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Dedicated to the memory of Zdeněk Frolík

Abstract. In this paper, we give a negative answer to a problem about metrization which was posed by J. Nagata in [5], and we discuss several related problems.

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A function $g : \mathbb{N}\times X \to 2^X$ is called a *g-function* if $g(n,x)$ is an open neighborhood of x for all $n \in \mathbb{N}$ and $x \in X$. A g-function g is said to be *decreasing* if $g(n+1,x) \subseteq g(n,x)$ for all $n \in \mathbb{N}$ and $x \in X$. Let $g^1(n,x) = g(n,x)$ and let $g^{i+1}(n,x) = \bigcup\{g(n,y) : y \in g^i(n,x)\}$ for $i \in \mathbb{N}$.

The following problem was posed by J. Nagata ([5], problem after Theorem 9) : *Is a T_1 space X metrizable if X has a g-function which satisfies the following conditions:*

(1) *If $x \in g^2(n,x_n)$ for each $n \in \mathbb{N}$, then $x_n \to x$;*
(2) *For all $n \in \mathbb{N}$ and $Y \subseteq X$, $\mathrm{Cl}Y \subseteq \bigcup\{g^2(n,y) : y \in Y\}$?*

In the following, we answer this problem negatively and discuss some related problems.

Example 1. *A non-metrizable Moore space X with a g-function which satisfies conditions* (1) *and* (2).

Let $X = \mathbb{R} \times (\{1/n : n \in \mathbb{N}\} \cup \{0\})$, and topologize X with the following modification of the tangent disc topology (see [6] pages 101-103): all the points of $\mathbb{R} \times \{1/n : n \in \mathbb{N}\}$ are isolated. For a point of the form $(r,0)$, let $D_n(r,0) = \{(a,b) \in X : d((a,b),(r,1/n)) < 1/n\}$ and let $\{D_n(r,0) \cup \{(r,0)\} : n \in \mathbb{N}\}$ be a neighborhood base of $(r,0)$; here d denotes the Euclidean distance of $\mathbb{R}^2$. It is easy to see that X is a non-normal Moore space.

We define a g-function on X as follows:

$$g(n,(a,b)) = \begin{cases} \{(a,b)\} \cup D_{2n}(a,b) & \text{if } b = 0; \\ \{(a,0)\} \cup D_n(a,0) & \text{if } b = 1/n; \\ \{(a,b)\} & \text{if } b > 1/n; \\ \{(a',b') \in X : d((a',b'),(a,b)) < 1/n \text{ and } b' > 0\} & \text{if } 0 < b < 1/n. \end{cases}$$

The author would like to thank Dr. H. Junnila for his help and useful suggestions. Actually, Example 2 of this paper is due to him.

The proof that g is a g-function which satisfies conditions (1) and (2) is not difficult and we leave it for the reader. ■

The following example shows that a non-metrizable space can even have a decreasing g-function satisfying conditions (1) and (2).

Example 2. *A non-metrizable Moore space X with a decreasing g-function which satisfies conditions* (1) *and* (2).

Let $X = \mathbf{R}^2$. We give a topology on X as follows: All the points which belong to $X \setminus (\mathbf{R} \times \{0\})$ are isolated; for $(r,0) \in \mathbf{R} \times \{0\}$, set $B_n(r,0) = \{(a,b) \in X : b = |r-a| < 1/n\} \cup \{(r,b) : -1/n < b < 0\}$ and let $\{B_n(r,0) : n \in \mathbf{N}\}$ be a neighborhood base of $(r,0)$. It is trivial to check that X is a Moore space. Since the closed subspace $X' = \{(a,b) \in X : b \geq 0\}$ is R. W. Heath's V-space ([**3**] Example 1) which is not normal, X is not a metrizable space.

We define a g-function on X as follows:

$$g(n,(a,b)) = \begin{cases} B_n(a,b) & \text{if } b = 0; \\ \{(a,b)\} & \text{if } |b| \geq 1/n; \\ \{(a,b),(a-b,-b),(a+b,-b)\} & \text{if } 0 < b < 1/n; \\ B_n(a,0) & \text{if } -1/n < b < 0. \end{cases}$$

Then it is easy to check that g is a decreasing g-function which satisfies conditions (1) and (2). ■

A space with a g-function which satisfies condition (1) is a σ-space ([**4**]) and it is not difficult to show that a space with a g-function satisfying conditions (1) and (2) is a first countable space. Since a Moore space is a first countable σ-space, one could ask whether a regular space with a g-function which satisfies conditions (1) and (2) is a Moore space. By virtue of the following example, the answer to this question is also negative.

Example 3. *A stratifiable space X with a decreasing g-function which satisfies conditions* (1) *and* (2) *such that X is not a Moore space.*

Let $X = \mathbf{R}^2$. We give a topology on X as follows: All the points of $X \setminus (\mathbf{R} \times \{0\})$ are isolated. For $(r,0) \in \mathbf{R} \times \{0\}$, let $B_n(r,0) = \{(a,b) \in X : |a-r| < 1/n, |b| < 1/n\} \setminus \{(r,b) : 0 < b < 1/n\}$ and let $\{B_n(r,0) : n \in \mathbf{N}\}$ be a neighborhood base of $(r,0)$.

It is easy to show that X is a stratifiable space. The subspace $X' = \{(a,b) \in X : b \geq 0\}$ of X is not metrizable ([1] Example 9.1) and hence not developable, so X is not developable. We define a g-function on X as follows:

$$g(n,(a,b)) = \begin{cases} B_n(a,b) & \text{if } b = 0; \\ \{(a,b)\} & \text{if } |b| \geq 1/n; \\ \{(a,b),(a,-b)\} & \text{if } 0 < b < 1/n; \\ B_n(a,0) \setminus \{(a,0)\} & \text{if } -1/n < b < 0. \end{cases}$$

It is not difficult to show that g is a decreasing g-function which satisfies conditions (1) and (2). ■

Remark 1. Assume that X has a g-function which satisfies the following condition:

(3) For each $Y \subseteq X$, $\mathrm{Cl}Y \subseteq \bigcup\{g(n,y) : y \in Y\}$.

We can always find a decreasing g-function g' on X such that g' also satisfies condition (3) and $g'(n,x) \subseteq g(n,x)$ for all $n \in \mathbb{N}$ and $x \in X$. For example, we can take $g'(n,x) = \bigcap\{g(i,x) : 1 \leq i \leq n\}$. Hence, in many results which involve a g-function satisfying condition (3) and certain other conditions, whether this g-function is decreasing is not important. Examples 4 and 5 show that for a g-function which satisfies condition (2), the situation is quite different.

Example 4. *A Moore space X with a g-function which satisfies conditions* (1) *and* (2) *such that X has no decreasing g-function which satisfies the same conditions.*

We prove that the space X of Example 1 has no decreasing g-function satisfying (1) and (2).

Assume that X has such a g-function. We first prove that for all $r \in \mathbb{R}$ and $m,n \in \mathbb{N}$, we have that $(r,0) \in \bigcup\{g(n,(a,b)) : (a,b) \in D_m(r,0)\}$. Otherwise, there exist $r \in \mathbb{R}$ and $m,n \in \mathbb{N}$ such that $(r,0) \notin \bigcup\{g(n,(a,b)) : (a,b) \in D_m(r,0)\}$. By condition (1), there exists $k \in \mathbb{N}$ such that $(r,0) \notin \bigcup\{g(k,(a,b)) : (a,b) \notin D_m(r,0) \cup \{(r,0)\}\}$. Let $j = \max\{n,k\}$. Then $(r,0) \notin \bigcup\{g(j,(a,b)) : (a,b) \in X \setminus \{(r,0)\}\}$, and hence $(r,0) \notin \bigcup\{g^2(j,(a,b)) : (a,b) \in X \setminus \{(r,0)\}\}$. By condition (2), $\mathrm{Cl}(X \setminus \{(r,0)\}) \subseteq \bigcup\{g^2(j,(a,b)) : (a,b) \in X \setminus \{(r,0)\}\}$. As a consequence, $(r,0) \notin \mathrm{Cl}(X \setminus \{(r,0)\})$, which is a contradiction.

For all $n,k \in \mathbb{N}$, let $R_{n,k} = \{r \in \mathbb{R} : D_k(r,0) \subseteq g(n,(r,0))\}$. Note that, for each $n \in \mathbb{N}$, we have that $\bigcup_{k\in\mathbb{N}} R_{n,k} = \mathbb{R}$. Since every interval of $\mathbb{R}$ is of the second category, we can find inductively, for $n \in \mathbb{N}$, a closed interval $[c_n,d_n]$ and a $k(n) \in \mathbb{N}$, such that $R_{n,k(n)}$ is dense in $[c_n,d_n]$ and $[c_{n+1},d_{n+1}] \subseteq [c_n,d_n]$. Pick a $r_0 \in \bigcap_{n\in\mathbb{N}} [c_n,d_n]$. Note that, for $n \in \mathbb{N}$, $r_0 \in \mathrm{Cl}(R_{n,k(n)} \setminus \{r_0\})$. It follows that

$$D_{k(n)}(r_0,0) \subseteq$$
$$\subseteq \bigcup\{D_{k(n)}(r,0) : r \in R_{n,k(n)},\ r \neq r_0\} \subseteq \bigcup\{g(n,(r,0)) : r \in R_{n,k(n)},\ r \neq r_0\}.$$

Choose $(a_n,b_n) \in D_{k(n)}(r_0,0)$ such that $(r_0,0) \in g(n,(a_n,b_n))$ and choose $r_n \in R_{n,k(n)}$, $r_n \neq r_0$, such that $(a_n,b_n) \in g(n,(r_n,0))$ for each $n \in \mathbb{N}$. Then by condition (1) we have that $(r_n,0) \to (r_0,0)$, a contradiction. ■

Remark 2. Theorem 7 of [5] states that a T_1*-space X is metrizable if and only if X has a g-function which satisfies condition (2) and the following condition:*

(4) *For each $x \in X$ and each neighborhood U of x, there exists $n \in \mathbb{N}$ such that*

$$x \notin Cl(\bigcup\{g(n,y) : y \in X \setminus U\}).$$

In the proof of this theorem, it was assumed that the g-function appearing in the theorem is decreasing. The following example shows that the assumption of "decreasing" should be included in the statement of theorem.

Example 5. *A non-metrizable stratifiable space with a g-function satisfying conditions* (2) *and* (4).

The space is X' of Example 3. We define a g-function on X' as follows:

$$g(n,(a,b)) = \begin{cases} B'_n(a,b) & \text{if } b = 0; \\ \{(a,b)\} & \text{if } b \geq 1/n; \\ \{(a,b)\} \cup B'_n(a-\frac{2}{n},0) \cup B'_n(a+\frac{2}{n},0) & \text{if } 0 < b < 1/n, \end{cases}$$

where $B'_n(a,b)$ denotes the intersection of X' and $B_n(a,b)$ of Example 3.

It is easy to see that g satisfies condition (2). Note that if $B'_{4n}(r,0) \cap g(4n,(a,b)) \neq \emptyset$, then $(a,b) \in B'_n(r,0)$; it follows from this that g satisfies condition (4). ∎

Remark 3. In [5], the proof of Theorem 8 is based on Theorem 7. But Theorem 8 holds without the assumption that the g-function involved is decreasing. This is because if a g-function g satisfies condition (1) of Theorem 2 of [5], then the g-function g^2 satisfies the same condition and Theorem 8 of [5] follows from Proposition 1 of [8] or Theorem 3 of [2].

Even though Nagata's problem has negative solution, the following result holds:

Proposition. *A T_1 space X is metrizable if and only if X has a decreasing g-function which satisfies, for some $k \in \mathbb{N}$, the following conditions:*

(5) *If $x \in g^{k+1}(n, x_n)$ for each $n \in \mathbb{N}$, then $x_n \to x$;*

(6) *For each $Y \subseteq X$, $\mathrm{Cl}Y \subseteq \bigcup\{g^k(n,y) : y \in Y\}$ for each $n \in \mathbb{N}$.*

PROOF : The "*only if*" part is obvious. We prove the "*if*" part.

Assume that X has a decreasing g-function which satisfies (5) and (6). We first prove that g satisfies the following condition:

(∗) If $x_n \to x$ and $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$, then $y_n \to x$.

Assume that $x_n \to x$ and $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$. Then $x \in \mathrm{Cl}\{x_n : n \in \mathbb{N}\} \subseteq \bigcup\{g^k(m, x_n) : n \in \mathbb{N}\}$ for each $m \in \mathbb{N}$. Since g is decreasing, we can choose a subsequence $\{x_{n_m} : m \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$ such that $x \in g^k(m, x_{n_m})$ for each $m \in \mathbb{N}$, and hence $x \in g^{k+1}(m, y_{n_m})$ for each $m \in \mathbb{N}$. By condition (5), $y_{n_m} \to x$. Note that conditions (5) and (6) imply that X is first countable; by [8] Lemma 4, g satisfies condition (∗).

Let $g'(n,x) = g^k(n,k)$ for all $n \in \mathbb{N}$ and $x \in X$. Since g satisfies condition (∗), g' also satisfies (∗). By condition (6), we obtain that $\mathrm{Cl}Y \subseteq \bigcup\{g'(n,y) : y \in Y\}$ for all $Y \subseteq X$ and $n \in \mathbb{N}$. By virtue of Proposition 1 of [8] or Theorem 3 of [2], X is metrizable. ∎

For $k = 1$, the result above coincides with Theorem 9 of [5].

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