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On isomorphisms between σ-ideals on ω_1

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Abstract. Two σ-ideals on ω_1 are called n-isomorphic if there is an isomorphism between them expressible as a composition of n involutions. It is proved that any two isomorphic σ-ideals in ω_1 are 2-isomorphic. Further, we consider σ-ideals with bases of cardinality ω_1 and study the cases when two σ-ideals are isomorphic, 1-isomorphic, properly 2-isomorphic.

Keywords: σ-ideal, isomorphism, composition of involutions

Classification: 04A05

0. Introduction.

We use the standard set-theoretical notation (see e.g. [**K**]). The phrase "a σ-ideal on ω_1" will mean that we speak of a σ-ideal $\mathcal{I} \subseteq \mathcal{P}(\omega_1)$ such that $\omega_1 \notin \mathcal{I}$ and $\{x\} \in \mathcal{I}$ for all $x \in \omega_1$. A subfamily $\mathcal{F}$ of $\mathcal{I}$ is called a base of $\mathcal{I}$ if each member of $\mathcal{I}$ is contained in a member of $\mathcal{F}$. If the unions of all countable subfamilies of a family $\mathcal{H} \subseteq \mathcal{P}(\omega_1)$ form a base of $\mathcal{I}$, we say that $\mathcal{I}$ is generated by $\mathcal{H}$. For $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{P}(\omega_1)$, we denote by $\mathcal{F}_1 \oplus \mathcal{F}_2$ the family of all sets $E \subseteq \omega_1$ such that $E \subseteq A_1 \cup A_2$ for some $A_i \in \mathcal{F}_i$, $i = 1, 2$. Observe that if $\mathcal{I}$ nd $\mathcal{J}$ are σ-ideals on ω_1 and $\omega_1 \notin \mathcal{I} \oplus \mathcal{J}$, then $\mathcal{I} \oplus \mathcal{J}$ is the σ-ideal generated by $\mathcal{I} \cup \mathcal{J}$.

Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals on ω_1. We say that:

(i) $\mathcal{I}$ and $\mathcal{J}$ are orthogonal if $\mathcal{I} \oplus \mathcal{J} = \mathcal{P}(\omega_1)$ or, equivalently, if there are $A \in \mathcal{I}$ and $B \in \mathcal{J}$ such that $A \cup B = \omega_1$;

(ii) $\mathcal{I}$ has property (P^*) if the complement of each member of $\mathcal{I}$ contains an uncountable member of $\mathcal{I}$ (see [**Ba**]; cf. also condition (γ), Theorem H4 in [**M**]);

(iii) $\mathcal{I}$ and $\mathcal{J}$ are isomorphic if there is a bijection $f : \omega_1 \to \omega_1$ such that, for the bijection $f^* : \mathcal{P}(\omega_1) \to \mathcal{P}(\omega_1)$ given by $f^*(E) = f[E]$ (the image of E) for $E \in \mathcal{P}(\omega_1)$, we have $f^*[\mathcal{I}] = \mathcal{J}$; then f is called an isomorphism between $\mathcal{I}$ and $\mathcal{J}$ (cf. [**BTW**]);

(iv) a bijection $f : X \to X$ is an involution if $f = f^{-1}$ (or, equivalently, if $f \circ f$ is the identity);

(v) $\mathcal{I}$ and $\mathcal{J}$ are n-isomorphic, where n is a positive integer, if there are involutions $f_i : \omega_1 \to \omega_1$ ($i = 1, 2 \dots, n$) such that $f_1 \circ f_2 \circ \dots \circ f_n$ is an isomorphism between $\mathcal{I}$ and $\mathcal{J}$;

(vi) $\mathcal{I}$ and $\mathcal{J}$ are properly n-isomorphic if they are n-isomorphic and either $n = 1$ or $\mathcal{I}$ and $\mathcal{J}$ are not $(n-1)$-isomorphic;

(vii) $\mathcal{I}$ and $\mathcal{J}$ are strongly isomorphic if they are n-isomorphic for some n.

Obviously, if $\mathcal{I}$ and $\mathcal{J}$ are strongly isomorphic, the are isomorphic. Concerning the converse implication, the problem arises whether every bijection $f : \omega_1 \to \omega_1$ can be composed from a finite number of involutions. The author would like to thank Professor P. Simon for bringing him the solution which considerably simplified the previous version of the paper (he does not know where the fact stated in the solution can be found).

Proposition 0.1. *For any set X, every bijection $f : X \to X$ is composable from two involutions.*

PROOF : At first, consider the case when f is a shift from the set Z of all integers onto Z, given by $f(n) = n+1$. Then $f = h \circ g$ where g and h are involutions given by $g(n) = -n$ and $h(n) = -n+1$. Next, consider a general case. If $f(x) = y$, we write briefly $x \to y$. We shall find a set $T \subseteq X$ and a partition of X into sets $A(x)$, for $x \in T$, such that $x \in A(x)$ for any x, and every $A(x)$ is either of the form $A(x) = \{x_0, \dots, x_n\}$ where $n \in \omega$ and

$$x_0 \to x_1 \to \cdots \to x_n \to x_0, \tag{1}$$

or of the form $A(x) = \{x_i : i \in Z\}$ where

$$\cdots \to x_{-1} \to x_0 \to x_1 \to \dots \tag{2}$$

(in both cases, x_i's are distinct). This can be done inductively. Consider any $x_0 \in X$. Let $x_0 \in T$ and define a sequence $x_0 \to x_1 \to \cdots \to x_i \to \dots$ as long as possible to have all x_i's distinct. Then we get either (1), if the procedure if finite, or (2), if it is infinite. All terms of (1) or of (2) form $A(x_0)$. Next, if possible, consider any $x_* \in X \setminus A(x_0)$. Let $x_* \in T$ and define $A(x_*)$ analogously as $A(x_0)$. The procedure will stop if the union of all constructed $A(x)$'s is X. Observe that (1) can be written in the form

$$\cdots \to x_0 \to x_1 \to \cdots \to x_n \to x_0 \to x_1 \to \cdots \to x_n \to x_0 \to \dots . \tag{1'}$$

For each $x \in T$, the function $f \mid A(x)$ is a bijection on $A(x)$ and can be expressed as $h^{(x)} \circ g^{(x)}$ where $g^{(x)}$ and $h^{(x)}$ are defined for (1') or (2), analogously as g and h for the shift on Z. It is easy to verify that $g^{(x)}$ and $h^{(x)}$ are well-defined involutions on $A(x)$. Finally, observe that the mappings g and h on X such that $g \mid A(x) = g^{(x)}$ and $h \mid A(x) = h^{(x)}$ for $x \in T$ are involutions and $f = h \circ g$. ∎

From Proposition 0.1 we get

Corollary 0.2. *Two σ-ideals $\mathcal{I}$ and $\mathcal{J}$ on ω_1 are isomorphic if and only if they are strongly isomorphic. Moreover, if $\mathcal{I}$ and $\mathcal{J}$ are isomorphic, they are 2-isomorphic.*

An interesting fact on isomorphisms between σ-ideals is contained in the Sierpiński–Erdös theorem (see [S], [E], [O], [M]) concerning Lebesgue null sets and meager sets on the real line $\mathbf{R}$. Note that Continuum Hypothesis (CH) is assumed there. If $\mathbf{R}$ is replaced by ω_1, then CH may be omitted and the general version of the theorem is the following (cf. [O]):

Theorem 0.3. *Any two σ-ideals on ω_1 with bases of cardinality ω_1 and with property (P^*) are isomorphic. If, additionally, they are orthogonal, then they are 1-isomorphic.*

Note that a σ-ideal on ω_1 with a base of cardinality ω_1 and with property (P^*) has a nice characterization. Before we give it, let us recall some definitions.

A family $\mathcal{F} \subseteq \mathcal{P}(\omega_1)$ is called almost disjoint (in abbr. a.d.) on ω_1 if $|A| = \omega_1$ for each $A \in \mathcal{F}$, and $|A \cap B| < \omega_1$ for any distinct $A, B \in \mathcal{F}$. It is known that there is no maximal (with respect to inclusion) a.d. family on ω_1 of cardinality ω_1. So, there are a.d. families on ω_1 of cardinalities $> \omega_1$ and, among them, the maximal family obtained by Zorn's lemma. For details, see [K]. By a partition of ω_1 we mean a family of pairwise disjoint subsets of ω_1 with the union equal to ω_1.

Proposition 0.4. *Let $\mathcal{I}$ be a σ-ideal on ω_1. The following statements are equivalent:*

(a) *$\mathcal{I}$ has (P^*) and a base of cardinality ω_1,*
(b) *$\mathcal{I}$ is generated by a partition of ω_1 into ω_1 uncountable sets,*
(c) *$\mathcal{I}$ is generated by an a.d. family of cardinality ω_1,*
(d) *$\mathcal{I}$ has (P^*) and is generated by a family of ω_1 uncountable sets.*

PROOF : The only nontrivial implication among (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) is the first one and it was established in [S] (see also [O]). ∎

Note that the implication (a) $\Rightarrow$ (b) is a crucial point in the proof of 0.3.

Theorem 0.3 suggests several problems concerning isomorphisms between σ-ideals and, among them, the following questions for σ-ideals $\mathcal{I}$ and $\mathcal{J}$ on ω_1 with bases of cardinality ω_1:

1° Assume that $\mathcal{I}$ and $\mathcal{J}$ have not (P^*). When are they isomorphic, properly n- isomorphic $(n = 1, 2)$?
2° Assume that $\mathcal{I}$ and $\mathcal{J}$ have (P^*) and are not orthogonal. When are they properly n-isomorphic $(n = 1, 2)$?

In the paper we try to give the answers.

Notice that σ-ideals generated by a.d. families on ω_1 of cardinality $> \omega_1$ are natural examples of σ-ideals without bases of cardinality ω_1. Other examples can be derived from [BTW]. It seems that there are many various nonisomorphic σ-ideals without bases of cardinality ω_1 (see e.g. Theorem 5.10 in [BTW]) and this can enable one to describe the corresponding equivalence classes in a simple way. Note that some new conditions guaranteeing isomorphisms between σ-ideals on ω_1 which have no bases of cardinality ω_1 are obtained in [P].

1. Simpler cases.

At first, note that if $A \subseteq \omega_1$ and $\mathcal{I}$ is a σ-ideal such that among subsets of A only countable sets belong to $\mathcal{I}$, then $\mathcal{I} \cap \mathcal{P}(A) = [A]^{<\omega_1}$.

Lemma 1.1. *Let $\mathcal{I}$ be a σ-ideal on ω_1 with an uncountable member and with a base of cardinality ω_1. Then $\mathcal{I}$ has not (P^*) if and only if*

$$\mathcal{I} = \mathcal{P}(A) \oplus [\omega_1 \setminus A]^{<\omega_1}$$

for some $A \subseteq \omega_1$ *such that* $|A| = |\omega_1 \setminus A| = \omega_1$.

PROOF : Necessity. If $\mathcal{I}$ has not (P^*), there is $A \in \mathcal{I}$ such that among subsets of $\omega_1 \setminus A$ only countable sets belong to $\mathcal{I}$. Of course, $\omega_1 \setminus A$ is uncountable (otherwise $\omega_1 \in \mathcal{I}$). The set A is also uncountable, since, in the opposite case, $\mathcal{I}$ would consist of countable sets only. Thus it follows that $\mathcal{I}$ is of the desired form.

Sufficiency. The set A belongs to $\mathcal{I}$ and its complement contains no uncountable member of $\mathcal{I}$. Thus $\mathcal{I}$ has not (P^*). Let $\omega_1 \setminus A = \{x_\alpha : \alpha < \omega_1\}$. The family $\{A\} \cup \{\{x_\nu : \nu < \alpha\} : \alpha < \omega_1\}$ forms a base of $\mathcal{I}$. ∎

Proposition 1.2 (cf. Theorem H4 in [M]). *Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals on ω_1 with bases of cardinality ω_1. Then $\mathcal{I}$ and $\mathcal{J}$ are isomorphic if and only if exactly one of the cases holds:*

(1) $\mathcal{I} = \mathcal{J} = [\omega_1]^{<\omega_1}$,
(2) $\mathcal{I} = \mathcal{P}(A) \oplus [\omega_1 \setminus A]^{<\omega_1}$, $\mathcal{J} = \mathcal{P}(B) \oplus [\omega_1 \setminus B]^{<\omega_1}$ *for some* $A \subseteq \omega_1$ *and* $B \subseteq \omega_1$ *such that* $|A| = |\omega_1 \setminus A| = |B| = |\omega_1 \setminus B| = \omega_1$,
(3) $\mathcal{I}$ *and* $\mathcal{J}$ *have* (P^*).

PROOF : Sufficiency.. In case (1), the assertion is trivial. In case (2), choose bijections $g : A \to B$ and $h : \omega_1 \setminus A \to \omega_1 \setminus B$. Then $f : \omega_1 \to \omega_1$ equal to g on A and to h on $\omega_1 \setminus A$ is the desired isomorphism. In case (3), recall the proof of the first statement in Theorem 0.3 (see [S]). By Proposition 0.4, we get partitions $\{X_\alpha : \alpha < \omega_1\}$ and $\{Y_\alpha : \alpha < \omega_1\}$ of ω_1 into uncountable sets, generating $\mathcal{I}$ and $\mathcal{J}$, respectively. Consider bijections $f_\alpha : X_\alpha \to Y_\alpha$ for $\alpha < \omega_1$ and let $f : \omega_1 \to \omega_1$ be equal to f_α on $X_\alpha, \alpha < \omega_1$. Thus f is the desired isomorphism.

Necessity. Assume that (1) does not hold. Thus both $\mathcal{I}$ and $\mathcal{J}$ possess uncountable members. Assume that (3) does not hold. Then at least one of the σ-ideals $\mathcal{I}$ and $\mathcal{J}$ has not (P^*). If neither of them has (P^*), then (2) holds by Lemma 1.1. So assume that, for example, $\mathcal{I}$ has not (P^*) and $\mathcal{J}$ has (P^*). By Lemma 1.1, we get $\mathcal{I} = \mathcal{P}(A) \oplus [\omega_1 \setminus A]^{<\omega_1}$ for the respective A. Suppose that f is an isomorphism between $\mathcal{I}$ and $\mathcal{J}$. Since $A \in \mathcal{I}$, the set $B = f[A]$ belongs to $\mathcal{J}$. By (P^*), choose an uncountable $E \subseteq \omega_1 \setminus B$ belonging to $\mathcal{J}$. Of course, $E = f[D]$ for some $D \subseteq \omega_1 \setminus A$. Since f is an isomorphism between $\mathcal{I}$ and $\mathcal{J}$, we have $D \in \mathcal{I}$, hence D is countable. Thus E is countable, a contradiction. ∎

Proposition 1.2 describes, in fact, the equivalence classes (given by (1), (2) and (3)) when the relation "to be isomorphic" is restricted to the set of all σ-ideals on ω_1 with bases of cardinality ω_1. Next, one can ask the question when these σ-ideals are properly n-isomorphic for $n = 1, 2$. To answer it, we consider cases (1), (2) and (3) separately.

In case (1), the situation is trivial: the σ-ideal $[\omega_1]^{<\omega_1}$ is 1-isomorphic to itself. Next, let us study case (2).

Proposition 1.3. *Assume that $\mathcal{I}$ and $\mathcal{J}$ fulfil* (2). *Then*

(α) *if either* $|A \triangle B| < \omega_1$ *or* $|A \setminus B| = |B \setminus A| = \omega_1$, *then $\mathcal{I}$ and $\mathcal{J}$ are 1-isomorphic;*
(β) *if either* $|A \setminus B| < \omega_1 = |B \setminus A|$ *or* $|B \setminus A| < \omega_1 = |A \setminus B|$, *then $\mathcal{I}$ and $\mathcal{J}$ are properly 2-isomorphic.*

PROOF of (α): If $|A\Delta B| < \omega_1$, then $\mathcal{I} = \mathcal{J}$ and the identity is the desired isomorphism. So, assume that $|A \setminus B| = |B \setminus A| = \omega_1$. Consider a bijection $g : A \setminus B \to B \setminus A$ and define $f : \omega_1 \to \omega_1$ by the formula

$$f = \begin{cases} g & \text{on } A \setminus B \\ g^{-1} & \text{on } B \setminus A \\ \text{the identity} & \text{on } \omega_1 \setminus (A\Delta B). \end{cases}$$

This is the desired isomorphism.

PROOF of (β): From 0.2 it follows that $\mathcal{I}$ and $\mathcal{J}$ are 2-isomorphic. However, we now give a short proof. Put $C = \omega_1 \setminus (A \cup B)$. From (2) and the assumptions of (β) we easily deduce that $|C| = |\omega_1 \setminus C| = \omega_1$. Define $\mathcal{K} = \mathcal{P}(C) \oplus [\omega_1 \setminus C]^{<\omega_1}$. Since $|A \setminus C| = |C \setminus A| = \omega_1 = |B \setminus C| = |C \setminus B|$, therefore, by ($\alpha$), there are isomorphisms f and g such that $f^*[\mathcal{I}] = \mathcal{K}$, $f = f^{-1}$ and $g^*[\mathcal{K}] = \mathcal{J}$, $g = g^{-1}$, respectively. The composition $g \circ f$ guarantees that $\mathcal{I}$ and $\mathcal{J}$ are 2-isomorphic. To show that they are properly 2-isomorphic, suppose that there is an isomorphism h between $\mathcal{I}$ and $\mathcal{J}$ such that $h = h^{-1}$. Let, for instance, $|A \setminus B| < |B \setminus A| = \omega_1$. Since $A \cap B \in \mathcal{I} \cap \mathcal{J}$, we have $h[A \cap B] \in \mathcal{I} \cap \mathcal{J}$. From the definitions of $\mathcal{I}$ and $\mathcal{J}$ it follows that $|h[A \cap B] \setminus (A \cap B)| < \omega_1$. Since $h \circ h$ is the identity, we have

$$(*) \qquad |(A \cap B) \setminus h[A \cap B]| < \omega_1.$$

From $B \in \mathcal{J}$ we get $h[B] \in \mathcal{I}$, and thus $|h[B] \setminus A| < \omega_1$. Moreover, $|h[B] \setminus (A \cap B)| < \omega_1$ since $|A \setminus B| < \omega_1$. Now, by ($*$), we have $|h[B] \setminus h[A \cap B]| < \omega_1$, which is impossible because $|h[B] \setminus h[A \cap B]| = |h[B \cap A]| = |B \setminus A| = \omega_1$. ∎

Proposition 1.3 gives a complete characterization of properly n-isomorphic pairs of σ-ideals (for $n = 1, 2$) in case (2) of 1.2. It seems more difficult to obtain a respective result for case (3). This will be discussed in the next section.

2. Strong isomorphisms of normal σ-ideals.

In the sequel, σ-ideals on ω_1 with the property (P^*) and with bases of cardinality ω_1 will be called normal.

Theorem 0.3 shows that orthogonality is a sufficient condition for two normal σ-ideals to be 1-isomorphic. However, this condition is not necessary, which follows from the next two propositions.

Proposition 2.1. *If $\mathcal{I}$ and $\mathcal{J}$ are orthogonal normal σ-ideals, then $\mathcal{I} \cap \mathcal{J}$ is normal.*

PROOF : Let $\{X_\alpha : \alpha < \omega_1\}$ and $\{Y_\alpha : \alpha < \omega_1\}$ be the respective partitions of ω_1, associated with $\mathcal{I}$ and $\mathcal{J}$ by Proposition 0.4 (b). Since $\mathcal{I}$ and $\mathcal{J}$ are orthogonal, we can choose these partitions so that $X_0 \cup Y_0 = \omega_1$ and $X_0 \cap Y_0 = \emptyset$. Then it is easily seen that $\mathcal{I} \cap \mathcal{J}$ is generated by the partition $\{X_\alpha : 0 < \alpha < \omega_1\} \cup \{Y_\alpha : 0 < \alpha < \omega_1\}$. Hence the assertion follows from Proposition 0.4. ∎

Proposition 2.2. *For each normal σ-ideal $\mathcal{I}$, there is a normal σ-ideal $\mathcal{J}$ such that $\mathcal{I} \cap \mathcal{J} = [\omega_1]^{<\omega_1}$, and $\mathcal{I}$ and $\mathcal{J}$ are 1-isomorphic.*

PROOF : Let $\{X_\alpha : \alpha < \omega_1\}$ be the respective partition of ω_1, associated with $\mathcal{I}$ by Proposition 0.4. Assume that $X_\alpha = \{x^\alpha_\gamma : \gamma < \omega_1\}$, $\alpha < \omega_1$. Define $Y_\gamma = \{x^\alpha_\gamma :$

$\alpha < \omega_1\}$, $\gamma < \omega_1$, and let $\mathcal{J}$ be the σ-ideal generated by the partition $\{Y_\gamma : \gamma < \omega_1\}$. Then $\mathcal{J}$ is normal by Proposition 0.4 and it is easily seen that $\mathcal{I} \cap \mathcal{J} = [\omega_1]^{<\omega_1}$. The involution $f : \omega_1 \to \omega_1$, given by $f(x^\alpha_\gamma) = x^\gamma_\alpha$ for $\langle \alpha, \gamma \rangle \in \omega_1 \times \omega_1$, is the desired isomorphism between $\mathcal{I}$ and $\mathcal{J}$. ∎

Let us give a simple criterion for two normal σ-ideals to be properly 2-isomorphic.

Proposition 2.3. *If $\mathcal{I}$ and $\mathcal{J}$ are normal σ-ideals such that $\mathcal{I} \subsetneqq \mathcal{J}$, then $\mathcal{I}$ and $\mathcal{J}$ are properly 2-isomorphic.*

PROOF : By Corollary 0.2, the σ-ideals $\mathcal{I}$ and $\mathcal{J}$ are 2-strongly isomorphic However, let us give a direct proof. Let $\{X_\alpha : \alpha < \omega_1\}$ and $\{Y_\alpha : \alpha < \omega_1\}$ be the respective partitions of ω_1, associated with $\mathcal{I}$ and $\mathcal{J}$ by Proposition 0.4. By the assumption, we may suppose that $X_0 = Y_0$. Since $\omega_1 \times \omega_1$ and ω_1 are equipotent, we can partition X_0 into disjoint uncountable sets Z_α, $0 < \alpha < \omega_1$. Consider bijections $f_\alpha : Z_\alpha \to X_\alpha$ and $g_\alpha : Z_\alpha \to Y_\alpha$ for $0 < \alpha < \omega_1$. Define $f : \omega_1 \to \omega_1$ and $g : \omega_1 \to \omega_1$ by

$$f = \begin{cases} f_\alpha & \text{on } Z_\alpha, \quad 0 < \alpha < \omega_1 \\ f_\alpha^{-1} & \text{on } X_\alpha, \quad 0 < \alpha < \omega_1 \end{cases}$$

and

$$g = \begin{cases} g_\alpha & \text{on } Z_\alpha, \quad 0 < \alpha < \omega_1 \\ g_\alpha^{-1} & \text{on } Y_\alpha, \quad 0 < \alpha < \omega_1 . \end{cases}$$

Thus f and g are involutions and $(g \circ f)^* [\mathcal{I}] = \mathcal{J}$ (cf. (iii) in Section 0).

To finish the proof, suppose that $h : \omega_1 \to \omega_1$ is an isomorphism between $\mathcal{I}$ and $\mathcal{J}$ such that $h = h^{-1}$. Then we have $\mathcal{J} = h^* [\mathcal{I}] \subsetneqq h^* [\mathcal{J}] = \mathcal{I}$, a contradiction. ∎

Problem 2.4. Characterize the set of all 1-isomorphic (or properly 2-isomorphic) normal σ-ideals on ω_1.

Finally, note one more property of normal σ-ideals.

Proposition 2.5. *If $\mathcal{I}$ and $\mathcal{J}$ are nonorthogonal normal σ-ideals, then $\mathcal{I} \oplus \mathcal{J}$ is normal.*

PROOF : By Proposition 0.4, let us associate with $\mathcal{I}$ and $\mathcal{J}$ the respective partitions $\{X_\alpha : \alpha < \omega_1\}$ and $\{Y_\alpha : \alpha < \omega_1\}$. It suffices to find a respective partition $\{Z_\alpha : \alpha < \omega_1\}$ for $\mathcal{I} \oplus \mathcal{J}$. Since $\mathcal{I}$ and $\mathcal{J}$ are not orthogonal, we have

$$(**) \qquad \text{for any } A \in \mathcal{I} \oplus \mathcal{J}, \text{ there are } \alpha < \omega_1 \text{ and } \beta < \omega_1 \text{ such that } |(X_\alpha \cup Y_\beta) \setminus A| = \omega_1.$$

Fix a well-ordering $\prec$ of $\omega_1 \times \omega_1$ isomorphic to the natural ordering of ω_1. Define $Z_0 = X_0 \cup Y_0$. Next, assume that $0 < \nu < \omega_1$, and that the sets $Z_\gamma \in \mathcal{I} \oplus \mathcal{J}$ for $\gamma < \nu$ are defined. By $(**)$, choose the first (with respect to $\prec$) pair $\langle \alpha_\nu, \beta_\nu \rangle \in \omega_1 \times \omega_1$ for which $|(X_{\alpha_\nu} \cup Y_{\beta_\nu}) \setminus \bigcup_{\gamma < \nu} Z_\gamma| = \omega_1$. Put

$$Z_\nu = \bigcup_{\langle \alpha, \beta \rangle \preceq \langle \alpha_\nu, \beta_\nu \rangle} (X_\alpha \cup Y_\beta) \setminus \bigcup_{\gamma < \nu} Z_\gamma.$$

This ends the induction. It easily follows that the sets Z_α, $\alpha < \omega_1$ form the required partition. ∎

References

[Ba] Balcerzak M., *Some properties of ideals of sets in Polish spaces*, preprint.

[BTW] Baumgartner J.E., Taylor A.D., Wagon S., *Structural properties of ideals*, Dissert. Math. **197** (1982).

[E] Erdös P., *Some remarks on the set theory*, Ann. Math. **44** (1943), 643–646.

[K] Kunen K., *Set Theory*, North Holland, 1980.

[M] Morgan J.C.II, *Point Set Theory*, Marcel Dekker Inc., 1990.

[O] Oxtoby J.C., *Measure and Category*, Springer–Verlag, 1971.

[P] Plewik Sz., *Isomorphic ideals*, preprint.

[S] Sierpiński W., *Sur la dualité entre la première catégorie et la mesure nulle*, Fund. Math. **22** (1934), 276–280.

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