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Some remarks on preservation of topological products

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Abstract. We study the product preservation property by means of a topological epireflector $\mathbf{r} : \mathbf{TOP} \to \mathbf{R}$. Results are obtained with respect to the (bi,initial) factorization of $\mathbf{r}$ and the category $\mathbf{r}$–**COMP** defined by $\mathbf{r}$. We generalize a result of Ishii [3], concerning the Tychonoff reflector.

Keywords: epireflector, preservation of products, r–compact spaces

Classification: 54B10, 54B30, 18A40, 18B30, 54D30

In this note we deal with the property of preservation of topological products by means of an epireflector $\mathbf{r} : \mathbf{TOP} \to \mathbf{R}$. As a first result we show that $\mathbf{r}$ always preserves finite products and arbitrary Hausdorff products of spaces in the bireflective hull $\mathbf{B}(\mathbf{R})$ of $\mathbf{R}$ in **TOP**. Hence, the product preservation for $\mathbf{r}$ depends essentially on the first factor of the usual (bi,initial) factorization of $\mathbf{r}$. This generalizes and motivates the situation one has, e.g., with the T_1–modification of topologies, which preserves finite products of symmetric spaces, and other similar reflectors.

It is known ([2]; Appl.3) that, in case $\mathbf{R}$ is a category of compact Hausdorff spaces, then $\mathbf{r}$ preserves all products. Then, it does make sense to consider the largest subcategory of **TOP** whose objects are all spaces X having compact reflection $\mathbf{r}(X)$. This is the category $\mathbf{r}$–**COMP**, introduced and studied in [6], where its objects are characterized by means of special filters and covers.
$\mathbf{r}$–**COMP** is closed under finite products but, in general, it is not closed hereditary. Moreover, it turns out that $\mathbf{r}$–**COMP** is closed under products exactly when $\mathbf{r}$ preserves them.

We refer to [1] for all undefined concepts and unproved facts.
Let **TOP** (resp. $\mathbf{TOP_2}$) denote the category of all topological spaces (resp. Hausdorff spaces) and continuous maps. $\mathbf{R}$ will be an epireflective subcategory of **TOP** with reflector

$$\mathbf{r} : \mathbf{TOP} \to \mathbf{R}.$$

Then, for every topological space X there is an onto reflection map

$$\mathbf{r}_X : X \to \mathbf{r}(X)$$

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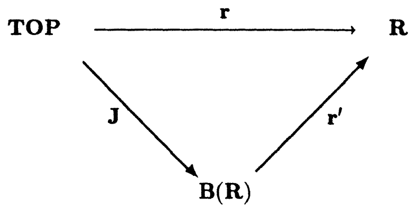
which is initial with respect to every other map $f : X \to R, R \in \mathbf{R}$.
The functor r admits a canonical decomposition

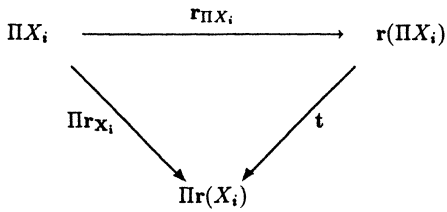


where:

(i) $\mathbf{B}(\mathbf{R})$ is the bireflective hull of $\mathbf{R}$ in **TOP**, with reflector J and reflection map $j_X : X \to X'$, being X' the space having the same underlying set as X and initial topology with respect to $\mathbf{r}_X$. j_X is underlined by the identify function. $\mathbf{B}(\mathbf{R}) = \{X' : X \in \mathbf{TOP}\}$.

(ii) $\mathbf{r}'$ is the restriction of $\mathbf{r}$ to $\mathbf{B}(\mathbf{R})$. Let us note that $\mathbf{r}(X) = \mathbf{r}(X') = \mathbf{r}'(X')$ holds, for every space X, hence we denote by $\mathbf{r}_{X'} : X' \to \mathbf{r}(X)$ the reflection map. Then, $\mathbf{r}_X = \mathbf{r}'_X \cdot j_X$.

Let $\{X_i : i \in I\}$ be any family of topological spaces and let ΠX_i be the product. By the universal property of the reflection there is a unique map t which makes the following diagram commutative:



One says that the reflector $\mathbf{r}$ preserves (finite) products whenever the map t is a homeomorphism, for every given (finite) family of topological spaces. In such a case we write $\mathbf{r}(\Pi X_i) = \Pi \mathbf{r}(X_i)$.

Proposition 1. $\mathbf{r}'$ *always preserves finite products.* $\mathbf{r}'$ *preserves all products whenever* $\mathbf{R} \subset \mathbf{TOP}_2$.

PROOF: Let $\{X'_i : i \in I\}$ be any family of spaces in $\mathbf{B}(\mathbf{R})$ and let $t' : \mathbf{r}(\Pi X'_i) \to \Pi \mathbf{r}(X'_i)$ be the unique map such that $t' \cdot \mathbf{r}_{\Pi X'_i} = \Pi \mathbf{r}_{X'_i}$. Since each $\mathbf{r}_{X'_i}$ is open and onto, and since $\mathbf{r}_{\Pi X'_i}$ is also onto, it follows that t' is open and onto. It is known ([**2**], 2.2 and 3.3) that t' must be also injective in case I is finite or $\mathbf{r}(\Pi X'_i)$ is a Hausdorff space. This completes the proof. ∎

Let us recall that the topology on X' is given by those subsets $A \subset X$ such that $A = \mathbf{r}_X^{-1}(B)$, for B open in $r(X)$. We call such sets $\mathbf{r}$–*open* sets of X. A is $\mathbf{r}$–open in X iff $\mathbf{r}_X(A)$ is open in $\mathbf{r}(X)$ and $A = \mathbf{r}_X^{-1}\mathbf{r}_X(A)$.

Let now X and Y be two arbitrary topological spaces and let $A \subset X, B \subset Y$ be **r**–open subsets. Then $A \times B$ is **r**–open in $X \times Y$; in other words, the identity function

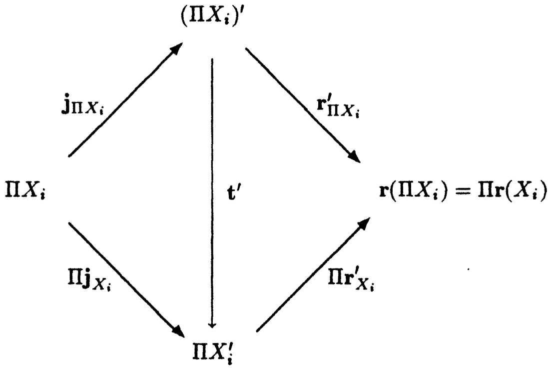
$$t' : (X \times Y)' \to X' \times Y'$$

is always continuous. However it is not a homeomorphism in general, that is an **r**–open subset of $X \times Y$ is not always expressible as union of "rectangular" **r**–open sets. In fact, it is easily seen that t' is a homeomrphism iff J preserves the product of X' and Y'.

Proposition 2. **r** *preserves finite products iff J does. The analogous assertion holds for arbitrary products, whenever* $\mathbf{R} \subset \mathbf{TOP}_2$

PROOF: If J preserves finite products, then, by the proposition above and the fact that $\mathbf{r} = \mathbf{r}' \cdot J$, the assertion follows. The same in case of arbitrary products and $\mathbf{R} \subset \mathbf{TOP}_2$.

Conversely, let $\{X_i : i \in I\}$ be a given family of spaces and consider the following commutative diagram



We know that t' is the identity on the underlying sets, hence it is sufficient to show that t' is also an open map. Let U be a basic open set in $(\Pi X_i)'$, that is U is an **r**–open set in ΠX_i; then there is an open set $V \subset \mathbf{r}(\Pi X_i) = \Pi\mathbf{r}(X_i)$ such that $U = \mathbf{r}^{-1}_{\Pi X_i}(V) = (t')^{-1}(\Pi\mathbf{r}'_{X_i})^{-1}(V)$. It follows that $t'(U) = (\Pi\mathbf{r}'_{X_i})^{-1}(V)$ is open in $\Pi X'_i$. ■

Remarks.

It is well known that the T_1–modification of topological spaces preserves the finite products of symmetric spaces (i.e. spaces in which $x \in cl\{y\}$ implies $y \in cl\{x\}$). Such spaces form the bireflective hull of the category $\mathbf{TOP}_1$ of T_1 spaces in **TOP**. Similarly, let **R** be one of the following categories: $\mathbf{TOP}_2$ (= Hausdorff spaces), $\mathbf{TOP}_3$ (=regular Hausdorff spaces), **TYCH** (= completely regular Hausdorff spaces), 0–**dim** (= zero–dimensional Hausdorff spaces), **FHAUS** (=functionally Hausdorff spaces), **URY** (= Urysohn spaces). Then, the corresponding modification functor $\mathbf{r}$ preserves, respectively, all products of spaces in $\mathbf{B}(\mathbf{TOP}_2)$ = (HO–spaces of K. Csaszar), $\mathbf{B}(\mathbf{TOP}_3)$ = (regular spaces), $\mathbf{B}(\mathbf{TYCH})$ = (completely regular spaces), $\mathbf{B}(\mathbf{0\text{-}dim})$ = (zero–dimensional spaces), $\mathbf{B}(\mathbf{FHAUS})$ = (spaces X such that, for $x, y \in X, x \in cl\{y\}$ iff no continuous map $f : X \to \mathbf{R}$ separates them), $\mathbf{B}(\mathbf{URY})$= (spaces X such that given $x, y \in X, x \in cl\{y\}$ iff there exist no disjoint closed neighborhoods of them).

Let $\tau : \mathbf{TOP} \to \mathbf{TYCH}$ be the Tychonoff modification functor. The τ–open sets of a space X are those subsets that are union of cozero–sets of X [**3**]. Morita [**5**] has shown that, for spaces X and Y, the equality $\mathbf{r}(X \times Y) = \mathbf{r}(X) \times \mathbf{r}(Y)$ holds iff every cozero–set of $X \times Y$ is a union of rectangular cozero–sets.

In [**6**] the category **r–COMP** of **r**– compact spaces was introduced; its objects are exactly those topological spaces X having compact reflection $\mathbf{r}(X)$ (no separation axiom is assumed).
r–COMP is closed under finite products ([**6**]; Th.3.6) and has the property that it is closed under arbitrary products exactly when **r** preserves them, as shown by the following

Theorem 1. *Let $\{X_i : i \in I\}$ be a family of* **r**–*compact spaces and assume that* $\mathbf{R} \subset \mathbf{TOP}_2$. *The following statements are equivalent:*

(i) $\mathbf{r}(\Pi X_i) = \Pi \mathbf{r}(X_i)$.
(ii) ΠX_i *is* **r**–*compact.*

PROOF: (i)→(ii) is immediate. As for (ii)→(i) it suffices to note that the map $t : \mathbf{r}(\Pi X_i) \to \Pi \mathbf{r}(X_i)$ is a bijective map (by Th.1.1) whose domain is compact and whose range is Hausdorff. ∎

r–compact spaces can be characterized by means of special filters and covers; we recall some facts from [**6**], for sake of completeness.

Given a topological space X and a subset $A \subset X$, the **r**–*interior* of A is defined to be the set $\mathrm{int}_{\mathbf{r}}(A) = X - \mathbf{r}_X^{-1}\,\mathrm{cl}(\mathbf{r}_X(A))$, where cl denotes ordinary closure in $\mathbf{r}(X)$.

An **r**–*cover* of X is an open cover $\{U_i : i \in I\}$ such that $X = \cup_{i \in I} \mathrm{int}_{\mathbf{r}} U_i$. A filter F in X is an **r**–*filter* whenever $\cup_{F \in \mathcal{F}} \mathrm{cl}(F) = \cup_{F \in \mathcal{F}} \mathrm{cl}_{\mathbf{r}}(F)$. Where $\mathrm{cl}_{\mathbf{r}}(F) = \mathbf{r}_X^{-1}\,\mathrm{cl}(\mathbf{r}_X(F))$.

Theorem 2. *[6]*
The following statements are equivalent for a space X:

(i) *X is* **r**–*compact.*
(ii) *Every* **r**–*cover of X admits a finite subcover.*
(iii) *Every closed* **r**–*filter in X has adherent points.*

Although **r–COMP**is obviously **r**–closed hereditary, it is in general neither hereditary nor closed ereditary.

Let $\tau : \mathbf{TOP} \to \mathbf{TYCH}$ be the Tychonoff reflector; in such a case τ–**COMP** contains properly the class of *w–compact* spaces defined in [**3**] and, hence, that of compact spaces. A topological space X is said to be w–compact whenever, given any family of closed subsets of X, each containing a non empty cozero set, and having the finite intersection property, it has empty intersection.

In ([**3**]; Prop.2.3) a τ–compact, not w–compact space is constructed. Consider the example ([**3**]; Pag.178) where I is the unit interval and $A = \{1/n : n \in \mathbf{N}\}$. Let σ be the topology on I obtained by modifying the usual one in that the basic neighborhoods of 0 do not contain points of A. Then (I, σ) is a τ–compact space, but A is not.

Finally, let us observe that τ–**COMP** is closed under arbitrary products; in fact, every τ–open set in a product ΠX_j, is union of rectangular τ–open sets ([**3**]; Th.1.8 and its proof). It follows that our theorem 1, applied to τ, generalizes properly the corresponding Th.1.8 in the paper of Ishii.

REFERENCES

[1] Herrlich H., "Topologische Reflexionen und Coreflexionen," L.N. 78 Springer Verlag, 1968.
[2] Hušek M., de Vries J., *Preservation of functors close to reflectors*, Top.Appl. **27** (1987), 171-189.
[3] Ishii T., *On the Tychonoff functor and w-compactness*, Top.Appl. **11** (1980), 173-187.
[4] Marny Th., *On epireflective subcategories of topological categories*, Gen.Top.Appl. **10** (1979), 175-181.
[5] Oka S., *Tychonoff functor and product spaces*, Proc.Japan.Acad. **54** (1978), 97-100.
[6] Stramaccia L., *Classes of spaces defined by an epireflector*, Proc. 3rd Nat.Conf. on Topology, Trieste 1986, Suppl.Rend.Circ.Mat.Palermo, Serie II **18** (1988), 423-432.

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