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Note to the paper of Fučík and Mawhin

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Abstract. In [2], the existence of periodic solutions to the periodically perturbed n-th order scalar differential equation was proved. In the special case (namely, if the nonlinear damping term in the equation is omitted), the assumptions under which the existence result holds can be essentially weakened.

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Although the main emphasis in [2] is focussed on the vector case, the following statement has been proved there for the scalar equation

$$x^{(n)} + \sum_{j=1}^{n-1} a_j x^{(n-j)} + g(x)x' + h(x) = p(t) \tag{1}$$

with real constant coefficients a_j, $j = 1, \dots, n-1$, where $g(x)$, $h(x)$ and $p(t)$ are continuous functions for all values of their arguments and $p(t)$ is everywhere ω-periodic.

Theorem [2]. *Even-order equation* (1) *admits an ω-periodic solution, provided*

(i) $(-1)^k a_{2k} \geq 0$ *for* $k = 1, \dots, [(n-1)/2]$ ([.] *means "entière"*),

(ii) $\exists h(\infty) := \lim_{x\to\infty} h(x) < \infty$ *and* $\exists h(-\infty) := \lim_{x\to-\infty} h(x) > -\infty$: $h(-\infty) < h(x) < h(\infty)$ *for all* x,

(iii) $\exists R$ *(a positive constant):* $\{h(x) - \frac{1}{\omega}\int_0^\omega p(t)\,dt\}\operatorname{sgn} x > 0$ *for* $|x| > R$.

The purpose of this paper is to show how the assumptions of Theorem 2 can be weakened if we consider its special case $g = 0$. More precisely, we will prove that condition (i) can be weakened to

$$\sum_{k=1}^{[(n-1)/2]} (-1)^{k+1} a_{2k}^{+} \left(\frac{\omega}{2\pi}\right)^{2k} < 1, \text{ where } a_{2k}^{+} := \begin{cases} a_{2k} & \text{for } a_{2k} > 0 \\ 0 & \text{for } a_{2k} \leq 0, \end{cases} \tag{2}$$

(ii) can be reduced to the boundedness requirement of $h(x)$, and (iii) can be substituted by the reverse inequality

$$\{h(x) - \frac{l}{\omega}\int_0^\omega p(t)\,dt\}\operatorname{sgn} x < 0 \qquad \text{for } |x| > R.$$

Moreover, such an improved result can be still extended, without any modification, to odd-order equation (1), while the first assumption together with the second one can be replaced by the growth restriction

$$(3)\qquad \sum_{k=1}^{[(n-1)/2]} (-1)^{k+1} a_{2k}^{+} \left(\frac{\omega}{2\pi}\right)^{2k} + \sup_{x\in(-\infty,\infty)} |h'(x)| \left(\frac{\omega}{2\pi}\right)^{n} < 1.$$

Hence, we can give for $g(x) \equiv 0$ the following generalization of Theorem [2].

Theorem. *Let us assume that $g(x) \equiv 0$. Furthermore, let (3) or, in case that $h(x)$ is an everywhere bounded function, (2) be satisfied together with*

$$(4)\quad \{h(x) - \frac{1}{\omega}\int_0^\omega p(t)\,dt\}\operatorname{sgn} x > 0 \ \text{or} \ \{h(x) - \frac{1}{\omega}\int_0^\omega p(t)\,dt\}\operatorname{sgn} x < 0 \qquad \text{for } |x| > R,$$

where R is a suitable positive constant. Then equation (1) admits an ω-periodic solution.

PROOF : We apply the standard version of the Leray-Schauder alternative (see e.g. [3]), consisting in the verification of the uniform a priori estimates of all solutions, and their derivatives up to the $(n-2)$th order including, to the one-parameter family of equations

$$(1_\mu)\qquad x^{(n)} + \mu\sum_{j=1}^{n-1} a_j x^{(n-j)} + \mu h(x) + (1-\mu)cx = \mu p(t) \quad \mu \in (0,1\rangle,$$

satisfying the boundary conditions

$$(\mathrm{P})\qquad x^{(i)}(0) = x^{(i)}(\omega) \qquad \text{for } i = 0,1,\dots,n-1,$$

while all roots of binomial $\lambda^n + c$ with a suitable real constant $c \neq 0$ are different from the integer multiples of $2\pi i/\omega$.

Since the last requirement can be always satisfied for a sufficiently small $|c|$, we can restrict ourselves to showing the a priori estimates only.

Hence, let $x(t)$ be a solution of $(1_\mu)-(P)$. Substituting $x(t)$ into (1_μ), multiplying the obtained identity by $x'(t)$ and integrating from 0 to ω, we get

$$\int_0^\omega x^{(n)2}(t)\,dt + \mu \sum_{k=1}^{[(n-1)/2]} (-1)^k a_{2k} \int_0^\omega x^{(n-k)2}(t)\,dt +$$
$$+ \mu\int_0^\omega h(x(t))x^{(n)}(t)\,dt + (l-\mu)c^*(-1)^{n/2}\int_0^\omega x^{(n/2)2}(t)\,dt =$$
$$= \mu\int_0^\omega p(t)x^n(t)\,dt, \text{ where } c^* := \begin{cases} c & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.}\end{cases}$$

Applying furthermore the well-known Schwarz and the Wirtinger inequalities (see e.g. [3]), we arrive at the relation

$$\int_0^\omega x^{(n)2}(t)\,dt \le$$

$$\le \left\{ \sum_{k=1}^{[(n-1)/2]} (-1)^{k+1} a_{2k}^{+} \left(\frac{\omega}{2\pi}\right)^{2k} + (H' + |c^*|)\left(\frac{\omega}{2\pi}\right)^{n} \right\} \int_0^\omega x^{(n)2}(t)\,dt +$$

$$+ \sqrt{\omega}(H+P)\left\{\int_0^\omega x^{(n)2}(t)\,dt\right\}^{\frac{1}{2}},$$

where $H' = 0$ for $H := \sup_{x\in(-\infty,\infty)} |h(x)| < \infty$,
otherwise $H = 0$ for $H' : \sup_{x\in(-\infty,\infty)} |h'(x)| < \infty$, i.e.

$$\int_0^\omega x^{(n)2}(t)\,dt \le \omega(H+P)^2/\Omega^2 := D_n^2,$$

where (cf. (2), (3))

$$\Omega := 1 - \left\{ \sum_{k=1}^{[(n-1)/2]} (-1)^{k+1} a_{2k}^{+} \left(\frac{\omega}{2\pi}\right)^{2k} + (H' + |c^*|)\left(\frac{\omega}{2\pi}\right)^{n} \right\} > 0$$

for a sufficiently small $|c|$.

Since points $t_j \in (0,\omega)$ exists, according to Rolle's theorem, such that $x^{(j)}(t_j) = 0$ for $j = 1,\dots,n-1$, we obtain from there (using again the Schwarz and the Wirtinger inequalities) that

$$|x^{(j)}(t)| \le \int_0^\omega |x^{(j+1)}(t)|\,dt \le \sqrt{\omega}\left\{\int_0^\omega x^{(j+1)2}(t)\,dt\right\}^{\frac{1}{2}} \le$$

$$\le \sqrt{\omega}\left(\frac{\omega}{2\pi}\right)^{n-j-1}\left\{\int_0^\omega x^{(n)2}(t)\,dt\right\}^{\frac{1}{2}} \le \sqrt{\omega}\left(\frac{\omega}{2\pi}\right)^{n-j-1} D_n := D^j$$

for $j = 1,\dots,n-1$.

Now, substituting $x(t)$ into (1_μ), integrating the obtained identity from 0 to ω, and multiplying it by $\operatorname{sgn} x$, we would come for $\min_{t\in\langle 0,\omega\rangle} |x(t)| > R$, under the choice of c in order

$$c\left\{h(x) - \frac{1}{\omega}\int_0^\omega p(t)\,dt\right\} \operatorname{sgn} x > 0$$

to be satisfied for $|x| > R$, to the inequality

$$\int_0^\omega |h(x(t)) - \frac{1}{\omega}\int_0^\omega p(t)\,dt|\,dt \le 0,$$

a contradiction to (4). Therefore, $\min_{t\in\langle 0,\omega\rangle} |x(t)| \le R$, and consequently (cf. (5))

$$|x(t)| \le \int_0^\omega |x'(t)|\,dt \le \omega D^1.$$

This completes the proof. ■

Remark. Modifying slightly our approach, we can get a similar result in case of time depending coefficients – see [1].

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