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Abstract. A system of two partial quasilinear integrodifferential equations (hyperbolic and parabolic) is studied. The proofs of the existence, uniqueness of the weak solution and its continuous dependence on the right-hand side and on the initial functions are given.

Keywords: Rothe's method, evolution systems

Classification: 65M20, 58D25

1. Introduction.

This paper is motivated by the study of some problems from the theory of coupled dynamical linear thermoelasticity (see [1]–[4], [8], [10]–[11],[13]...). The whole article is written in terms of abstract Hilbert spaces H, Y where $H \cap Y$ is dense in H and Y. One system of two quasilinear evolution differential equations (hyperbolic and parabolic), the right-hand sides of which contain Volterra operators, is studied. The proof of the existence of a weak solution is done by Rothe's method (method of lines or discretization in time) using the technique developed in [6]–[7], [12].

The semidiscrete approximate solution is defined and the rate of convergence $0(\Delta t^{1/2})$ of Rothe's functions in the spaces $C(J,H)$ and $L_2(J,Y)$ is established.

2. Notations and preliminaries.

Let H, H_1, Y, Y_1 be real abstract Hilbert spaces with norms $|\ \ |, |\ \ |_1, \|\ \ \|, \|\ \ \|_1$, where $H \cap Y$ is dense in H and Y. Denote by S_t the interval $\langle -q, t\rangle$ for $t \in J = \langle 0, T\rangle$ where $T < \infty, q \in (0, \infty)$. In the following we work in the function spaces of the types $C(J,X), L_\infty(J,X), L_2(J,X), H^k(J,X)$ where X is a Banach space, the basic properties of which can be found in [9]. By $\to$ ($\rightharpoonup$) is denoted the strong (weak) convergence. Let $\langle z, w\rangle_H, \langle u, v\rangle_Y$ be the continuous pairings for $z \in H_1, w \in H, u \in Y_1, v \in Y$.

If X, Y are Banach spaces, $\alpha \in (0,1\rangle$ then:

- By $\mathrm{Lip}_\alpha(X,Y)$ is denoted the set of all functions $g : X \to Y$ satisfying

$$\|g(u) - g(v)\|_Y \le C\|u - v\|_X^\alpha \qquad \forall u, v \in X.$$

For $\alpha = 1$ the notation $\mathrm{Lip}(X,Y) \equiv \mathrm{Lip}_1(X,Y)$ is used.

- By $\mathrm{Lip}(J \times X, Y)$ is denoted the set of all functions $g : J \times X \to Y$ satisfying

$$\|g(t,u) - g(t',v)\|_Y \le C\left(|t - t'|\,[1 + \|u\|_X + \|v\|_X] + \|u - v\|_X\right)$$
$$\forall t, t' \in J; \forall u, v \in X.$$

Definition 2.1. (see [5]) The operator $E: L_\infty(J,X) \to L_\infty(J,X)$ (X is a Banach space) is said to be a Volterra operator in X iff

$$[u(s) = v(s) \text{ for a.e. } s \in S_t, t \in J] \Rightarrow$$
$$[E(u)(s) = E(v)(s) \text{ for a.e. } s \in \langle 0,t\rangle].$$

Let $E : \mathrm{Lip}(S_T, H) \to \mathrm{Lip}(S_T, H)$ resp. $F : \mathrm{Lip}(S_T, Y) \to \mathrm{Lip}(S_T, Y)$ be Volterra operator in H resp. Y and $G : L_\infty(J,Y) \to L_\infty(J,Y), I : L_\infty(J,H) \to L_\infty(J,H)$ be in the form

$$R(z)(t) = \int_0^t K(t,s)z(s)\,ds, \quad R = G, I; K \in L_\infty(J \times J). \tag{2.2}$$

Let us fix $f \in \mathrm{Lip}(J \times H^3 \times Y^2, Y_1), e \in \mathrm{Lip}(J \times Y \times H^3 \times Y^2, H_1), \mu : J \to Y_1$ and the continuous bilinear forms $p(t;z,w)$, $a_1(t;u,v)$, $a_2(t;u,v)$, $b(t;u,v)$, $d(t;z,v)$, $g(t;u,v)$, $\rho_1(t;u,w)$, $g_1(u,v)$ for $z,w \in H$ and $u,v \in Y (t \in J)$. The notation $r^{(k)}(t;x,y)$ is used for $\partial_t^k r(t;x,y)$.

We consider the following problem:

PC–1. To find u, v such that

(i) $u \in \mathrm{Lip}_{1/2}(J,Y) \cap \mathrm{Lip}(S_T,H), \partial_t u \in L_2(J,Y) \cap L_\infty(J,H)$

(ii) $v \in \mathrm{Lip}(S_T, Y \cap H), \partial_t v \in L_\infty(J, Y \cap H) \cap \mathrm{Lip}(S_T,H), \partial_t^2 v \in L_\infty(J,H)$

(iii) the following identity is satisfied:

$$\begin{gathered} p(t;\partial_t u(t),\varphi) + a_1(t;u(t),\varphi) = \rho_1(t;\partial_t v(t),\varphi) + g_1(\partial_t v(t),\varphi) + \\ +\langle f(t,E(u)(t),E(v)(t),E(\partial_t v)(t),F(v)(t),G(u)(t)),\varphi\rangle_Y + \\ +\langle e(t,u(t),E(u)(t),E(v)(t),E(\partial_t v)(t),F(v)(t),G(u)(t)),\varphi\rangle_H, \\ p(t;\partial_t^2 v(t),\phi) + b(t;\partial_t v(t),\phi) + a_2(t;v(t),\phi) = \\ = d(t;v(t),\phi) + d(t;I(u)(t),\phi) + d(t;I(v)(t),\phi) + \\ +g(t;G(v)(t),\phi) - \rho_1(t;\phi,u(t)) - g_1(\phi,u(t)) + \langle \mu(t),\phi\rangle_Y + \\ +\langle e(t,u(t),E(u)(t),E(v)(t),E(\partial_t v)(t),F(v)(t),G(u)(t)),\phi\rangle_H \\ \forall \varphi,\phi \in Y \cap H \text{ for a.e. } t \in J \end{gathered} \tag{2.3}$$

(iv) $u = \alpha, v = \beta, \partial_t v = \gamma$ in $S_0 = \langle -q, 0\rangle$ where $\alpha \in \mathrm{Lip}(S_0,H), \beta \in \mathrm{Lip}(S_0, Y \cap H)$ and $\gamma \in \mathrm{Lip}(S_0,H)$.

Remark 2.4. Integral kernels K in operators G, I may be different in different terms of (2.3) but the ones in bilinear forms d,g must fulfil $D^\omega K \in L_\infty(J \times J)$ (D is the total differential, $\omega \geq 2$ will be determined) and the ones in other terms must satisfy only $D^{\omega-1}K \in L_\infty(J \times J)$. For the simplicity we consider that (2.24) holds for all integral kernels K.

Remark 2.5. Bilinear forms p, d; functions e; operators E, F may be different at any two places of their occurrence in (2.3) but they must fulfil the relations (2.6)–(2.23). Without loss of generality we shall keep the notation from (2.3).

Let us consider the following conditions ($\forall t \in J; \omega, \lambda \geq 0$ will be determined; $\forall z, w \in H; \forall u, v \in Y; \forall y \in Y \cap H$):

(2.6) $$p(t; z, w) = p(t; w, z)$$

(2.7) $$p(t; z, z) \geq C_1 |z|^2$$

(2.8) $$|p^{(k)}(t; z, w)| \leq C|z||w| \qquad k = 0, \ldots, \lambda$$

(2.9) $$a_1(t; y, y) \geq C_1 \|y\|^2 - C|y|^2$$

(2.10) $$|a_1^{(k)}(t; u, v)| \leq C\|u\|\|v\| \qquad k = 0, \ldots, \omega$$

(2.11) $$a_2(t; u, v) = a_2(t; u, v)$$

(2.12) $$a_2(t; y, y) \geq C_1 \|y\|^2 - C|y|^2$$

(2.13) $$|a_2^{(k)}(t; u, v)| \leq C\|u\|\|v\| \qquad k = 0, \ldots, \omega$$

(2.14) $$b^{(1)}(t; u, v) = b^{(1)}(t; v, u)$$

(2.15) $$|b^{(k)}(t; u, v)| \leq C\|u\|\|v\| \qquad k = 0, \ldots, \omega$$

(2.16) $$b(t; y, y) \geq -C|y|^2$$

(2.17) $$\exists \alpha \in (0,1) : \alpha a_2(t; y, y + b^{(1)}(t; y, y) \geq -C|y|^2$$

(2.18) $$|g^{(k)}(t; u, v)| \leq C\|u\|\|v\| \qquad k = 0, \ldots, \omega$$

(2.19) $$|g_1(u, v)| \leq C\|u\|\|v\|$$

(2.20) $$|d^{(k)}(t; z, u)| \leq C|z|\|u\| \qquad k = 0, \ldots, \omega$$

(2.21) $$|\rho_1^{(k)}(t; , u, z)| \leq C\|u\||z| \qquad k = 0, \ldots, \omega$$

(2.22) $$|E(x)(t) - E(x)(t')| \leq |t - t'|\theta(\|x\|_{C(S_t, H)})(1 + \|\partial_t x\|_{L_\infty(S_t, H)})$$
$$\forall t, t' \in J, t' < t; \theta \in C(R_+, R_+); \forall x \in \mathrm{Lip}(S_T, H)$$

(2.23) $$|F(x)(t) - F(x)(t')| \leq |t - t'|\theta(\|x\|_{C(S_t, Y)})(1 + \|\partial_t x\|_{L_\infty(S_t, Y)})$$
$$\forall t, t' \in J; t' < t; \theta \in C(R_+, R_+); \forall x \in \mathrm{Lip}(S_T, Y)$$

(2.24) $$D^\omega K \in L_\infty(J \times J)$$

(2.25) $$\mu \in H^\omega(J, Y_1)$$

(compatibility condition)
for $U_0 = \alpha(0)$, $V_0 = \beta(0)$, $V_1 = \gamma(0) \in Y \cap H$ exist $U_1, V_2 \in H$ such that

$$p(0; U_1, \varphi) + a_1(0; U_0, \varphi) = \rho_1(0; V_1, \varphi) + g_1(V_1, \varphi) +$$
$$+ \langle f(0, E(\alpha)(0), E(\beta)(0), E(\gamma)(0), F(\beta)(0), 0), \varphi \rangle_Y +$$
$$+ \langle e(0, U_0, E(\alpha)(0), E(\beta)(0), E(\gamma)(0), F(\beta)(0), 0), \varphi \rangle_H,$$

(2.26) $$p(0; V_2, \phi) + b(0; V_1, \phi) + a_2(0; V_0, \phi) = d(0; V_0, \phi) +$$
$$+ \langle \mu(0), \phi \rangle_Y - \rho_1(0; \phi, U_0) - g_1(\phi, U_0) +$$
$$\langle e(0, U_0, E(\alpha)(0), E(\beta)(0), E(\gamma)(0), F(\beta)(0), 0), \varphi \rangle_H$$
$$\forall \varphi, \phi \in Y \cap H.$$

Remark 2.27. The function θ may be different in both inequalities (2.22) and (2.23). $C, \varepsilon, C_\varepsilon$ denote the generic positive constants which do not depend on n and which are not necessarily the same at any two places (ε is a small constant and $C_\varepsilon = C(\varepsilon^{-1})$).

For a given positive integer n the following notation is introduced ($i = 1, \dots n; \tau = T/n; t_i = i\tau$):

$$w_i = w(t_i), \qquad \delta w_i = (w_i - w_{i-1})/\tau$$

(where w is an arbitrary function),

(2.28)
$$u_n(t) = \begin{cases} \alpha(t) & t \in S_0 \\ u_{i-1} + (t - t_{i-1})\delta u_i & t_{i-1} < t \le t_i; \quad i = 1, \dots n \end{cases}$$

(2.29)
$$v_n(t) = \begin{cases} \beta(t) & t \in S_0 \\ v_{i-1} + (t - t_{i-1})\delta v_i & t_{i-1} < t \le t_i; \quad i = 1, \dots n \end{cases}$$

(2.30)
$$\overline{u}_n(t) = \begin{cases} \alpha(t) & t \in S_0 \\ u_i & t_{i-1} < t \le t_i; \quad i = 1, \dots n \end{cases}$$

(2.31)
$$\overline{v}_n(t) = \begin{cases} \beta(t) & t \in S_0 \\ v_i & t_{i-1} < t \le t_i; \quad i = 1, \dots n \end{cases}$$

(2.32)
$$V_n^{(1)}(t) = \begin{cases} \gamma(t) & t \in S_0 \\ \delta v_{i-1} + (t - t_{i-1})\delta^2 v_i & t_{i-1} < t \le t_i; \quad i = 1, \dots n \end{cases}$$

(2.33)
$$\overline{V}_n^{(1)}(t) = \begin{cases} \gamma(t) & t \in S_0 \\ \delta v_i & t_{i-1} < t \le t_i; \quad i = 1, \dots n \end{cases}$$

(2.34)
$$\widetilde{u}_{i-1} = \widetilde{u}_{i-1,n}(t) = \begin{cases} \alpha(t) & t \in S_0 \\ U_0 = \alpha(0) & t \in \langle 0, \tau \rangle \\ u_{j-1} + (t - t_j)\delta u_j & t \in \langle t_j, t_{j+1} \rangle; j = 1, \dots, i-1 \\ u_{i-1} & t \in \langle t_i, T \rangle \end{cases}$$

The functions $\widetilde{v}_{i-1}$ resp. $\widetilde{\delta v}_{i-1}$ are defined analogously as $\widetilde{u}_{i-1}$ but instead α will be β resp.γ.

(2.35) $$R_n(w)(t) = R(w)(t_i) \quad t \in (t_{i-1}, t_i\rangle; \quad R = E, F, G, I$$

(2.36) $$r_n(t, \xi) = r(t_i, \xi) \quad t \in (t_{i-1}, t_i\rangle; r = p, a_1, a_2, b, g, \rho_1, e, f, \mu.$$

(2.37) $$\widetilde{w}_n = \widetilde{w}_{n-i,n} \quad w = u, v, \delta v.$$

The key idea for solving PC–1 is to replace the t–derivative by a difference quotient of a given step τ. In this way we obtain an elliptic problem we are able to solve successively for $i = 1, \dots, n$. Using these solutions u_i, v_i we construct piecewise linear (Rothe's) function s $u_n(t)$, $v_n(t)$ (see (2.28), (2.29)) as an approximate semidiscrete solution of PC–1.

3. A priori estimates.

Let us consider this semidiscrete problem:

PD–1. To find $u_i, v_i \in Y \cap H (i = 1, \dots, n)$ such that

(i) $u_0 = U_0$, $v_0 = V_0$, $\delta v_0 = V_1$, $\quad U_0, V_0, V_1 \in Y \cap H$

(ii) the relation (3.1) is satisfied:

$$
\begin{gathered}
p(t_i; \delta(u_i, \varphi) + a_1(t_i; u_i, \varphi) = \langle f_i, \varphi \rangle_Y + \langle e_i, \varphi \rangle_H + \\
\rho_1(t_i; \delta v_i, \varphi) + g_1(\delta v_i, \varphi), \\
p(t_i; \delta^2 v_i, \phi) + b(t_i; \delta v_i, \phi) + a_2(t_i; v_i, \phi) = \\
= d(t_i; I_i u, \phi) + d(t_i; I_i v, \phi) + g(t_i; G_i v, \phi) + \langle \mu_i, \phi \rangle_Y + \\
+ d(t_i; v_{i-1}, \phi) + \langle e_i, \phi \rangle_H - \rho_1(T_i; \phi, u_i) - g_1(\phi, u_i) \\
\forall \varphi, \phi \in Y \cap H
\end{gathered}
\tag{3.1}
$$

where

$$
\begin{aligned}
f_i &= f(t_i, E_i u, E_i v, E_i \delta v, F_i v, G_i u) \\
e_i &= e(t_i, u_{i-1}, E_i u, E_i v, E_i \delta v, F_i v, G_i u) \\
R_i z &= R(\widetilde{z}_{i-1})(t_i) \quad \text{for } R = E, F, G, I.
\end{aligned}
$$

It is easy to see that PD–1 is solvable for every $i = 1, \dots, n$ under the conditions of Theorem 4.18 and applying Lax–Milgram lemma.

An important step in our approach is to obtain some suitable a priori estimates for u_n, v_n. Let us state the following very useful identities and inequalities the proofs of which are straightforward and so they are left to the reader.

$$
|E_i z|^2 \le C \left(1 + \max_{1 \le k \le i} |z_k|^2 \right) \tag{3.2}
$$

$$
|\delta E_i z|^2 \le C\theta^2 \left(\max_{1 \le k \le i} |z_k| \right) \left(1 + \max_{1 \le k \le i} |\delta z_k|^2 \right) \tag{3.3}
$$

$$
\|F_i z\|^2 \le C \left(1 + \max_{1 \le k \le i} \|z_k\|^2 \right) \tag{3.4.}
$$

$$
\|\delta F_i z\|^2 \le C\theta^2 \left(\max_{1 \le k \le i} \|z_k\| \right) \left(1 + \max_{1 \le k \le i} \|\delta z_k\|^2 \right) \tag{3.5}
$$

$$
\|G_i z\|^2 \le C \left(1 + \sum_{k=1}^{i} \|z_k\|^2 \tau \right) \tag{3.6}
$$

$$\|\delta G_i z\|^2 \le C\left(1+\sum_{k=1}^{i}\|z_k\|^2\tau+\|z_{i-1}\|^2+\|z_{i-2}\|^2\right)\le \le C\left(1+\sum_{k=1}^{i}(\|z_k\|^2+\|\delta z_k\|^2)\tau\right) \tag{3.7}$$

$$\|\delta^2 G_i z\|^2 \le C\left(1+\sum_{k=1}^{i}\|z_k\|^2\tau+\|z_{i-1}\|^2+\|\delta z_{i-2}\|^2+\|\delta z_{i-1}\|^2\right) \tag{3.8}$$

$$|I_i z|^2 \le C\left(1+\sum_{k=1}^{i}|z_k|^2\tau\right) \tag{3.9}$$

$$|\delta I_i z|^2 \le C\left(1+\sum_{k=1}^{i}|z_k|^2\tau+|z_{i-1}|^2+|z_{i-2}|^2\right)\le \le C\left(1+\sum_{k=1}^{i}(|z_k|^2+|\delta z_k|^2)\tau\right) \tag{3.10}$$

$$|\delta^2 I_i z|^2 \le C\left(1+\sum_{k=1}^{i}|z_k|^2\tau+|z_{i-1}|^2+|\delta z_{i-2}|^2+|\delta z_{i-1}|^2\right) \tag{3.11}$$

(Young's inequality)

$$|ab| \le \varepsilon a^2 + C_\varepsilon b^2 \quad \forall a,b\in R; \forall\varepsilon\in(0,1) \tag{3.12}$$

$$\left(\sum_{i=1}^{n} a_i\right)^2 \le n\sum_{i=1}^{n} a_i^2 \quad \forall a_i \in R \tag{3.13}$$

(per partes formula for pairing)

$$\sum_{k=1}^{m}\langle z_k, w_k - w_{k-1}\rangle = \langle z_m, w_m\rangle - \langle z_0, w_0\rangle - \sum_{k=1}^{m}\langle \delta z_k, w_{k-1}\rangle\tau \tag{3.14}$$

(per partes formula for bilinear form)

$$\sum_{k=1}^{m} r(t_k; z_k, w_k - w_{k-1}) = r(t_m; z_m, w_m) - r(0; z_0, w_0) - -\sum_{k=1}^{m}\left(r(t_{k-1};\delta z_k, w_{k-1}) + \delta r(t_k; z_k, w_{k-1})\right)\tau \tag{3.15}$$

(per partes formula for symmetric bilinear form)

$$\sum_{k=1}^{m} r(t_k; z_k - z_{k-1}, z_k) = \Big(r(t_m; z_m, z_m) - r(0; z_0, z_0) + +\sum_{k=1}^{m}\left(r(t_k;\delta z_k,\delta z_k)\tau^2 - \delta r(t_k; z_{k-1}, z_{k-1})\tau\right)\Big)/2. \tag{3.16}$$

The main tools by the proofs will be a suitable choice of a test function, Young's and Hölder's inequalities.

Lemma 3.17. *Let the assumptions of Theorem 4.18 be fulfilled. Then*

$$|u_j| + |\delta v_j| + \|v_j\| + \sum_{i=1}^{j} \|u_i\|^2\tau + |v_j| \le C$$

for $j = 1, \ldots, n$ *and* $\tau < \tau_0$.

PROOF: Putting $\varphi = u_i\tau, \phi = \delta v_i\tau$ in (3.1) and then summing up for $i = 1, \ldots, j$ we have

$$\sum_{i=1}^{j} (p(t_i; \delta u_i, u_i) + a_1(t_i; u_i, u_i))\,\tau = \sum_{i=1}^{j} (\langle f_i, u_i\rangle_Y +$$
$$+\langle e_i, u_i\rangle_H + \rho_1(t_i; \delta v_i, u_i) + g_1(\delta v_i, u_i))\,\tau,$$

(3.18)
$$\sum_{i=1}^{j} (p(t_i; \delta^2 v_i, \delta v_i) + b(t_i; \delta v_i, \delta v_i)+$$
$$a_2(t_i; v_i, \delta v_i))\,\tau = \sum_{i=1}^{j} (d(t_i; v_{i-1} + I_i u + I_i v, \delta v_i) + g(t_i; G_i v, \delta v_i)+$$
$$\langle \mu(t_i), \delta v_i\rangle_Y + \langle e_i, \delta v_i\rangle_H - \rho_1(t_i; \delta v_i, u_i) - g_1(\delta v_i, u_i))\,\tau.$$

Using (3.2)–(3.16), the following estimates can be obtained

$$C_1/2|u_j|^2 + (C_1 - \varepsilon)\sum_{i=1}^{j} \|u_i\|^2\tau \le C_\varepsilon\left(1 + \sum_{i=1}^{j} A_i\tau\right) +$$
$$+\sum_{i=1}^{j} (\rho_1(t_i; \delta v_i, u_i) + g_1(\delta v_i, u_i))\,\tau,$$

(3.19)
$$C_1/2|\delta v_j|^2 + (C_1/2 - \varepsilon)\|v_j\|^2 \le \varepsilon\sum_{i=1}^{j} \|u_i\|^2\tau + C_\varepsilon\left(1 + \sum_{i=1}^{j} A_i\tau\right) -$$
$$-\sum_{i=1}^{j} (\rho_1(t_i; \delta v_i, u_i) + g_1(\delta v_i, u_i))\,\tau,$$

where

$$A_i = \max_{1\le k\le i} |u_k|^2 + \max_{1\le k\le i} |\delta v_k|^2 + \max_{1\le k\le i} \|v_k\|^2 + \sum_{k=1}^{i} \|u_k\|^2\tau.$$

Summing up the both inequalities in (3.19) it yields (for sufficiently small ε)

$$|u_j|^2 + |\delta v_j|^2 + \|v_j\|^2 + \sum_{i=1}^{j} \|u_i\|^2\tau \le C\left(1 + \sum_{i=1}^{j} A_i\tau\right),$$

thus moreover

$$A_j \le C\left(1 + \sum_{i=1}^{j} A_i\tau\right).$$

The required result is obtained by applying Gronwall's lemma. ■

Lemma 3.20. *Let the assumptions of Theorem 4.18 be satisfied. Then*

$$|\delta u_j| + \|\delta v_j\| + \|u_j\| + \sum_{i=1}^{j} \|\delta u_i\|^2\tau + |\delta^2 v_j| \le C$$

for $j = 1,\dots,n$ *and* $\tau \le \tau_0$.

PROOF: After subtracting (3.1) from (3.1) for $i, i-1$ ($i \ge 2$, for $i = 1$ the compatibility condition is used), setting $\varphi = \delta u_i, \phi = \delta^2 v_i$ and summing up for $i = 1,\dots,j$ we get

$$\begin{aligned}
&\sum_{i=1}^{j} \big(p(t_i;\delta u_i,\delta u_i) - p(t_{i-1};\delta u_{i-1},\delta u_i) + a_1(t_i;u_i,\delta u_i) - \\
(3.21_1)\qquad & -a_1(t_{i-1};u_{i-1},\delta u_i)\big) = \sum_{i=1}^{j} \big(\langle f_i - f_{i-1},\delta u_i\rangle_Y + \langle e_i - e_{i-1},\delta u_i\rangle_H + \\
&+\rho_1(t_i;\delta v_i,\delta u_i) - \rho_1(t_{i-1};\delta v_{i-1},\delta u_i) + g_1(\delta v_i - \delta v_{i-1},\delta u_i)\big),
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{j} \big(p(t_i;\delta^2 v_i,\delta^2 v_i) - p(t_{i-1};\delta^2 v_{i-1},\delta^2 v_i) + b(t_i;\delta v_i,\delta^2 v_i) - \\
&-b(t_{i-1};\delta v_{i-1},\delta^2 v_i) + a_2(t_i;v_i,\delta^2 v_i) - \\
(3.21_2)\qquad &-a_2(t_{i-1};v_{i-1},\delta^2 v_i)\big) = \sum_{i=1}^{j} \big(g(t_i;G_i v,\delta^2 v_i) + \\
&+d(t_i;v_{i-1} + I_i u + I_i v,\delta^2 v_i) - d(t_{i-1};v_{i-2} + I_{i-1}u + I_{i-1}v,\delta^2 v_i) + \\
&+\left[\langle \delta\mu(t_i),\delta^2 v_i\rangle_Y + \langle \delta e_i,\delta^2 v_i\rangle_H - g_1(\delta^2 v_i,\delta u_i)\right]\tau - \\
&-g(t_{i-1};G_{i-1}v,\delta^2 v_i) - \rho_1(t_i;\delta^2 v_i,u_i) + \rho_1(t_{i-1};\delta^2 v_i,u_{i-1})\big).
\end{aligned}$$

By a standard argument (analogously as in [7, lemma 3.16]) using (3.2)–(3.16) the following can be obtained

$$\begin{aligned}
&C_1/2|\delta u_j|^2 + (C_1 - \varepsilon)\sum_{i=1}^{j}\|\delta u_i\|^2\tau \le C_\varepsilon\left(1 + \sum_{i=1}^{j} B_i\tau\right) + \\
&\sum_{i=1}^{j}\big(\rho_1(t_i;\delta^2 v_i,\delta u_i) + g_1(\delta^2 v_i,\delta u_i)\big)\tau, \\
&\big(C_1|\delta^2 v_j|^2 + a_2(t_j;\delta v_j,\delta v_j) + \delta b(t_j;\delta v_j,\delta v_j)\big)/2 + \\
&\sum_{i=1}^{j}\big(a_2(t_i;\delta^2 v_i,\delta^2 v_i) + \delta b(t_i;\delta^2 v_i,\delta^2 v_i)\big)\tau^2 \le \varepsilon\|\delta v_j\|^2 + \\
&+\varepsilon\sum_{i=1}^{j}\|\delta u_i\|^2\tau + C_\varepsilon\left(1 + \sum_{i=1}^{j} b_i\tau\right) -
\end{aligned}$$

$$- \sum_{i=1}^{j} \left(\rho_1(t_i;\delta^2 v_i, \delta u_i) + g_1(\delta^2 v_i, \delta u_i)\right) \tau,$$

where

$$B_i = \max_{1\le k\le i} |\delta u_k|^2 + \max_{1\le k\le i} |\delta^2 v_k|^2 + \max_{1\le k\le i} \|\delta v_k\|^2 + \sum_{k=1}^{i} \|\delta u_k\|^2 \tau.$$

Summing up the both above inequalities, applying (2.17) (for sufficiently small ε) it yields

$$B_j \le C\left(1 + \sum_{i=1}^{j} B_i \tau\right).$$

Hence the application of Gronwall's lemma is sufficient to complete the proof. ■

4. Existence and continuous dependence.

In this section the compactness of u_n, v_n in some function spaces is proved. This fact is exploited by the proof of existence. Uniqueness of solution is a consequence of continuous dependence on the right–hand side and on the initial functions.

The a priori estimates from Lemmas 3.17, 3.20 can be rewritten in this form:

$$|\partial_t V_n^{(1)}(t)| \le C \text{ for a.e. } t \in J; \quad \|\overline{V}_n^{(1)}(t)\|_{Y\cap H} \le C \quad \forall t \in J \tag{4.1}$$

$$\|v_n(t)\|_{Y\cap H} + \|\overline{v}_n(t)\|_{Y\cap H} \le C \quad \forall t \in J \tag{4.2}$$

$$|V_n^{(1)}(t) - V_n^{(1)}(t')| \le C|t-t'| \quad \forall t,t' \in J \tag{4.3}$$

$$|V_n^{(1)}(t) - \overline{V}_n^{(1)}(t)| \le C/n \quad \forall t \in J \tag{4.4}$$

$$\|v_n(t) - v_n(t')\| \le C|t-t'| \quad \forall t,t' \in J \tag{4.5}$$

$$\|v_n(t) - \overline{v}_n(t)\| \le C/n \quad \forall t \in J \tag{4.6}$$

$$\|v_n - \widetilde{v}_n\|_{C(S_T,Y)\cap C(S_T,H)} \le C/n \tag{4.7}$$

$$\|\widetilde{\delta v}_n - V_n^{(1)}\|_{C(S_T,H)} \le C/n \tag{4.8}$$

$$|\partial_t u_n(t)| + \|\partial_t u_n\|_{L_2(J,Y)} \le C \quad \text{for a.e. } t \in J \tag{4.9}$$

$$\|u_n(t)\|_{Y\cap H} + \|\overline{u}_n(t)\|_{Y\cap H} \le C \quad \forall t \in J \tag{4.10}$$

$$|u_n(t) - u_n(t')| \le C|t-t'| \quad \forall t,t' \in J \tag{4.11}$$

$$\|u_n(t) - u_n(t')\| \le C|t-t'|^{1/2} \quad \forall t,t' \in J \tag{4.12}$$

$$|u_n(t) - \overline{u}_n(t)| + |\widetilde{u}_n(t) - \overline{u}_n(t)| \le C/n \quad \forall t \in J \tag{4.13}$$

$$\|u_n - \overline{u}_n\|_{L_2(J,Y)} + \|\widetilde{u} - \overline{u}_n\|_{L_2(J,Y)} \le C/n \tag{4.14}$$

The identity (3.1) can be written in this way (the variable t is omitted):

$$p_n(t;\partial_t u_n, \varphi) + a_{1,n}(t;\overline{u}_n, \varphi) = \langle f_n, \varphi\rangle_Y + \langle e_n, \varphi\rangle_H + \rho_{1,n}(t;\overline{V}_n^{(1)}, \varphi) + g_1(\overline{V}_n^{(1)}, \varphi), \tag{4.15}$$

$$p_n(t;\partial_t V_n^{(1)},\phi)+b_n(t;\overline{V}_n^{(1)},\phi)+a_{2,n}(t;\overline{v}_n,\phi)=$$
$$=d_n(t;\widetilde{v}_n(t-\tau)+I_n(\widetilde{u}_n)+I_n(\widetilde{v}_n),\phi)+g_n(t;G_n(\widetilde{v}_n),\phi)+$$
$$+\langle\mu_n,\phi\rangle_Y++\langle e_n,\phi\rangle_H-\rho_{1,n}(t;\phi,\overline{u}_n)-g_1(\phi,\overline{u}_n)$$
$$\forall\varphi,\phi\in Y\cap H;\ \text{for a.e. } t\in J,$$

where

$$f_n=f_n(t,E_n(\widetilde{u}_n)(t),E_n(\widetilde{v}_n)(t),E_n(\widetilde{\delta u_n})(t),F_n(\widetilde{v}_n)(t),G_n(\widetilde{u}_n)(t))$$

and e_n is defined analogously.

Lemma 4.16. *Let the conditions of the Theorem 4.18 be satisfied. Then*

(i) $\exists v\in \mathrm{Lip}(S_T,Y\cap H)$ *such that*

$$\partial_t v\in L_\infty(J,Y)\cap C(J,H),\partial_t^2 v\in L_\infty(J,H)$$
$$\|v_n-v\|^2_{C(J,Y)}+\|V_n^{(1)}-\partial_t v\|^2_{C(J,H)}\le C/n,$$

(ii) $\exists u\in \mathrm{Lip}_{1/2}(J,Y)\cap \mathrm{Lip}(S_T,H)$ *such that*

$$\partial_t u\in L_2(J,Y)\cap L_\infty(J,H)$$
$$\|u_n-u\|^2_{C(S_T,H)}+\|u_n-u\|^2_{L_2(J,Y)}\le C/n$$
$$u_n\to u \text{ in } C(J,Y) \text{ and } \partial_t u_n\rightharpoonup\partial_t u \text{ in } L_2(J,Y\cap H)$$

(u_n,v_n denote the subsequences of u_n,v_n).

PROOF: Let us subtract (4.15) from (4.15) for $n=r,n=s$ where $\varphi=\overline{u}_r-\overline{u}_s,\phi=\overline{V}_r^{(1)}-\overline{V}_s^{(1)}$. Using (4.1)–(4.14) we estimate

$$1/2\partial_t p(t;u_r-u_s,u_r-u_s)+a_1(t;\overline{u}_r-\overline{u}_s,\overline{u}_r-\overline{u}_s)\le\varepsilon\|\widetilde{u}_r-\widetilde{u}_s\|^2+$$
$$C_\varepsilon(r^{-1}+s^{-1}+A_{rs}(t))+g_1(\overline{V}_r^{(1)}-\overline{V}_s^{(1)},\overline{u}_r-\overline{u}_s)+$$
$$+\rho_1(t;\overline{V}_r^{(1)}-\overline{V}_s^{(1)},\overline{u}_r-\overline{u}_s),$$

$$1/2\left(\partial_t p(t;V_r^{(1)}-V_s^{(1)},V_r^{(1)}-V_s^{(1)})+\partial_t a_2(t;v_r-v_s,v_r-v_s)\right)\le\tag{4.17}$$
$$\le\varepsilon\|\widetilde{u}_r-\widetilde{u}_s\|^2+C_\varepsilon(r^{-1}+s^{-1}+A_{rs}(t))+$$
$$+\partial_t d(t;v_r-v_s,v_r-v_s)+\partial_t d(t;I(u_r-u_s),v_r-v_s)+$$
$$+\partial_t d(t;I(v_r-v_s),v_r-v_s)+\partial_t g(t;G(v_r-v_s),v_r-v_s)-$$
$$-\rho_1(t;\overline{V}_r^{(1)}-\overline{V}_s^{(1)},\overline{u}_r-\overline{u}_s)-g_1(\overline{V}_s^{(1)}-\overline{V}_s^{(1)},\overline{u}_r-\overline{u}_s),$$

where

$$A_{rs}(t)=\max_{\langle 0,t\rangle}|u_r-u_s|^2+\max_{\langle 0,t\rangle}|V_r^{(1)}-V_s^{(1)}|^2+\max_{\langle 0,t\rangle}\|v_r-v_s\|^2+$$
$$+\int_0^t\|\overline{u}_r-\overline{u}_s\|^2d\xi.$$

Summing up the both inequalities in (4.17) and integrating it over $(0,t)$ the following can be obtained (for sufficiently small ε)

$$A_{rs}(t) \le C\left(r^{-1} + s^{-1} + \int_0^t A_{rs}(\xi)\,d\xi\right).$$

Hence Gronwall's lemma gives us

$$\|u_r - u_s\|_{C(S_T,H)} + \|u_r - u_s\|_{L_2(J,Y)} \to 0$$
$$\|v_r - v_s\|_{C(J,Y)} + \|V_r^{(1)} - V_s^{(1)}\|_{C(J,H)} \to 0$$

for $r, s \to \infty$.

The rest of the proof is a consequence of the a priori estimates (4.1)-(4.14). ∎

Theorem 4.18. *Suppose* $f \in \operatorname{Lip}(J \times H^3 \times Y^2, Y_1)e \in \operatorname{Lip}(J \times Y \times H^3 \times Y^2, H_1)$, $E \in \operatorname{Lip}(C(S_T,H), C(J,H))$, $F \in \operatorname{Lip}(C(S_T,Y), C(J,Y))$. *Moreover* $\alpha(0), \beta(0), \gamma(0) \in Y \cap H$ *and (2.2); (2.6)-(2.10) for* $\omega = \lambda = 1$*; (2.11)-(2.26) for* $\omega = 2$ *are satisfied. Then there exists a solution of PC-1.*

PROOF: The idea of the proof is the following. Integrating (4.15) over $(0,t)$ for $t \in J$, using lemma 4.16 and (4.1)-(4.14), taking the limit as $n \to \infty$ it is easy to see that the pair u, v from lemma 4.16 is the solution of PC-1. We demonstrate it only for one member of (4.15)

For a continuous bilinear form p it holds

$$|p_n(t; z, w) - p(t; z, w)| \le C|z||w|.$$

Now, using $\partial_t u_n \to \partial_t u$ in $L_2(J,H)$, by virtue of the fact that $\int_0^t p(s; z, w)\,ds$ is the continuous bilinear form in $L_2(J,H) \times L_2(J,H)$, we obtain

$$\int_0^t p_n(s; \partial_s u_n(s), \varphi)\,ds \to \int_0^t p(s; \partial_s u(s), \varphi)\,ds. \quad ∎$$

Let us consider PC-1 but instead of $E(F, G, I, K, e, f, d, g, \mu, p, a_1, a_2, b, \rho_1, g_1, \alpha, \beta, \gamma)$ is the same symbol with bar. This problem is denoted by $\overline{\text{PC-1}}$. Let the conditions of Theorem 4.18 be satisfied for PC-1, $\overline{\text{PC-1}}$ and there exist $q \in C(J, R_+), q_0 \in R_+$ such that $(\forall t \in J; \forall z, w \in H; \forall x, y \in Y)$:

(4.19) $$|p(t; z, w) - \overline{p}(t; z, w)| \le q(t)\,(1 + |z||w|)$$

(4.20) $$|r(t; x, y) - \overline{r}(t; x, y)| \le q(t)\,(1 + \|x\|\|y\|)$$

for $r = a_1, a_2, b, g, g^{(1)}$

(4.21) $$|r(t; z, x) - \overline{r}(t; z, x)| \le q(t)\,(1 + |z|\|x\|)$$

for $r = d, d^{(1)}$

(4.22) $\|e(t,\xi) - \overline{e}(t,\xi)\|_{H_1} \le q(t)\,(1 + \|\xi\|_X)$

$\forall \xi \in X = Y \times H^3 \times Y^2$

(4.23) $\|f(t,\xi) - \overline{f}(t,\xi)\|_{Y_1} \le q(t)\,(1 + \|\xi\|_X)$

$\forall \xi \in X = H^3 \times Y^2$

(4.24) $\|\mu^{(\omega)}(t) - \overline{\mu}^{(\omega)}(t)\|_{Y_1} \le q(t) \quad \omega = 0,1$

(4.25) $|E(z)(t) - \overline{E}(z)(t)| \le q(t)\left(1 + \max_{\langle 0,t\rangle} |z|\right)$

(4.26) $|F(x)(t) - \overline{F}(x)(t)| \le q(t)\left(1 + \max_{\langle 0,t\rangle} \|x\|\right)$

(4.27) $\max_{s\in\langle 0,t\rangle} |K(t,s) - \overline{K}(t,s)| \le q(t)$

(4.28) $\max_{s\in\langle 0,t\rangle} |\partial_t(K(t,s) - \overline{K}(t,s))| \le q(t)$

(for integral kernels in $(2.3)_2$

(4.29) $|\rho_1(t;x,z) - \overline{\rho}_1(t;x,z)| \le q(t)\,(1 + \|x\|\|z\|)$

(4.30) $|g_1(x,y) - \overline{g}_1(x,y)| \le q_0\,(1 + \|x\|\|y\|)\,.$

Remark 4.31. The function q may be different in every estimate (4.19)–(4.29).

Theorem 4.32. *Let u, v resp. $\overline{u}, \overline{v}$ be the solution of PC–1 resp. $\overline{\text{PC–1}}$ for $\alpha(0) = U_0, \beta(0) = V_0, \gamma(0) = V_1$ resp. $\overline{\alpha}(0) = \overline{U}_0, \overline{\beta}(0) = \overline{V}_0, \overline{\gamma}(0) = \overline{V}_1$. Moreover the relations (4.19)–(4.30) are satisfied. Then*

$$\max_{\langle 0,t\rangle} |u - \overline{u}|^2 + \max_{\langle 0,t\rangle} \|v - \overline{v}\|^2 + \max_{\langle 0,t\rangle} |\partial_t(v - \overline{v})|^2 + \int_0^t \|u - \overline{u}\|^2\,ds \le$$
$$Ce^{Ct}\left[|U_0 - \overline{U}_0|^2 + \|V_0 - \overline{V}_0\|^2 + |V_0 - \overline{V}_0|^2 + |V_1 - \overline{V}_1|^2 + \int_0^t \left(q_0 + q(s) + q^2(s)\right)\,ds\right].$$

PROOF: The continuous dependence result can be obtained by a standard way and so we only sketch the key idea.

Let us subtract (2.3) from the analogous identity in $\overline{\text{PC–1}}$ for $\varphi = u - \overline{u}$ and $\phi = \partial_t(v - \overline{v})$. Integrating it over $(0,t)$; using the a priori estimates for both solutions u, v and $\overline{u}, \overline{v}$; applying Gronwall's lemma we conclude the proof. ∎

Consequence 4.33. The solution of PC–1 is unique.

PROOF: Trivial. ∎

The compatibility condition (2.26) plays an important part in our proofs. There arise some interesting questions: "Is (2.26) too restrictive or not?", "How many initial functions exist for which (2.26) is satisfied?". The following lemma answers us the questions

Let us consider this situation: $Y \hookrightarrow H$ is dense ($\hookrightarrow$ denotes the continuous imbedding), $Y_1 = Y^*$, $H_1 = H^* = H$ (X^* being the dual space to X) and the duality between $\varphi \in Y$ and $f_n \in Y^*$ coincides with the scalar product $(.,.)$ in H in this sense:

$$(h,\varphi) = \langle f_h, \varphi\rangle_Y \quad \forall \varphi \in Y.$$

Let us fix the continuous bilinear form $\rho(t; z, w)$ for $z \in Y, w \in H$ and $\mu \in H(J, Y^*), \nu \in H(J, H^*)$. We shall deal with (4.34) instead of (2.3) in PC–1.

$$\begin{aligned} p(t;\partial_t u,\varphi) + a_1(t;u,\varphi) &= g(t;G(u),\varphi) + g(t;G(v),\varphi) + \\ &+\langle \mu,\varphi\rangle_Y + (\nu,\varphi), \\ p(t;\partial_t^2 v,\phi) + b(t;\partial_t v,\phi) + a_2(t;v,\phi) &= d(t;I(u),\phi) + \\ g(t;G(v),\phi) + \rho(t;G(u),\phi) &+ \langle \mu,\phi\rangle_Y + (\nu,\phi) \\ &\forall \varphi,\phi \in Y. \end{aligned} \tag{4.34}$$

Let us denote the sets:

$$R = \{u_z \in Y : a_1(0;u_z,\varphi) = \langle \mu(0),\varphi\rangle_Y + (\nu(0),\varphi) - p(0;z,\varphi), \forall \varphi, z \in Y\}.$$
$$S = \{v_{z,w} \in Y : a_2(0;v_{z,w},\phi) = \langle \mu(0),\phi\rangle_Y + (\nu(0),\phi) - p(0;z,\phi) - b(0;w,\phi), \forall \phi, z, w \in Y\}.$$

Lemma 4.35. *Let (2.2), (2.6)-(2.8), (2.10)-(2.11), (2.13), (2.15), (2.18), (2.20), (4.36)-(4.38) for $\omega = \lambda = 0$ be fulfilled. Moreover let $\mu \in H(J, Y^*), \vartheta \in H(J, H^*)$.*

$$|\rho(t;z,w)| \le C\|z\||w| \quad \forall z \in Y, \forall w \in H \tag{4.36}$$
$$a_1(0;z,w) = a_1(0;w,z) \quad \forall z, w \in Y \tag{4.37}$$
$$a_i(0;z,z) \ge C_1\|z\|^2 \; i = 1,2 \quad \forall z \in Y \tag{4.38}$$

Then the sets R, S are dense in Y.

PROOF: We demonstrate the assertion only for the set R (for S it is analogous).
Let us fix $z \in Y$. Then for

$$\pi_z(\varphi) = \langle \mu(0),\varphi\rangle_Y + (\nu(0),\varphi) - p(0;z,\varphi), \quad \varphi \in Y$$

there exists a uniquely determined $u_z \in Y$ such that

$$a_1(0;u_z,\varphi) = \pi_z(\varphi).$$

Let $y, \varphi \in Y$.
For $y \in Y \quad \exists f_y \in Y^* : a_1(0;y,\varphi) = \langle f_y,\varphi\rangle_Y$.
For $f_y \in Y^* \quad \exists g_y \in Y^* : \langle f_y,\varphi\rangle_Y = \langle \mu(0),\varphi\rangle_Y + (\nu(0),\varphi) - \langle g_y,\varphi\rangle_Y$.
For $g_y \in Y^* \quad \exists g \in Y^* \exists h \in H : \|g_y - g\|_* \le \varepsilon, \langle g,\varphi\rangle_Y = (h,\varphi)$ ($\|\cdot\|_*$ being a norm

in Y^*).
For $h \in H$ $\exists \widetilde{h} \in H : (h, \varphi) = p(0; \widetilde{h}, \varphi)$.
For $\widetilde{h} \in H$ $\exists r \in Y : |\widetilde{h} - r| \le \varepsilon$.
For $r \in Y$ $\exists w_r \in R : a_1(0; w_r, \varphi) = \pi_r(\varphi)$.
Thus we have

$$a_1(0; y - w_r, \varphi) = p(0; r, \varphi) - \langle g_y, \varphi\rangle_Y = p(0; r - \widetilde{h}, \varphi) + \langle g - g_Y, \varphi\rangle_Y,$$

from which for $\varphi = y - w_r$ we conclude

$$\|y - w_r\| \le \varepsilon$$

i.e. the set R is dense in Y. ■

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