Alexander Abian
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# SOLVABILITY OF INFINITE SYSTEMS OF LINEAR EQUATIONS

ALEXANDER ABIAN
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Let $S$ be a finite or infinite system of polynomial equations over a field $F$. It is not true in general that if every proper subsystem of $S$ has a solution in $F$ then $S$ has a solution in $F$. For instance, the system $S$ of polynomial equations

(1) $$(a - x)\, y_a - 1 = 0 \qquad \text{with } a \in F$$

is such that every proper subsystem of it has a solution in $F$, however, the entire system has no solution in $F$. Indeed, if $P$ is a proper subsystem of (1) such that, say, the equation $(b - x)\, y_b - 1 = 0$ does not appear in $P$, then a solution of $P$ is given by $x = b$ and $y_a = (a - b)^{-1}$ which exists since $a \neq b$. Nevertheless, the entire system (1) has no solution in $F$ since if (1) had a solution in $F$ with, say, $x = r$, then the equation $(r - x)\, y_r - 1 = 0$ would have no solution in $F$.

In sharp contrast to the above is the case of a system of linear equations over a field. As shown below any system (finite or infinite) $L$ of linear equations over a field $F$ has a solution in $F$ under a weaker assumption; namely, the assumption that every finite subsystem of $L$ has a solution in $F$.

**Theorem.** *Let $(L_i = 0)_{i\in E}$ be a (not necessarily finite) system of linear equations $L_i = 0$ over a field $F$. Then the system $(L_i = 0)_{i\in E}$ has a solution in $F$ if and only if every finite subsystem of it has a solution in $F$.*

Proof. Clearly, if the entire system has a solution in $F$ then every finite subsystem of it has a solution in $F$.

Thus, in what follows we suppose that every finite subsystem of $(L_i = 0)_{i\in E}$ has a solution in $F$ and we prove that the entire system has a solution in $F$.

Let $(x_i)_{i\in U}$ be the set of all the variables appearing in the system $(L_i = 0)_{i\in E}$. Moreover, let $(L_i)_{i\in V}$ be the set of all linear polynomials in variables $(x_i)_{i\in U}$ with coefficients in $F$, including the constant polynomials, i.e., $F \subseteq (L_i)_{i\in V}$.

Next, let $(L_i)_{i\in Q}$ be the subspace of $(L_i)_{i\in V}$ generated by the set of vectors $(L_i)_{i\in E}$. We prove that:

(1) $$r \notin (L_i)_{i\in Q} \qquad \textit{for every nonzero element } r \text{ of } F$$

Assume on the contrary that $r \in (L_i)_{i\in Q}$. Hence, $r$ is equal to a finite linear combination of vectors belonging to $(L_i)_{i\in E}$, i.e.

$$0 \neq r = \sum_{i\in N} r_i L_i \qquad \textit{for a finite subset } N \textit{ of } E \tag{2}$$

where $r_i \in F$ for every $i \in N$. However, by our supposition, the finite system $(L_i = 0)_{i\in N}$ has a solution in $F$. Thus, there exists a substitution by elements of $F$ of the variables appearing in $(L_i)_{i\in N}$ such that the right-hand side of the equality in (2) is equal to zero. But this contradicts (2). Hence, our assumption is false and (1) is established.

Now, let $(L_i)_{i\in B}$ be a basis for the subspace $(L_i)_{i\in Q}$ and let 1 be the multiplicative unit of $F$. From (1) and the axiom of choice it follows that $\{1\} \cup (L_i)_{i\in B}$ can be enlarged to a basis, say,

$$\{1\} \cup (L_i)_{i\in B} \cup (L_i)_{i\in D}. \tag{3}$$

for the entire vector space $(L_i)_{i\in V}$.

Finally, based on (3), we consider the linear mapping $f$ from $(L_i)_{i\in V}$ onto $F$ defined by:

$$f(1) = 1 \qquad \textit{and} \qquad f(L_i) = 0 \qquad \textit{for every} \qquad i \in (B \cup D) \tag{4}$$

Since $(L_i)_{i\in B}$ is a basis for $(L_i)_{i\in Q}$, we see that $f(L_i) = 0$ for every $i \in E$. But then from the linear additivity of $f$, it follows that $r_i = f(x_i)$ for every $i \in U$ gives a solution (in $F$) of the entire system $(L_i = 0)_{i\in E}$, as desired.

*A. Abian*
*Iowa State University*
*Ames, Iowa 50010*
*U.S.A.*