b_1, k_2) = d(b_1, p)$ and G contains an odd cycle, or a$b_1 - z_1 - p - k_2$ path is a $b_1 - k_2$ geodesic and $z_1 \in P_2$. Both alternatives lead to a contradiction, whence $b_1 \in V \backslash P_2$; similarly we see that $a_1 \in V \backslash P_2$.

By repeating the consideration above for $[y, k_1]$, we obtain a prime convex P_1 containing x, y and k_1, and its counterpart, the convex $V \backslash P_1$, contains the points a_2, b_2 and p. If $d(x, z_1) = d(b_2, z_2)$, then $d(x, z_2) \neq d(b_1, z_1)$ and by the isomorphism of (ii), the triple b_1, a_1, p does not satisfy (3) although $d(b_1, p) + d(a_1, p) < d(x, p) + d(y, p)$. A similar contradiction is obtained also in the case $d(x, z_1) \neq d(b_2, z_2)$, and thus the original assumption is false. Hence (3) holds for every triple $p, x, y \in V$.

Let p be an arbitrary point of G. As mentioned above, the conditions (1)-(3) imply that G can be translated into a meetsemilattice S_p with p as its least element and with the order relation: $a \leq b \iff [p, a] \subset [p, b]$. Now we must show that every meetsemilattice S_p is a lattice L_p.

(4) Let x and y be two maximal elements of S_p without an upper bound. We choose from all pairs x, y of S_p the pair, for which the sum $d(x \wedge y, x) + d(x \wedge y, y)$ is the greatest. There is a cycle containing y and p; if not, then the cutline of G divides G into two pieces the separation of which by means of prime convexes produces certainly two nonisomorphic subgraphs. Let Q be a minimal cycle containing y and p and R a minimal cycle containing x and p. Note that $Q \neq R$, because the element $x \wedge y$ exists in S_p. Assume that the number of points $n(Q)$

satisfies the relation: $n(Q) \leq n(R)$. Let an $x \wedge y - y$ geodesic be $x \wedge y = z_0, z_1, z_2, ..., z_m = y$. Because $n(Q) \leq n(R)$, we have $d(x \wedge y, y) \leq d(x \wedge y, x)$. Let P be a prime convex containing p, x and $x \wedge y$ but not z_1. Then, by the isomorphism of (ii), there is in $V \backslash P$ the image $u = \varphi(x)$, and clearly $x \wedge u = x \wedge y$, $d(x \wedge y, u) + d(x \wedge y, x) > d(x \wedge y, x) + d(x \wedge y, y)$ and there is no upper bound for x and u. This is a contradiction, and thus the property (4) holds for G.

Let q be a point of G, $k \neq q$ a point adjacent to q, and P a prime convex separating q and k ($q \in P$ and $k \notin P$). Then the isomorphism of (ii) guarantees the order isomorphism between the lattices L_q and L_k, and thus every two lattices L_a and L_b derived from G are order isomorphic, if a and b are adjacent. Let the least element of H_1 in (ii) be a, that of H_2 be b, and let $a = c(1), c(2), ..., c(m) = b$ be an $a - b$ geodesic in G. By the observation above, $L_{c(i)}$ is order isomorphic to $L_{c(i+1)}$ for $i = 1, ..., m-1$, and thus $H_1 = L_a$ is order isomorphic to $L_b = H_2$. This completes the proof.

Every Boolean graph is a highly symmetric graph as the conditions (A) and (B) show. An open problem is, whether the condition (C) is dependent of (A) and (B) (i.e. is every highly symmetric graph a Boolean graph)? We have not yet found any highly symmetric graph without the Boolean property (C).

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