Comment.Math.Univ.Carolinae 30,4(1989)675-690

Hankel integral transforms of a new space of generalized functions of slow growth

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Abstract. In this paper the Hankel integral transformation is extended to a new space of generalized functions of slow growth by employing the method of adjoints. Many of the proofs are made by using the Mellin integral transformation.

Keywords: Hankel transformation, testing function space, generalized functions

Classification: 46F12

1. Introduction.

After Schwartz's [12] extension of the Fourier transform to generalized functions, the extension of classical integral transformations to generalized functions have comprised an active and interesting area of research (see, for example, L.S. Dube and J.N.Pandey [3], E.L. Koh and A.H. Zemanin [7], A.H. Zemanian [18] and J.M. Mendez [9]).

The Hankel integral transformations defined by

$$(1) \qquad h_{\mu,1}\{f(x)\}(y) = \int_0^\infty xJ_\mu(xy)f(x)\,dx$$

and,

$$(2) \qquad h_{\mu,2}(f(x)\}(y) = y\int_0^\infty J_\mu(xy)f(x)\,dx$$

have been extended to spaces of generalized functions by I. Fenyo [6], J.M. Mendez [9] and J.J. Betancor [1], amongst others.

Our main objective in this paper is to define the Hankel transformations (1) and (2) on new spaces of generalized functions of slow growth.

We introduce two spaces of testing functions denoted by $H_{\mu,1}$ and $F_{\mu,1}$ and it is proved that the following diagram of isomorphisms

$$\begin{array}{ccc} F_{\mu,1} & \xrightarrow{K_\mu} & F_{\mu,1} \\ \mathcal{M}\uparrow & & \uparrow\mathcal{M} \\ H_{\mu,1} & \xrightarrow{h_{\mu,1}} & H_{\mu,1} \end{array}$$

is commutative provided that $\mu > 0$, where $\mathcal{M}$ denotes the usual Mellin integral transformation and the mapping K_μ is defined by

$$K_\mu(\Phi) = 2^{s-1}\frac{\Gamma((s+\mu)/2)}{\Gamma(1+(\mu-s)/2}\phi(2-s), \quad (\phi \in F_{\mu,1}).$$

We also establish that, for $\mu > 0$, $h_{\mu,2}$ is an automotorphism on the space $H_{\mu,2}$ of testing functions of structure similar to $H_{\mu,1}$.

The Hankel transformations $h_{\mu,1}$ and $h_{\mu,2}$ satisfy the mixed Parseval equation ([**9**])

$$\int_0^\infty h_{\mu,1}\{f\}(y)g(y)\,dy = \int_0^\infty f(x)h_{\mu,2}\{g\}(x)\,dx. \tag{3}$$

According to J.M. Mendey [9] the generalized transform $h'_{\mu,1}f$ of $f \in H'_{\mu,2}$ is defined, for $\mu > 0$, through the relation

$$\langle h'_{\mu,1}f, \phi\rangle = \langle f, h_{\mu,2}\phi\rangle, \quad \phi \in H_{\mu,2}. \tag{4}$$

The generalized $h'_{\mu,2}$ transformation is defined on $H'_{\mu,1}$ as the adjoint of the classical transform $h_{\mu,1}$ on $H'_{\mu,1}$, provided that $\mu > 0$. More exactly, if $f \in H'_{\mu,1}$ then $h'_{\mu,2}f$ is given by

$$\langle h'_{\mu,2}f, \phi\rangle = \langle f, h_{\mu,1}\phi\rangle, \quad \phi \in H_{\mu,1}. \tag{5}$$

Notice that definitions (4) and (5) appear as generalizations of the mixed Parseval equation (3).

The notation and terminology of this work will follow that of [**18**]. I denotes the open interval $(0,\infty)$. $\langle f(t), \phi(t)\rangle$ denotes the number assigned to some element $\phi(t)$ in a testing function space by a member f of the dual space. $D(I)$ is the space of infinitely differentiable functions defined on I having compact support. The topology of $D(I)$ is that which makes its dual the space $D'(I)$ of the Schwartz distributions. $E(I)$ denotes the space of infinitely differentiable functions defined on I and $E'(I)$ is the space of distributions on I whose supports are compact subset of I.

2. The spaces $H_{\mu,1}$ and $F_{\mu,1}$ of testing functions and its duals.

In this section we introduce the testing functions spaces $H_{\mu,1}$ and $F_{\mu,1}$ and its main properties are studied.

2.1. The space $H_{\mu,1}$. Let μ be a real number. $H_{\mu,1}$ denotes the space of all complex valued smooth functions $\phi(t)$ on $0 < t < \infty$ such that for each nonnegative integer k, $D^k\phi(t)$ is of rapid descent as $x \to \infty$ (i.e. $D^k\phi(t)$ tends to zero faster than any power of $\frac{1}{x}$ as $x \to \infty$) and on which the functionals $\gamma^\mu_{m,k}$ defined by

$$\gamma^\mu_{m,k}(\phi) = sup_{x\in I}|x^m B^k_\mu \phi(x)| \quad , \quad m,k = 0,1,2,\ldots$$

assume finite values, where $B_\mu = x^{-\mu-1}Dx^{2\mu+1}Dx^{-\mu}$.

$H_{\mu,1}$ is a linear space under the pointwise addition on functions and their multiplication by complex number. Each $\gamma^\mu_{m,k}$ is clearly a seminorm on $H_{\mu,1}$ and $\gamma^\mu_{m,0}$ is a norm, for every $m \in \mathbb{N}$. Consequently, $\{\gamma^\mu_{m,k}\}_{m,k=0}^\infty$ is a multinorm on $H_{\mu,1}$. We assign to $H_{\mu,1}$ the topology generated by $\{\gamma^\mu_{m,k}\}_{m,k=0}^\infty$ thereby making it a countably multinormed space.

The dual space $H'_{\mu,1}$ consists of all continuous linear functionals of $H_{\mu,1}$. $H'_{\mu,1}$ is also a linear space to which assign the weak topology generated by the multinorm $\{\eta_\phi\}$, where $\eta_\phi(f) = |\langle f, \phi\rangle|$ and ϕ ranges through $H_{\mu,1}$.

It is obvious that the space $D(I)$ is contained in $H_{\mu,1}$ and the topology of $D(I)$ is stronger than that inducted on it by $H'_{\mu,1}$. Hence the restriction of any $f \in H'_{\mu,1}$ to $D(I)$ is in $D'(I)$.

One can easily prove that if $f(t)$ is a locally and absolutely integrable function on I, then $f(t)$ generates a member of $H'_{\mu,1}$ through the definition

$$\langle f, \phi\rangle = \int_0^\infty f(t)\phi(t)\,dt \quad , \quad \phi \in H_{\mu,1}.$$

We shall now derive a structure formula for an element of $H'_{\mu,1}$.

Proposition 1. *Let f be in $H'_{\mu,1}$. Then there exist essentially bounded measurable functions $g_{m,k}(x)$ defined on $x > 0$ for $m,k = 0,1,2,\dots r$, where r is some nonnegative integer depending on f such that*

$$f = \sum_{m.k=0}^{r} xB^k_\mu(x^{m-1}Dg_{m,k}(x)).$$

PROOF : Let f be in $H'_{\mu,1}$. In virtue of [18, Theorem 1.8-1] there exist a nonnegative integer r and a positive constant C such that

$$(6) \qquad |\langle f, \phi\rangle| \le C \max_{0\le m,k\le r} \gamma^\mu_{m,k}(\phi), \quad \phi \in H_{\mu,1}.$$

Moreover

$$(7) \qquad x^m B^k_\mu \phi(x) = -\int_x^\infty D(t^m B^f_\mu \phi(t))\,dt, \quad x > 0$$

because $\lim_{x\to\infty} x^m B^k_\mu\phi(x) = 0$.

Hence, by (6) and (7), one follows

$$(8) \qquad |\langle f, \phi\rangle| \le C \max_{0\le m,k\le r} \|D(t^m B^k_\mu \phi(t))\|_1, \quad \phi \in H_{\mu,1}$$

where $\|\cdot\|_1$ denotes the norm in the space $L_1(0,\infty)$ of equivalence classes of absolutely integrable measurable functions.

If we define the mapping

$$J : H_{\mu,1} \to JH_{\mu,1} \subset (L_1(0,\infty))^{(r+1)^2} = L_1(0,\infty) \times \overset{(r+1)^2}{\cdots} \times L_1(0,\infty)$$
$$\phi \to J\phi = (D(t^m B_\mu^k \phi(t)))_{m,k=0}^r$$

then, in view of (8), the mapping

$$T : JH_{\mu,1} \to \mathbb{C}$$
$$(D(t^m B_\mu^k \phi(t)))_{m,k=0}^r \to \langle f, \phi \rangle$$

is continuous, when $JH_{\mu,1}$ is endowed with the topology induced in it by $L_1(0,\infty)$.

In virtue of the Hahn-Banach theorem the mapping T can be continuously extended to $(L_1(0,\infty))^{(r+1)^2}$.

On the other hand, since the dual of $L_1(0,\infty)$ is equivalent to $L_\infty(0,\infty)$ (see F.Treves [15]), there exist essentially bounded measurable functions $h_{m,k}$, $m, k = 0, \dots, r$ such that

$$\langle f, \phi \rangle = \langle \sum_{m,k=0}^{r} h_{m,k}(x), D(x^m B_\mu^k \phi(x)) \rangle =$$
$$= \langle -\sum_{m,k=0}^{r} x B_\mu^k (x^{m-1} D h_{m,k}(x)), \phi(x) \rangle, \quad \phi \in H_{\mu,1}.$$

By making $g_{m,k} = -h_{m,k}$ $(m, k = 0, \dots, r)$ the proof of this proposition is completed. ∎

2.2. The space $F_{\mu,1}$. Let $\mu > 0$ and $0 < \varepsilon < 1$. $F_{\mu,1}^\varepsilon$ is the space of all holomorphic functions ϕ on the domain $G = \mathbb{C} - \{-\mu - 2k \mid k \in \mathbb{N}\}$ having at most simple poles in $-\mu, -\mu - 2, \mu - 4, \dots$, such that

$$\sigma_{m,k}^{\mu,\varepsilon}(\Phi) = \sup_{s \in V_\varepsilon(m,k)} |(s)_{\mu,k} \Phi(s)| < \infty, \quad (m, k = 0, 1, \dots)$$

where

$$V_\varepsilon(m,k) = \{ s \in \mathbb{C} / -2k + m + \varepsilon \le \Re s \le -2k + m + 1 + \varepsilon \}$$

and

$$(s)_{\mu,k} = \begin{cases} 1, & \text{if } k = 0 \\ (s+\mu)(s+\mu+2)\dots(s+\mu+2k-2)(s-\mu)(s-\mu+2)\dots & \\ \qquad \dots(s-\mu+2k-2), & \text{if } k \le 1. \end{cases}$$

The $\sigma_{m,k}^{\mu,\varepsilon}$ are norms on $F_{\mu,1}^\varepsilon$. Moreover, $F_{\mu,1}^\varepsilon$ is understood to posses the topology generated by the multinorm $\{\sigma_{m,k}^{\mu,\varepsilon}\}_{m,k=0}^\infty$, thus, $F_{\mu,1}^\varepsilon$ is a countably multinormed space.

We can prove, by making use the maximum modulus theorem for the holomorphic functions, that the space $F_{\mu,1}^\varepsilon$ is, actually, independent of ε. Hence, in the sequel we will denote by $F_{\mu,1}$, $\sigma_{m,k}^\mu$, $V(m,k)$ to $F_{\mu,1}^\varepsilon$, $\sigma_{m,k}^{\mu,\varepsilon}$ $V_\varepsilon(m,k)$, respectively.

We now list some properties of the space $F_{\mu,1}$.

Proposition 2. *$F_{\mu,1}$ is complete and therefore it is a Frechet space.*

PROOF : Let $\{\Phi_\nu\}_{\nu,1}^\infty$ be a Cauchy sequence in $F_{\mu,1}$ and let Ω denote an arbitrary compact subset of $G = \mathbb{C} - \{-\mu - 2k : k \in \}$.

We can easily prove that

$$\mathbb{C} = (\bigcup_{n=0}^{\infty} V(n,0)) \cup (\bigcup_{n=1}^{\infty} V(0,n)) \cup (\bigcup_{n=1}^{\infty} V(1,n)).$$

Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\Omega \subset (\bigcup_{n=1}^{n_0} V(0,n)) \cup (\bigcup_{n=1}^{n_0} V(1,n)).$$

Let $W = \{(m,k) : (k = 0, m = 0, \ldots, n_0) \text{ or } (m \in \{0,1\}, k = 1, \ldots, n_0)\}$.

Assume now without loss of generality that $0 < \varepsilon < \min(1,\mu)$. Then, there exists a constant $A > 0$ satisfying

$$(9) \qquad |(s)_{\mu,k}| \geq A,\ \forall s \in V(m,k) \cap \Omega \quad \text{and} \quad (m,k) \in W.$$

Therefore, given an $\eta > 0$,

$$(10) \qquad \sup_{s \in V(m,k)} |(s)_{\mu,k}(\Phi_\nu(s) - \Phi_{\nu'}(s))| < \eta$$

for every $\nu, \nu' \geq \nu_0$ for some $\nu_0 \in \mathbb{N}$.

From (9) and (10) we can deduce that $\{\Phi_\nu\}_{\nu=1}^\infty$ is uniformly regular on Ω.

Hence, in virtue of the completeness of the space of holomorphic functions of G (see J.B. Conway [2], pp. 151), there exists an analytic function Φ on G such that $\{D^k\Phi_\nu\}_{\nu=1}^\infty$ converges uniformly on Ω to $D^k\Phi$, for each $k \in \mathbb{N}$.

By using standard arguments we can prove that $\{\Phi_\nu\}_{\nu=1}^\infty$ converges to Φ in the space $F_{\mu,1}$. The proof is completed. ■

Therefore $F_{\mu,1}$ is a space of testing functions of G. $F'_{\mu,1}$, the dual of $F_{\mu,1}$, is a linear space to which we assign the usual (weak) topology. $F'_{\mu,1}$ is a space of generalized functions.

Proposition 3. *If n is an even nonnegative integer, then $F_{\mu+n,1} \subset F_{\mu,1}$, and the topology of $F_{\mu+n,1}$ is stronger than the one induced on it by $F_{\mu,1}$.*

PROOF : To see this assertion we assume that $\mu + 2 \neq \varepsilon + k$, for each integer k.

Let $\Phi \in F_{\mu+2,1}$.

We must analyze the expression $\sup_{s \in V(m,k)} |(s)_{\mu,k}\Phi(s)|$ with $m, k \in \mathbb{N}$.

If we define

$$V^-(m,k) = \begin{cases} V(m,k) & \text{if} \quad \mu + 2 \notin V(m,k) \\ V(m,k) - D(\mu+2;r) & \text{if} \quad \mu + 2 \in V(m,k) \end{cases}$$

where $D(\mu+2;r)=\{: |z-\mu-2|<r\}\subset V(m,k)$, by applying the maximum modulus theorem ([2]), then

$$\begin{aligned}&\sup_{s\in V(m,k)} |(s)_{\mu,k}\Phi(s)| = \\ (11)\qquad &= \sup_{s\in V^-(m,k)} |(s)_{\mu+2,k}\Phi(s)\frac{(s+\mu)(s-\mu+2k-2)}{(s-\mu-2)(s+\mu+2k)}| \leq \\ &\leq C \sup_{s\in V(m,k)} |(s)_{\mu+2,k}\Phi(s)|\end{aligned}$$

for a suitable positive constant C.

Hence, Φ in $F_{\mu,1}$. Moreover (11) implies that the topology of $F_{\mu+2,1}$ is stronger than that induced in it by $F_{\mu,1}$.

Therefore, our assertion is true for $n=2$. The general case follows by induction on n. ∎

We can conclude that the restriction of $f\in F'_{\mu,1}$ to $F_{\mu+n,1}$ is in $F'_{\mu+n,1}$, provided that n is an even nonnegative integer.

2.3. The Mellin integral transformations and the spaces $H_{\mu,1}$ and the spaces $F_{\mu,1}$.

In this paragraph we establish a paramount result of this paper: the Mellin transformation is an isomorphism from $H_{\mu,1}$ onto $F_{\mu,1}$, provided that $\mu>0$. This fact allows to prove a let of properties for $H_{\mu,1}$ in relation to other ones for $F_{\mu,1}$.

Theorem 1. *If $\mu>0$, then the Mellin integral transformation is an isomorphism from $H_{\mu,1}$ onto $F_{\mu,1}$.*

PROOF : Let $\phi\in H_{\mu,1}$. As it is well-known the Mellin transformation of ϕ is given by

$$\Phi(s)=(\mathcal{M}\phi)(s)=\int_0^\infty t^{s-1}\phi(t)\,dt. \tag{12}$$

It is easy to see that the integral in (12) is absolutely convergent provided that $Re\, s>0$.

Integrations by parts in (12) yield

$$\begin{aligned}\Phi(s)&=\left[\frac{t^s}{s+\mu}\phi(t)-\frac{t^{s+\mu+1}}{(s-\mu)(s+\mu)}Dt^{-\mu}\phi(t)\right]_{t\to 0^+}^{t\to\infty}+ \\ &+\frac{1}{(s-\mu)(s+\mu)}\int_0^\infty t^{s+1}B_\mu\phi(t)\,dt.\end{aligned} \tag{13}$$

From (13) we can deduce that if Re $s>\mu$ then

$$(s+\mu)(s-\mu)\Phi(s)=\int_0^\infty t^{s+1}B_\mu\phi(t)\,dt \tag{14}$$

because $D^k\phi(t)$ is of rapid descent as $x \to \infty$ for each $k \in \mathbb{N}$ and the function $x^{2\mu+1}Dx^{-\mu}\phi(x)$ is bounded on $0 < x < \infty$. (14) also holds when $s = \mu$.

The right hand side of (14) defines an analytic function when Re $s > -2$. Hence $\Phi(s)$ can be considered as an holomorphic function for Re $s > -2$ except at most in $s = -\mu$, where $\Phi(s)$ can have a simple pole.

By repeating above arguments one shows

$$(s)_{\mu,k}\Phi(s) = \int_0^\infty t^{s+2k-1}B_\mu^k\phi(t)\,dt \tag{15}$$

for $k \in \mathbb{N}$ and Re $s > -2k$. Moreover $\Phi(s)$ is an analytic function on C except in $s = -\mu, -\mu-2, -\mu-4, \ldots$ where Φ can have simple poles.

From (15) it is inferred

$$\sup_{s\in V(m,k)} |(s)_{\mu,k}\Phi(s)| \leq C\{\gamma_{m,k}^\mu(\phi) + \gamma_{m+2,k}^\mu(\phi)\}$$

for m and k belonging to $\mathbb{N}$ and C being a suitable positive constant independent of ϕ.

Hence the Mellin transformation defines a continuous mapping from $H_{\mu,1}$ in $F_{\mu,1}$ when $\mu > 0$.

Let now Φ be in $F_{\mu,1}$ and let c be a positive real number.

Define

$$\phi(t) = (\mathcal{M}^{-1}\Phi)(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s}\Phi(s)\,ds. \tag{16}$$

Notice that the integral in the right hand side of (16) is absolutely convergent and it is independent of $c > 0$. Moreover ϕ is an infinitely differentiable function.

By differentiating under the integral sign in (16), one has

$$t^m D^k\phi(t) = \frac{(-1)^k}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s+m-k}s(s+1)\ldots(s+k-1)\Phi(s)\,ds$$

from where, for every $m, k \in \mathbb{N}$ and $c > \max(\mu, m-k)$,

$$|t^m D^k\phi(t)| \leq M\sigma_{r,k+1}^\mu(\Phi)t^{-c+m-k} \tag{17}$$

for $t > 0$, where M is a positive constant no depending on t and r is a nonnegative integer such that

$$-2k - 2 + r + \varepsilon \leq c \leq -2k + r - 1 + \varepsilon$$

By (17), we get

$$\lim_{t\to\infty} t^m D^k\phi(t) = 0$$

or, in other words, $D^k\phi(t)$ is of rapid descent as $x \to \infty$ for every $k \in \mathbb{N}$.

By differentiating again under the integral sign in (16), one has

$$t^m B^k \phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s+m-2k} s_{\mu,k} \Phi(s) ds \tag{18}$$

for every $m, k \in \mathbb{N}$.

To show that $\mathcal{M}^{-1}$ is a continuous mapping from $F_{\mu,1}$ into $H_{\mu,1}$ we must discern two cases.

i) Assume $t \geq 1$ and choose $c > 0$ such that $-2k + m - c \leq 0$ and $c \neq \mu - 2k$. From (18) one follows

$$|t^m B_\mu^k \phi(t)| \leq \frac{1}{2\pi} t^{-2k+m-c} \sigma^\mu_{r+2,k+1}(\Phi) \int_{-\infty}^{+\infty} \frac{d\eta}{|(c + i\eta + \mu + 2k)(c + i\eta - \mu + 2k)|}$$

where $r \in \mathbb{N}$ and

$$-2k + r + \varepsilon \leq c \leq -2k + r + 1 + \varepsilon. \tag{19}$$

Hence,

$$\sup_{t \geq 1} |t^m B_\mu^k \phi(t)| \leq M \sigma^\mu_{r+2,k+1}(\Phi) \tag{20}$$

for a certain $M > 0$ and r is given by (19).

ii) Let now $0 < t < 1$. We can write

$$|t^m B_\mu^k \phi(t)| \leq |B_\mu^k \phi(t)| \quad \text{for every} \quad m, k \in \mathbb{N}$$

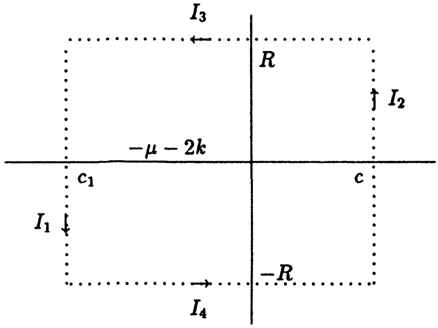
being

$$B_\mu^k \phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s-2k} (s)_{\mu,k} \Phi(s) ds \tag{21}$$

Let c_1 be a real number such that

$$-E[\mu] - 2k - 2 + \varepsilon \leq c_1 < -E[\mu] - 2k - 1$$

and consider the path $C = \sum_{i=1}^4 I_i$ of the following figure



Notice that the function $(s)_{\mu,k}\Phi(s)$ has at most a simple pole in $s=-\mu-2k$ at the inside of C. Hence, by Cauchy's residues theorem, we get

$$\frac{1}{2\pi i}\int\limits_{C}(s)_{\mu,k}\Phi(s)t^{-s-2k}ds = Res\{(s)_{\mu,k}\Phi(s)t^{-s-2k}; s=-\mu-2k\}.$$

Moreover the integral along I_3 and I_4 tends to zero as $R\to\infty$, because of our hypothesis on $\Phi(s)$. By combining these results we find that

$$\begin{aligned}B_\mu^k\phi(t) &= M_{\mu,k}t^\mu \lim_{s\to-\mu-2k}(s)_{\mu,k+1+E[\mu]}\Phi(s)+\\ &+\frac{1}{2\pi i}\int\limits_{c_1-i\infty}^{c_1+i\infty} t^{-s-2k}(s)_{\mu,k}\Phi(s)ds\end{aligned} \tag{22}$$

where

$$M_{\mu,k}=\lim_{s\to-\mu-2k}\frac{1}{\begin{aligned}&(s-\mu+2k)(s+\mu+2k+2)\dots\\ &\dots(s-\mu+2k+2E[\mu])(s+\mu+2k+2E[\mu])\end{aligned}}.$$

Since $t\in(0,1)$ and $-2k-2-E[\mu]+\varepsilon\le-\mu-2k\le-2k-E[\mu]+\varepsilon$, the following inequalities

$$\begin{aligned}|t^\mu M_{\mu,k}\lim_{s\to-\mu-2k}(s)_{\mu,k+1+E[\mu]}\Phi(s)| &\le |M_{\mu,k}|\{\sigma^\mu_{E[\mu],1+k+E[\mu]}(\Phi)+\\ &+\sigma^\mu_{E[\mu]+1,1+k+E[\mu]}(\Phi)\}\end{aligned} \tag{23}$$

$$|\frac{1}{2\pi i}\int\limits_{c_1-i\infty}^{c_1+i\infty} t^{-s-2k}(s)_{\mu,k}\Phi(s)ds| \le M_1\sigma^\mu_{E[\mu]+1,1+k+E[\mu]}(\Phi) \tag{24}$$

hold, for a certain $M_1 > 0$.

By combining (23) and (24) we can conclude

$$\sup_{t\in(0,1)} |B_\mu^k \phi(t)| \leq M_2\{\sigma^\mu_{E[\mu],1+k+E[\mu]}(\Phi)+ \tag{25}$$
$$|\sigma^\mu_{E[\mu]+1,1+k+E[\mu]}(\Phi)\}$$

where $M_2 > 0$ is independent of Φ.

Hence, by (25) and (20) one inferres

$$\gamma^\mu_{m,k}(\phi) \leq M_3\{\sigma^\mu_{E[\mu],1+k+E[\mu]}(\Phi) + \sigma^\mu_{E[\mu]+1,1+k+E[\mu]}(\Phi)+$$
$$\sigma^\mu_{r+2,k+1}(\Phi)\}$$

for suitable $M_3 > 0$ and $r \in \mathbb{N}$ is given by (19).

Therefore $\mathcal{M}^{-1}$ is a continuous mapping from $F_{\mu,1}$ into $H_{\mu,1}$.

Since $\mathcal{M}.\mathcal{M}^{-1}(\Phi) = (\Phi)$, for $\Phi \in F_{\mu,1}$ and $\mathcal{M}^{-1}.\mathcal{M}(\phi) = \phi$, for $\phi \in H_{\mu,1}$ (see I.N. Sneddon [13]), the proof of this theorem is finished. ∎

Some properties of $H_{\mu,1}$ can be now proved by using the Mellin transformation and according to Theorem 1. In effect, by virtue of Proposition 2, $H_{\mu,1}$ is a complete space and, hence a Frechet space, provided that $\mu > 0$. Moreover if $\mu > 0$ then $H_{\mu,1} \subset E(I)$ and the inclusion is continuous.

By invoking Proposition 3 we can state the following

Proposition 4. *If n is an even nonnegative integer then $H_{\mu+n,1}$ is stronger than that induced in it by $H_{\mu,1}$.*

2.4. Some operations on $H_{\mu,1}$ and $H'_{\mu,1}$. We now study some operators on the spaces $H_{\mu,1}$ and $H'_{\mu,1}$.

Proposition 5. *Let $\mu > 0$ and $\mu + \alpha > 0$. The mapping*

$$I_\alpha : H_{\mu,1} \to H_{\mu+\alpha,1}$$
$$\phi \to (I_\alpha\phi)(x) = x^\alpha\phi(x)$$

is an isomorphism. Its inverse is $I_{-\alpha}$.

PROOF : As it is well known

$$\mathcal{M}(I_\alpha\phi)(s) = \mathcal{M}(\phi)(s+\alpha)$$

ϕ being a suitable function. Hence, According to Theorem 1, I_α is an isomorphism from $H_{\mu,1}$ into $H_{\mu+\alpha,1}$ if, and only if, the mapping

$$\mathcal{Y}_\alpha : F_{\mu,1} \to F_{\mu+\alpha,1}$$
$$\Phi \to \mathcal{Y}_\alpha(\Phi)(s) = \Phi(s+\alpha)$$

is an isomorphism.

Let Φ be in $F_{\mu,1}$. Then $\mathcal{Y}_\alpha\Phi$ is an analytic function in $\mathbf{C} - \{-\mu - \alpha - 2k : k \in \mathbf{N}\}$. $\mathcal{Y}_\alpha\Phi$ can have at most simple poles in the points $-\mu - 2k$ with $k \in \mathbf{N}$.

On the other hand, for every $m, k \in \mathbf{N}$, one has

$$\sigma^{\mu+\alpha}_{m,k}(\mathcal{Y}_\alpha\Phi) = \sup_{s\in V(m,k)} |(s)_{\mu+\alpha,k}\Phi(s+\alpha)| = \tag{26}$$
$$= \sup_{s\in W(m,k)} |[s]_{\mu,k}\,\Phi(s)|$$

where

$$W(m,k) = \{s \in \mathbf{C} : -2k + m + \alpha + \varepsilon \le \operatorname{Re} s \le -2k + m + \alpha + \varepsilon + 1\}$$

and

$$[s]_{\mu,k} = \begin{cases} 1 & \text{if } k = 0 \\ (s+\mu)(s-2\alpha-\mu)\dots(s+\mu+2k-2)(s-\mu+2k-2-2\alpha & \text{if } k \ge 1. \end{cases}$$

By making use again the maximum modulus theorem we can deduce from (26) that

$$\sigma^{\mu+\alpha}_{m,k}(\mathcal{Y}_\alpha\Phi) \le M \sum_{i=1}^{3} \sigma^{\mu}_{m_i,k_i}(\Phi)$$

for certain $M > 0$ and $m_i,\ k_i \in \mathbf{N}$ $(i = 1, 2, 3)$.

Therefore $\mathcal{Y}_\alpha$ is a continuous linear mapping from $F_{\mu,1}$ into $F_{\mu+\alpha,1}$.

The proof can be easily finished. ∎

Note that from Propositions 4 and 5 we can inferred the following

Corollary 1. *If $P(x)$ is a polynomial then the mapping*

$$\begin{aligned} \mathcal{P} : H_{\mu,1} &\to H_{\mu,1} \\ \phi &\to \mathcal{P}(\phi)(x) = P(x^2)\phi(x) \end{aligned}$$

is linear and continuous.

We define the operators

$$T_\mu\phi(x) = x^\mu D x^{-\mu}\phi(x)$$
$$S_\mu\phi(x) = x^\mu \int_\infty^x t^{-\mu}\phi(t)\,dt$$
$$R_\mu\phi(x) = x^{-\mu-1} D x^{\mu+1}\phi(x)$$

By using arguments similar to the ones employed in Proposition 3 and 5 Theorem 1 it can be established

Proposition 6. *Let $\mu > 0$.*

a) *T_μ is an isomorphism from $H_{\mu,1}$ onto $H_{\mu+1,1}$ and its inverse is $S\mu$.*

b) *R_μ defines a continuous linear mapping from $H_{\mu+1,1}$ into $H_{\mu,1}$.*

c) *The operator B_μ is a continuous linear mapping from $H_{\mu,1}$ into itself.*

If we denote by T_μ^*, S_μ^*, R_μ^* and B_μ^* the adjoint operators of the classical ones T_μ, $S\mu$, R_μ and B_μ, according to well known results concerning to operators on duals of Frechet spaces (see A.H. Zemanian [**18**]), then

Proposition 7. *Let $\mu > 0$.*

a) *T_μ^* is an isomorphism from $H'_{\mu+1,1}$ onto $H'_{\mu,1}$ and S_μ^* is the inverse of T_μ^*.*

b) *R_μ^* is a continuous linear mapping from $H'_{\mu,1}$ into $H'_{\mu+1,1}$.*

c) *B_μ^* is a continuous linear mapping from $H'_{\mu,1}$ into itself.*

3.The Hankel transformation $h_{\mu,1}$ on $H_{\mu,1}$.

This section is devoted to describe the behaviour of the $h_{\mu,1}$-transformation on the space $H_{\mu,1}$ of testing functions. Previously we need to prove the following result

Lemma 1. *Let $\mu > 0$. The mapping*

$$K_\mu : F_{\mu,1} \to F_{\mu,1}$$
$$\Phi \to K_\mu(\Phi)(s) = 2^{s-1}\frac{\Gamma((\mu+s)/2)}{\Gamma(1+(\mu-s)/2)}\Phi(2-s)$$

is an automorphism.

PROOF : Let Φ be in $F_{\mu,1}$ and let $\varepsilon \in (0,1)$ such that

$$(\varepsilon + k : k \in \mathbb{Z}) \cap (\{-\mu - 2k : k \in \mathbb{N}\} \cup \{\mu + 2k : k \in \mathbb{Z}\}) = \emptyset.$$

Denote by $\psi(s) = K_\mu(\Phi)(s)$.

As it is known $\Gamma(s)$ is a meromorphic function that has simple poles at the points $s = 0, -1, -2, \ldots$. Hence $\psi(s)$ is also a meromorphic function having simple poles at $s = -\mu$, $-\mu - 2, \ldots$ at most.

Moreover, if $m, k \in \mathbb{N}$ then

$$\sigma^\mu_{m,k}(\Psi) = \sup_{s\in W_{(m,k)}} |[s]_{\kappa,k}\, \Phi(s) 2^{1-s}\frac{\Gamma(1+(\mu-s)/2)}{\Gamma((\mu+s)/2)}| \tag{27}$$

where

$$[s]_{\mu,k} = \begin{cases} 1 & \text{if } k = 0 \\ (\mu+2-s)(2-\mu-s)\ldots(\mu+2k-s)(2k-\mu-s) & \text{if } k \geq 1 \end{cases}$$

and

$$W_{(m,k)} = \{s \in \mathbb{C}; 2k - m + 1 - \varepsilon \leq \operatorname{Re} s \leq 2k - m + 2 - \varepsilon\}.$$

Since

$$C = (\bigcup_{n=0}^{\infty} V(n,0)) \cup (\bigcup_{n=1}^{\infty} V(0,n)) \cup (\bigcup_{n=1}^{\infty} V(1,n)).$$

there exists ε nonnegative integer k_1 such that

$$(28)\qquad \begin{aligned} W_{(m,k)} &\subset V(k_1,0) \cup V(k_1+1,0) \\ W_{(m,k)} &\subset V(0,k_1) \cup V(1,k_1+1) \\ W_{(m,k)} &\subset V(0,k_1+1) \cup V(1,k_1+1). \end{aligned}$$

On the other hand (see A.Erdelyi [5], 1.16(6)):

$$(29)\qquad |\Gamma(s)| \cong (2\pi)^{1/2} |\mathrm{Im}\ s|^{\mathrm{Re}\ s-(1/2)} \exp(-\frac{\pi}{2}|\mathrm{Im}\ s|) \qquad \text{as } |s| \to \infty.$$

Assume that $W_{(m,k)} \subset V(k_1,0) \cup V(k_1+1,0) = A(k_1)$ and chose $p \in \mathbf{N}$ such that

$$1 - Re(s) + 2k - 2p < -1, \quad \text{for} \quad s \in A(k_1).$$

Moreover, if $z_0 \in A(k_1)$ is a zero of the polynominal $(s)_{\mu,p}$ we choose a disc centered in z_0 and contained in $A(k_1)$. We proved in a similar way if $z_0 \in A(k_1)$ is a pole of the function $\Gamma(1+(\mu-s)/2)\,[s]_{\mu,k}$. Notice that such poles are not in $W_{(m,k)}$. We denote by $B(k_1)$ to the union of said discs provided that these ones exist and $A(k_1)^- = A(k_1) - B(k_1)$.

Now, by using the maximum modulus theorem we can deduce from (27) and (29)

$$\begin{aligned} \alpha^{\mu}_{m,k}(\psi) &\leq \sup_{s\in A(k_1)} - |\,[s]_{\mu,k}\, \Phi(s) 2^{1-s} \frac{\Gamma(1+(\mu-s)/2)}{\Gamma((\mu+s)/2)}| \leq \\ &\leq \sup_{s\in A(k_1)} - |\frac{[s]_{\mu,k}}{(s)_{\mu,p}} 2^{1-s} \frac{\Gamma(1+(\mu-s)/2)}{\Gamma((\mu+s)/2)}| \sup_{s\in A(k_1)} |(s)_{\mu,p}\Phi(s)| \leq \\ &\leq M(\sup_{s\in V(k_1+2p,2)} |(s)_{\mu,k}\Phi(s)| + \sup_{s\in V(k_1+12p,p)} |(s)_{\mu,p}\Phi(s)|) \end{aligned}$$

for a certain M positive constant.

A similar procedure can be followed in other cases in (28).

Therefore K_μ is a continuous linear mapping from $H_{\mu,1}$ into itself.

Since $K_\mu = K_\mu^{-1}$ the proof is finished. ∎

A main result of this paper is presented in the following.

Theorem 2:. *If $\mu > 0$, then $h_{\mu,1}$-transformation is an automorphism in the space $H_{\mu,1}$.*

PROOF : This theorem can be proved by making use of Theorem 1 and Lemma 1.

In effect, according to I.N.Sneddon [**13**],

$$\mathcal{M}(h_{\mu,1}\phi)(s) = 2^{s-1} \frac{\Gamma(1+(\mu-s)/2)}{\Gamma((\mu+s)/2)} \mathcal{M}(\phi)(2-s),\ \phi \in H_{\mu,1}$$

or, in other words,

$$h_{\mu,1}(\phi) = \mathcal{M}^{-1}.K_\mu.\mathcal{M}(\phi), \quad \phi \in H_{\mu,1}\ .$$

Therefore $h_{\mu,1}$ is an automorphism in the space $H_{\mu,1}$ provided that $\mu > 0$.

We now study certain operational rules for the $h_{\mu,1}$-transformation.

Proposition 8. *Let $\mu > 0$. If $\phi \in H_{\mu,1}$, then*

a) $h_{\mu+1,1}(t_\mu \Phi) = -yh_{\mu,1}(\phi)$
b) $T_\mu h_{\mu,1}(\phi) = h_{\mu+1,1}(-x\phi)$
c) $h_{\mu,1}(B_\mu \phi) = -y^2 h_{\mu,1}(\phi)$
d) $B_\mu h_{\mu,1}(\phi) = h_{\mu,1}(-x^2\phi)$.

If $\phi \in H_{\mu+1,1}$, then

e) $h_{\mu,1}(R_\mu \phi) = yh_{\mu+1,1}(\phi)$
f) $R_\mu h_{\mu+1,1}(\phi) = h_{\mu,1}(x\phi)$.

4. The Hankel transformation $h_{\mu,2}$ on the space $H_{\mu,2}$.

The $h_{\mu,1}$-transformation defined by (2) is related to the $h_{\mu,1}$-transformation through

$$h_{\mu,2}(\phi)(y) = yh_{\mu,1}(x^{-1}\phi(x))(y). \tag{30}$$

In view of the equality (30) we define the space $H_{\mu,2}$ of testing functions as follows: $\phi(x)$ is in $H_{\mu,2}$ if $\phi(x)x^{-1}$ belongs to $H_{\mu,1}$. $H_{\mu,2}$ is endowed with the topology generated by the family of seminorms $\{\eta^\mu_{m,k}\}^\infty_{m,k=0}$, where

$$\eta^\mu_{m,k}(\phi) = \gamma^\mu_{m,k}(x^{-1}\phi), \quad \phi \in H_{\mu,2}.$$

From Theorem 2 we can deduce.

Theorem 3. *If $\mu > 0$ then $h_{\mu,2}$ is an automorphism on the space $H_{\mu,2}$.*

Moreover the following operational rules for $h_{\mu,2}$ can be established.

Proposition 9. *Let $\mu > 0$. If $\phi \in H_{\mu,2}$, then*

a) $h_{\mu+1,2}(R^*_\mu \phi) = yh_{\mu,2}(\phi)$
b) $R^*_\mu h_{\mu,2}(\phi) = h_{\mu+1,2}(x\phi)$
c) $h_{\mu,2}(B^*_\mu \phi) = -y^2 h_{\mu,2}(\phi)$
d) $B^*_\mu h_{\mu,2}(\phi) = h_{\mu,2}(-x^2\phi)$.

If $\phi \in H_{\mu+1,2}$, then

e) $h_{\mu,2}(t^*_\mu \phi) = -yh_{\mu+1,2}(\phi)$
f) $T^*_\mu h_{\mu+1,2}(\phi) = h_{\mu,2}(-x\phi)$.

5. The generalized Hankel transformation.

According to the ideas developed by J.M.Mendez [9] and in analogy to the relation (3) we define the $h'_{\mu,1}$ transform $h'_{\mu,1}f$ of $f \in H'_{\mu,2}$ by

$$\langle h'_{\mu,1}f, \phi\rangle = \langle f, h_{\mu,2}\phi\rangle, \quad \phi \in H_{\mu,2}. \tag{31}$$

In words, we can say that the $h'_{\mu,1}$-transform $(h'_{\mu,1}f)$ of a generalized function $f \in H'_{\mu,2}$ is defined as the adjoint of the classical transformation $h_{\mu,2}$.

From Theorem 3 we can infer:

Theorem 4. *The $h'_{\mu,1}$- transformation is an automorphism onto the space $H'_{\mu,2}$ provided that $\mu > 0$.*

Notice that if f is in $H_{\mu,1}$ the f generates a regular distribution in the space $H'_{\mu,2}$ through

$$(32) \qquad \langle f, \phi \rangle = \int_0^\infty f(x)\phi(x)\,dx, \quad \phi \in H_{\mu,2}.$$

In effect, since $xf(x)$ is an absolutely integrable function on $(0, \infty)$ one has

$$|\langle f, \phi \rangle| \le \eta^{\mu}_{0,0}(\phi) \int_0^\infty |f(x)x|\,dx\,.$$

Therefore if $f \in H_{\mu,1}$ we can define $h_{\mu,1}f$ and $h'_{\mu,1}f$ the classical $h_{\mu,1}$-transform and the generalized $h'_{\mu,1}$-transform of f, respectively. However, according to (31), we have

$$\langle h'_{\mu,1}f, \phi \rangle = \langle f, h_{\mu,2}\phi \rangle, \quad \phi \in H_{\mu,2}$$

and in virtue of (32),

$$\langle h'_{\mu,1}f, \phi \rangle = \int_0^\infty f(x)h_{\mu,2}(\phi)(x)\,dx, \quad \phi \in H_{\mu,2}.$$

By invoking the mixed Parseval equation (3) one follows

$$\langle h'_{\mu,1}f, \phi \rangle = \int_0^\infty \phi(x)h_{\mu,1}(f)(x)\,dx = \langle h_{\mu,1}f, \phi \rangle, \qquad \phi \in H_{\mu,2}.$$

Hence $h'_{\mu,1}f$ and $h_{\mu,1}f$ coincide in the sense of equality in $H'_{\mu,2}$.
We now establish certain operational rules for the $h'_{\mu,1}$ transformation.

Proposition 10. *Let $\mu > 0$. If $f \in H'_{\mu,2}$, then*

a) $h'_{\mu+1,1}(T_\mu f) = -yh'_{\mu,1}(f)$
b) $T_\mu h'_{\mu,1}(f) = h'_{\mu+1,1}(-xf)$
c) $h'_{\mu,1}(B_\mu f) = -y^2 h'_{\mu,1}(f)$
d) $B_\mu h'_{\mu,1}(f) = h'_{\mu,1}(-x^2 f)$.

If $f \in H'_{\mu+1,2}$, then

e) $h'_{\mu,1}(R_\mu f) = yh'_{\mu+1,1}(f)$
f) $R_\mu h'_{\mu+1,1}(f) = h'_{\mu,1}(xf)$.

PROOF : We only prove a). The other operational rules can be proved in an analogous way.

Let $f \in H'_{\mu,2}$. According to Proposition 9, f), we have

$$\langle h'_{\mu+1,1}(T_\mu f), \phi\rangle = \langle T_\mu f, h_{\mu+1,2}\phi\rangle = \langle f, T^*_\mu h_{\mu+1,2}\phi\rangle =$$
$$=\langle f, h_{\mu,2}(-y\phi)\rangle = \langle -yh'_{\mu,1}f, \phi\rangle, \quad \text{for every} \quad \phi \in H_{\mu,2}.$$

∎

Note that the operational rules presented in Proposition 10 are extensions to generalized functions of those ones established in Proposition 8.

In a similar way we can define the generalized $h_{\mu,2}$-transformation on the space $H'_{\mu,1}$ provided that $\mu > 0$.

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(*Received* February 13,1989)