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An extension of Fan's fixed point theorem and equilibrium point of an abstract economy

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Abstract. A fixed point theorem more general than Fan's fixed point theorem is proved and using this fixed point theorem, the existence of an equilibrium point of an abstract economy given by preferences and an economy given by utility functions has been established.

Keywords: Abstract economy, equilibrium point, Nash equilibrium point, almost upper semicontinuous, upper semicontinuous, lower semicontinuous and quasiconcave function

Classification: Primary 47H10; Secondary 52A07, 49A40, 49B40, 90A06, 90A10, 90A14, 93D13

1. Introduction.

All topological spaces considered in this paper are assumed to be Hausdorff.

Following Debreu [2], Arrow and Debreu [1] and Shafer and Sonnenschein [12], we describe an abstract economy (or generalized game) with utility functions (or pay off) functions by $\mathcal{E} = \{X_\alpha, A_\alpha, U_\alpha : \alpha \in I\}$ where I is a finite or an infinite set of agents (or players) and for each $\alpha \in I, X_\alpha$ is the choice set (or strategy set), $A_\alpha : X = \prod_{\alpha \in I} X_\alpha \to 2^{X_\alpha}$ is the (budget) constraint correspondence, i.e. set valued mapping and $U_\alpha : X \to \mathbb{R}$ is the utility (or pay off) function and an economy with preference correspondences $\mathcal{E} = \{X_\alpha, A_\alpha, P_\alpha : \alpha \in I\}$ where I, X_α and A_α are as above and $P_\alpha : X \to 2^{X_\alpha}$ is the preference correspondence for the agent $\alpha \in I$. Before going any further we first make clear the symbols and notations to be used throughout this paper. X and $X_{-\alpha}$ will respectively denote the cartesian product $\prod_{\alpha \in I} X_\alpha$ and $\prod_{\substack{\beta \in I \\ \beta \neq \alpha}} X_\beta$ and $x_{-\alpha}$ will denote a generic element of $X_{-\alpha}$. We will also represent an element $x \in X$ by $\{x_\alpha\}$ where x_α is the projection of x onto X_α, i.e. x_α is the α-th co-ordinate of x. Thus we can write $x = \{x_\alpha\} = [x_\alpha, x_{-\alpha}]$ for each $\alpha \in I$.

A point $\overline{x} = \{\overline{x}_\alpha\}$ is called an *equilibrium point* of an abstract economy $\mathcal{E} = \{X_\alpha, A_\alpha, U_\alpha : \alpha \in I\}$ if for each $\alpha \in I$,

$$U_\alpha(\overline{x}) = U_\alpha[\overline{x}_\alpha, \overline{x}_{-\alpha}] = \sup_{z_\alpha \in A_\alpha(\overline{x})} U_\alpha[z_\alpha, \overline{x}_{-\alpha}].$$

It is worth noting that if $A_\alpha(x) = X_\alpha$ for each $x \in X$, the concept of an equilibrium point of the economy coincides with the well-known concept of Nash equilibrium point [9], for in the latter case, $U_\alpha(\overline{x}) = U_\alpha[\overline{x}_\alpha, \overline{x}_{-\alpha}] = \sup_{z_\alpha \in X_\alpha} U_\alpha[z_\alpha, \overline{x}_{-\alpha}]$ for each $\alpha \in I$. An economy $\mathcal{E} = \{X_\alpha, A_\alpha, U_\alpha : \alpha \in I\}$ can be expressed as an economy of the form $\{X_\alpha, A_\alpha, P_\alpha : \alpha \in I\}$ if for each $\alpha \in I$, we define the correspondence $P_\alpha : X \to 2^{X_\alpha}$ by $P_\alpha(x) = \{y_\alpha \in X_\alpha : U_\alpha([y_\alpha, x_{-\alpha}]) > U_\alpha(x)\}$ for each $x = \{x_\alpha\} \in X$.

Now suppose that for each $\alpha \in I, P_\alpha : X \to 2^{X_\alpha}$ is defined as above. Then it is clear that $\overline{x}$ is an equilibrium point of the economy $\mathcal{E} = \{X_\alpha, A_\alpha, U_\alpha : \alpha \in I\}$ if and only if $P_\alpha(\overline{x}) \cap A_\alpha(\overline{x}) = \emptyset$ and $\overline{x}_\alpha \in A_\alpha(\overline{x})$.

Thus if the economy instead of being given by utility functions is given by preference relations, we can define an equilibrium point of an abstract economy $\mathcal{E} = \{X_\alpha, A_\alpha, P_\alpha : \alpha \in I\}$ as a point $\overline{x} = \{x_\alpha\} \in X$ such that for each $\alpha \in I, \overline{x}_\alpha \in A_\alpha(\overline{x})$ and $P_\alpha(\overline{x}) \cap A_\alpha(\overline{x}) = \emptyset$.

The object of this paper is to extend the following two theorems to the case of locally convex topological vector spaces.

Theorem 1.1. (Debreu). *Let $\mathcal{E} = \{X_i, A_i, U_i\}_{i=1}^N$ be an abstract economy (a game) such that for each $i = 1, 2 \ldots, N$*

(i) *X_i is a nonempty compact convex subset of $\mathbb{R}^\ell$;*
(ii) *$A_i : X = \prod_{i=1}^N X_i \to 2^{X_i}$ is a continuous correspondence such that for each $x \in X, A_i(x)$ is nonempty and convex;*
(iii) *$U_i : X \to \mathbb{R}$ is continuous on X and quasiconcave in x_i (i.e. $U_i(\cdot, x_{-i})$ is quasiconcave for each x_{-i}).*

Then $\mathcal{E}$ has an equilibrium point.

Theorem 1.2. (Shafer and Sonnenschein). *Let $\{X_i, A_i, P_i\}_{i=1}^N$ be an abstract economy such that for each $i = 1, 2, \ldots, N$*

(i) *X_i is a nonempty compact convex subset of $\mathbb{R}^\ell$;*
(ii) *$A_i : X = \prod_{i=1}^N \to 2^{X_i}$ is a continuous correspondence such that for each $x \in X, A_i(x)$ is nonempty and convex;*
(iii) *$P_i : X \to 2^{X_i}$ has an open graph in $X \times X_i$ and for each $x = \{x_i\}_{i=1}^N, x_i \notin \operatorname{co}(P_i(x))$, where $\operatorname{co} A$ denotes the convex hull of A.*

Then $\mathcal{E}$ has an equilibrium point.

We should point out that our extension of Theorem 1.2 requires (1) I to be countable, (2) a strong irreflexivity of the preference correspondence, i.e. $x_\alpha \notin \overline{\operatorname{co}}\, P_\alpha(x)$ for each $x = \{x_\alpha\}$ (this seems to be unavoidable due to the pathological defect in the convex hull of a compact subset in an infinite dimensional space, i.e. the convex hull of a compact subset in an infinite dimensional space need not be compact, nor even closed, e.g. see Schaefer [11, p. 72]) and lastly—but most importantly—(3) an extension of Fan's fixed point theorem. Section 2 deals exclusively with the required extension of Fan's fixed point theorem which overcomes the difficulty arising out of the following fact: If $g : K \to 2^K$ is an upper semicontinuous set valued mapping where K is a nonempty compact convex subset of a locally convex topological vector space, then the mapping $f : K \to 2^K$ defined by $f(x) = \overline{\operatorname{co}}\, g(x)$ is not necessarily upper semicontinuous. We have not made any distinction between a correspondence and a set valued mapping. Rather we have used both to conform to the existing literature, e.g. in Section 2 we have preferred a set valued mapping to a correspondence.

Finally to see the relationship of our works with those of Mas–Collel [8] and Gale and Mas–Collel [5] we refer to Shafer and Sonnenschein [12].

2. An extension of Fan's fixed point theorem.

In this section we prove the fixed point theorem which is used in the next section to show the existence of an equilibrium point of an abstract economy given by preference correspondences and we also prove that this fixed point theorem includes the fixed point theorem of Fan as a special case.

Lemma 2.1. *We will need the following lemma proved by Fan* [3]. *Let X be a uniform space. Let A be a closed subset and B a compact subset of X. Then for any entourage U of the uniformity of X, there exists an entourage V such that*

$$V(A) \cap V(B) \subset U(A \cap B)$$

where $W(D) = \{x \in X : (x, a) \in W \text{ for some } a \in D\}$ for a subset D of X and an entourage W of the uniformity.

Let X and Y be two topological spaces. Then a set valued mapping $f : X \to 2^Y$ with closed values is said to be *upper semicontinuous* (*almost upper semicontinuous*) if for each $x_0 \in X$ and each open set U in Y with $f(x_0) \subset U$, there is an open set W with $x_0 \in W$ such that $f(x) \subset U(f(x) \subset \overline{U})$ for all $x \in W$. Trivially an upper semicontinuous mapping is almost upper semicontinuous.

A set valued mapping : $X \to 2^Y$ is said to be *lower semicontinuous* if for each $x_0 \in X$ we have: $y \in f(x_0)$ and $x_\delta \to x_0$ where $\{x_\delta : \delta \in D\}$ is a net in X and D is an ordered set imply the existence of a net $\{y_\delta : \delta \in D\}$ with $y_\delta \in f(x_\delta)$ and $y_\delta \to y$ (an equivalent definition in terms of open sets can be given). A set valued mapping $f : X \to 2^Y$ is said to be *continuous* if f is both upper semicontinuous and lower semicontinuous.

Theorem 2.1. *Let K be nonempty compact convex subset of a locally convex topological vector space E. Let $f : K \to 2^K$ be two set valued mappings such that*

(i) *f is almost upper semicontinuous and g is upper semicontinuous;*

and

(ii) *for each $x \in X, g(x) \neq \emptyset$ and* co $g(x) \subset f(x)$.

Then there exists a point $x_0 \in K$ such that $x_0 \in f(x_0)$.

Proof : For the sake of completness we will repeat the argument of Fan [3 wherever we find it necessary.

Let $\mathcal{B}$ be an open base of neighbourhoods of 0 of E such that each $V \in \mathcal{B}$ is convex and symmetric, i.e. $V = -V$. For each $V \in \mathcal{B}$, we define the sets

$$F_V = \{x \in K : x \in f(x) + \overline{V}\}$$

and

$$G_V = \{x \in K : x \in \text{co}\, g(x) + \overline{V}\},$$

where $\overline{V}$ denotes the closure of V.
Then by the condition of the theorem $G_V \subset F_V$ for each $V \in \mathcal{B}$. We first prove that $G_V \neq \emptyset$ for each $V \in \mathcal{B}$. To this end we consider a $V \in \mathcal{B}$ arbitrary but fixed. Since K is compact, there exists a finite number of points $x_1, x_2, \ldots, x_n$ in K such that $K \subset \bigcup_{i=1}^{n}(x_i + V)$. Let C be the closed convex hull of $\{x_1, x_2, \ldots, x_n\}$. For each $x \in C$, we define

$$g_V(x) = (g(x) + \overline{V}) \cap C.$$

Then clearly for each $x \in C, g_V(X)$ is a nonempty closed and hence compact subset of C. Now we prove that g_V is upper semicontinuous. Let $x_0 \in C$ and U be an open set in E such that $g_V(x_0) \subset U$. Since $g_V(x_0)$ is compact, we can find $V_1 \in \mathcal{B}$ such that $V_1 + g_V(x_0) \subset U$ (e.g. see Kelley and Namioka [6, 5.2 (vi), p. 35]). Now by Lemma 1.1 we can find $V_2 \in \mathcal{B}$ such that

$$(V_2 + g(x_0) + \overline{V}) \cap (V_2 + C) \subset V_1 + [(g(x_0) + \overline{V}) \cap C].$$

Then it follows that

$$(V_2 + g(x_0) + \overline{V}) \cap C \subset V_1 + g_V(x_0) \subset U.$$

Now by the upper semicontinuity of g, there exists a neighbourhood W of x_0 such that $g(x) \subset V_2 + g(x_0)$ for all $x \in W \cap K$.
Thus for all $x \in W \cap C$,

$$g_V(x) = (g(x) + \overline{V}) \cap C \subset [V_2 + g(x_0) + \overline{V}] \cap C \subset U.$$

Thus $g_V : C \to 2^C$ is an upper semicontinuous compact valued mapping. Now since C is a compact convex subset of finite dimensional subspace, the set valued mapping $h : C \to 2^C$ defined by $h(x) = \text{co } g_V(X), x \in C$ is upper semicontinuous and compact valued (e.g. see Nikaido [**10**, Theorems 4.8 and Corollary to Theorem 2.9]. Hence by Kakutani's fixed point theorem there is a point $x_0 \in C$ such that

$$x_0 \in h(x_0) = \text{co } g_V(x_0) = \text{co}\,[(g(x_0) + \overline{V}) \cap C] \subset (\text{co } g(x_0) + \overline{V}) \cap C,$$

i.e. $x_0 \in G_V$.
Since V is arbitrary, $G_V \neq \emptyset$ for each $V \in \mathcal{B}$. Hence F_V is nonempty for each $v \in \mathcal{B}$ as $G_V \subset F_V$.

In our next move, we prove that F_V is closed for each $V \in \mathcal{B}$. Again we consider a fixed but arbitrary $V \in \mathcal{B}$. We prove that $F_V^c = K \setminus F_V$ is open. Let $y \in F_V^c$. Then y is not contained in the closed set $f(y) + \overline{V}$. It is possible to find a $V' \in \mathcal{B}$ such that $(y + \overline{V}') \cap (f(y) + \overline{V} + \overline{V}') = \emptyset$. Now by almost upper semicontinuity of f, there is a $W \in \mathcal{B}$ such that $f(z) \subset \overline{f(y) + V'} \subset f(y) + \overline{V}'$ (as $f(y)$ is compact and $\overline{V}'$ is closed, $f(y) + \overline{V}'$ is closed, e.g. see Kelley and Namioka [**6**, 5.2 (vii), p. 35]) for all $z \in (y + W) \cap K$. We may assume $W \subset V'$. It then follows that for any $z \in (y+W) \cap K, z \notin f(z) + \overline{V}$, i.e. $z \notin F_V$ (for otherwise $z \in (y+W) \cap K \subset y + V'$ and $z \in f(z) + \overline{V} \subset f(y) + \overline{V} + \overline{V}'$ which leads to a contradiction). Thus we have proved that F_V^c is open. Since the finite intersection of members of $\mathcal{B}$ is again in $\mathcal{B}$, it follows that the family $\{F_V : V \in \mathcal{B}\}$ of closed sets has finite intersection property. Hence $\bigcap_{V \in \mathcal{B}} F_V \neq \emptyset$. Now it is easy to prove that for any point $x_0 \in \bigcap_{V \in \mathcal{B}} F_V, x_0 \in f(x_0)$. ∎

Lemma 2.2. *Let K be a nonempty compact convex subset of a locally convex topological vector space E and $g : K \to 2^K$ an almost upper semicontinuous set valued mapping. Then the set valued mapping $f : K \to 2^K$ defined by $f(x) = \overline{\mathrm{co}}\, g(x), x \in K$ is almost upper semicontinuous.*

PROOF : Let U be an open set containing $f(x)$. Since $f(x)$ is a compact subset of E, we can find a convex open neighbourhood N of 0 such that $f(x) + N \subset U$ (e.g. see Kelley and Namioka [**6**, 5.2. (vi), p. 35] and note that E is locally convex). Clearly $V = f(x) + N$ is convex open set containing $f(x)$ and $V \subset U$. Now since g is almost upper semicontinuous, there is an open set W containing x such that $g(y) \subset \overline{V}$ for every $y \in W \cap K$. Then as V is convex, $f(y) = \overline{\mathrm{co}}\, g(y) \subset \overline{V} \subset \overline{U}$ for each $y \in W \cap K$. ■

Corollary 2.1. *Let K be a nonempty compact convex subset of a locally convex topological vector space E and $g : K \to 2^K$ be an upper semicontinuous set valued mapping such that for each $x \in K, g(x)$ is a nonempty subset of K. Then there exists a point $x_0 \in K$ such that $x_0 \in \overline{\mathrm{co}}\, g(x_0)$.*

PROOF : We define the set valued mapping $f : K \to 2^K$ by $f(x) = \overline{\mathrm{co}}\, g(x), x \in K$. Then by Lemma 2.2, $f(x)$ is almost upper semicontinuous. Clearly the pair (f, g) satisfies all the conditions of Theorem 2.1. Hence there exists a point $x_0 \in K$ such that $x_0 \in f(x_0)$. ■

Corollary 2.2. (Fan's fixed point theorem). *Let K be a nonempty compact convex subset of a locally convex topological vector space. If $f : K \to 2^K$ is an upper semicontinuous set valued mapping such that for each $x \in K, f(x)$ is a nonempty convex subset of K, then there exists a point $x_0 \in K$ such that $x_0 \in f(x_0)$.*

PROOF : If we take $f = g$ in Theorem 1.1, the corollary is obtained. ■

Theorem 2.2. *Let X be a topological space and $\{y_\alpha : \alpha \in I\}$ be a family of compact spaces. If for each $\alpha \in I, f_\alpha : X \to 2^{Y_\alpha}$ is an almost upper semicontinuous (upper semicontinuous) set valued mapping and $Y = \prod_{\alpha \in I} Y_\alpha$, then the set valued mapping $f : X \to 2^Y$ defined by*

$$f(x) = \prod_{\alpha \in I} f_\alpha(x), \; x \in X$$

is almost upper semicontinuous (upper semicontinuous).

The case of upper semicontinuous mappings has been proved by Fan [**3**, Lemma 3].
PROOF : First note that for each $x \in X$, $f(x)$ is closed. Only a slight modification of the proof of the corresponding result for product of upper semicontinuous mappings (Fan [**3**]) is needed. We will indicate this modification for the first case when I is finite from which the changes required for the second case when I is infinite will be clear. Let $I = \{1, 2, \ldots, n\}$ and let U be an open set in $Y = \prod_{i=1}^n Y_i$ such that $f(x_0) = \prod_{i=1}^n f_i(x_0) \subset U$. Since $f_i(x_0)$ is compact for each i, we can, as in Fan, find an open set U_i in Y such that $f_i(x_0) \subset U_i \, (i = 1, 2, \ldots, n)$ and $\prod_{i=1}^n U_i \subset U$. Now as for each $i = 1, 2, \ldots, n$, f_i is almost upper semicontinuous,

there exists a neighbourhood W_i of x_0 such that $f_i(x) \subset \overline{U}_i$ for each $x \in W_i$. Thus $f(x) = \prod_{i=1}^n f_i(x) \subset \prod_{i=1}^n \overline{U}_i = \overline{\prod_{pi=1}^n U_i} \subset \overline{U}$ for each $x \in W = \bigcap_{i=1}^n W_i$. ■

3. Equilibrium point of abstract economy.

In this section we consider both kinds of economy described in Section 1, an abstract economy given by utility functions and an abstract economy given by preference relations and prove the existence of an equilibrium point for either case.

Theorem 3.1. *Let $\mathcal{E} = \{X_\alpha, A_\alpha, U_\alpha : \alpha \in I\}$ be an abstract economy such that for each $\alpha \in I$*

(i) *X_α is a nonempty compact and convex subset of a locally convex topological vector space E_α;*

(ii) *$A_\alpha : X = \prod_{\alpha \in I} X_\alpha \to 2^{X_\alpha}$ is continuous correspondence so that for each $x \in X$, $A_\alpha(x)$ is a nonempty compact and convex subset of X_α;*

(iii) *$U_\alpha : X \to \mathbf{R}$ is continuous and is quasiconcave in x_α.*

Then there is an equilibrium point $\overline{x} = \{\overline{x}_\alpha\} \in X$ of the economy, i.e. for each $\alpha \in I, U_\alpha(\overline{x}) = U_\alpha[\overline{x}_\alpha, \overline{x}_{-\alpha}] = \sup_{z_\alpha \in A(\overline{x}_\alpha)} U_\alpha[z_\alpha, \overline{x}_{-\alpha}]$.

PROOF : Let for each $\alpha \in I, F_\alpha(x) = \{y_\alpha \in X_\alpha : U_\alpha[y_\alpha, x_{-\alpha}]\} = \sup_{z_\alpha \in A_\alpha(x)} U_\alpha[z_\alpha, x_{-\alpha}]$. Since U_α is continuous and $A_\alpha(x)$ is compact, $F_\alpha(x) \neq \emptyset$ and clearly $F_\alpha(x) \subset A_\alpha(x)$. Thus for each $\alpha \in I, F_\alpha : X \to 2^{X_\alpha}$ is a set valued mapping. We will now prove that F_α has a closed graph. To this end we fix $\alpha \in I$ and let $\{(x^\delta, y_\alpha^\delta) : \delta \in D\}$ be a net in $X \times X_\alpha$ such that $x^\delta \to x$ and $y_\alpha^\delta \to y_\alpha$ and $y_\alpha^\delta \in F_\alpha(x^\delta)$, i.e. $U_\alpha[y_\alpha^\delta, x_{-\alpha}^\delta] = \sup_{z_\alpha \in A_\alpha(x^\delta)} U_\alpha[z_\alpha, x_{-\alpha}^\delta]$, i.e.

$$\text{(A)} \qquad U_\alpha[y_\alpha^\delta, x_{-\alpha}^\delta] \geq U_\alpha[z_\alpha, x_{-\alpha}^\delta] \quad \text{for all } z_\alpha \in A_\alpha(x^\delta)$$

Now since $y_\alpha^\delta \in F_\alpha(x^\delta) \subset A_\alpha(x^\delta)$ by upper semicontinuity of A_α, $y_\alpha \in A_\alpha(x)$. Now let $z_\alpha \in A_\alpha(x)$ be arbitrary. Then by lower semicontinuity of A_α, there is $z_\alpha^\delta \in A_\alpha(x^\delta)$ such that $z_\alpha^\delta \to z_\alpha$. But from (A), we have $U_\alpha[y_\alpha^\delta, x_{-\alpha}^\delta] \geq U_\alpha[z_\alpha^\delta, x_{-\alpha}^\delta]$. Taking limit we obtain $U_\alpha[y_\alpha, x_{-\alpha}] \geq U_\alpha[z_\alpha, x_{-\alpha}]$. Thus we have proved that $U_\alpha[y_\alpha, x_{-\alpha}] = \sup_{z_\alpha \in A_\alpha(x)} U_\alpha[z_\alpha, x_{-\alpha}]$, i.e. $y_\alpha \in F_\alpha(x)$. Hence F_α has a closed graph and therefore $F_\alpha(x)$ is a closed subset of X_α for each $x \in X$.

Next we prove that $F_\alpha(x)$ is convex for each $x \in X$ and each $\alpha \in I$. Let $x \in X$ and $\alpha \in I$ be arbitrarily fixed. Let $y_\alpha^1, y_\alpha^2 \in F_\alpha(x)$ and $\overline{y}_\alpha = \lambda y_\alpha^1 + \mu y_\alpha^2, \lambda, \mu \geq 0$ and $\lambda + \mu = 1$. Then $U_\alpha[y_\alpha^1, x_{-\alpha}] = \sup_{z_\alpha \in A_\alpha(x)} U_\alpha[z_\alpha, x_{-\alpha}] = U_\alpha[y_\alpha^2, x_{-\alpha}]$. Suppose that $\overline{y}_\alpha \notin F_\alpha(x)$. Then there will exist $u_\alpha \in A_\alpha(x)$ such that $U_\alpha[u_\alpha, x_{-\alpha}] > U_\alpha[\overline{y}_\alpha, x_{-\alpha}]$. Let $u_0 = [u_\alpha, x_{-\alpha}]$. Since U_α is quasiconcave in α-th co-ordinate, the set $B = \{z_\alpha \in X_\alpha : U_\alpha[z_\alpha, x_{-\alpha}] > U_\alpha(u_0)\}$ is convex. Also by the continuity of $U_\alpha, \overline{B} = \{z_\alpha \in X_\alpha : U_\alpha[z_\alpha, x_{-\alpha}] \geq U_\alpha(u_0)\}$ and is therefore convex. Now since $u_\alpha \in A_\alpha(x)$, it follows that $y_\alpha^1 \in \overline{B}$ and $y_\alpha^2 \in \overline{B}$. Hence $\overline{y}_\alpha \in \overline{B}$, i.e. $U_\alpha[\overline{y}_\alpha x_{-\alpha}] \geq U_\alpha(u_0) = U_\alpha[u_\alpha, x_{-\alpha}]$ which is a contradiction.

Thus we have proved that for each $\alpha \in I$, the set valued mapping $F_\alpha : X \to 2^{X_\alpha}$ has closed graph and has closed convex value for each $x \in X$. Hence for each $\alpha \in I$, F_α is upper semicontinuous and hence by Lemma 3 of Fan (1952) the set valued mapping $F : X \to 2^X$ defined by $F(x) = \prod_{\alpha \in I} F_\alpha(x), x \in X$ is upper semicontinuous and is evidently closed convex valued. Hence by fixed point theorem of Fan (here Corollary 2.2) there is a point $\overline{x} \in X$ such that $\overline{x} \in F(\overline{x})$. Now it is easy to see that this point $\overline{x}$ is an equilibrium point of the economy $\mathcal{E}$. ∎

Corollary 3.1. *Let $\{X_\alpha : \alpha \in I\}$ be a family of nonempty compact convex sets, each in locally convex topological vector space E_α. Let for each $\alpha \in I, U_\alpha : X = \prod_{\alpha\in I} X_\alpha \to \mathbf{R}$ be a continuous function such that U_α is quasiconcave in x_α. Then there is a Nash equilibrium point.*

PROOF : For each $\alpha \in I$, we define the set valued mapping $A_\alpha : X \to 2^{X_\alpha}$ by $A_\alpha(x) = A_\alpha[x_\alpha, x_{-\alpha}] = X_\alpha$. Clearly A_α is continuous. Hence the Corollary follows from Theorem 3.1. ∎

Remark 3.1.

More general result than Corollary 3.1 is known, for instance this result is known in Hausdorff topological space (e.g. see Ma [7] and Fan [4]).

Theorem 3.2. *Let $\mathcal{E} = \{X_i, A_i, P_i : i \in I\}$ be an abstract economy, where I is a countable set. Assume that for each $i \in I$,*

(a) *X_i is a nonempty compact and convex subset of a locally convex metrizable space E_i;*
(b) *$A_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ is a continuous correspondence such that for each $x \in X$, $A_i(x)$ is nonempty and convex;*
(c) *$P_i : X \to 2^{X_i}$ has an open graph in $X \times X_i$; and*
(d) *for each $x = \{x_i\} \in X, x_i \notin \overline{\text{co}}\, P_i(x)$.*

PROOF : As in Shafer and Sonnenschein (1975) we define for each $i \in I$ a continuous mapping $U_i : X \times X_i \to \mathbf{R}$ by $U_i(x, y_i) = \inf_{(u,z_i)\in G_i^c} \varrho_i(P_i(x, y_i), (u, z_i))$ where G_i^c is the complement of the graph G_i of P_i and ϱ_i is the metric in $X \times X_i$. Now for each $i \in I$, we define the set valued mapping $F_i : X \to 2^{X_i}$ by $F_i(x) = \{y_i \in X_i : U_i(x, y_i) = \sup_{z_i\in X_i} U_i(x, z_i)\}$.

Since U_i is continuous and A_i (being nonempty valued and upper semicontinuous) is nonempty compact valued correspondence, it follows that $F_i(x)$ is nonempty for each $x \in X$ and by similar argument as given in Theorem 3.1 we can show that F_i has closed graph and is therefore upper semicontinuous. Hence by Theorem 2.2 the set valued mapping $F : X \to 2^X$ defined by $F(x) = \prod_{i\in I} F_i(x), x \in X$ is upper semicontinuous. Thus by our own Corollary 2.1 there is a point $\overline{x} \in X$ such that $\overline{x} \in \overline{\text{co}}\, F(x) \subset \prod_{i\in I} \overline{\text{co}}\, F_i(x)$. Now we repeat the same argument of Shafer and Sonnenschein to show that $\overline{x}$ is an equilibrium point of the economy $\mathcal{E}$. Since $F_i(\overline{x}) \subset A_i(\overline{x})$ and $A_i(\overline{x})$ is closed and convex, $\overline{x} \in \prod_{i\in I} \overline{\text{co}}\, F_i(x) \subset \prod_{i\in I} A_i(\overline{x})$. Thus $\overline{x}_i \in A_i(\overline{x})$ for each $i \in I$ where $\overline{x} = \{\overline{x}_i\}$. It remains to show that for each $i \in I, P_i(\overline{x}) \cap A_i(\overline{x}) = \emptyset$. If $z_i \in P_i(\overline{x}) \cap A_i(\overline{x})$, then $U_i(\overline{x}, z_i) > 0$. This implies that $U_i(\overline{x}, y_i) > 0$ for all $y_i \in F_i(\overline{x})$. Hence $z_i \in P_i(\overline{x}) \cap A_i(\overline{x})$ implies that $F_i(\overline{x}) \subset P_i(\overline{x})$. Thus $\overline{x}_i \in \overline{\text{co}}\, F_i(\overline{x}) \subset \overline{\text{co}}\, P_i(\overline{x})$ which contradicts (d). ∎

Remark 3.2.

Since in a finite dimensional space, co A of a compact subset A is compact, the condition (d) reduces to (iii) of Shafer and Sonnenschein. Thus the theorem is indeed a generalization of their theorem.

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