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ON ASYMPTOTIC PROPERTIES OF CENTRAL DISPERSIONS OF THE k-TH KIND OF $y'' = q(t)y$, k = 1, 2, 3, 4

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1. We begin our discussion with introducing the definitions of central dispersions of all four kinds together with some properties of theirs—as given in the monograph [1]—which will be of need in the sequel, too. We introduce next some properties of the solutions of the differential equation

(q) $$y'' = q(t)\,y, \qquad q \in C_I^0,$$

with $I = (a, \infty)$. Throughout this paper (q) will be considered oscillatory for $t \to \infty$ (which means every nontrivial solution of (q) with infinitely many zeros on every interval (t_0, ∞), $t_0 \leqq a$). In all that follows, we shall always eliminate trivial solutions of (q).

Let n be a positive integer, $x \in I$ and y be a solution of (q), $y(x) = 0$. If $\varphi_n(x)$ $(\varphi_{-n}(x))$ is the n-th zero point of y lying to the right (to the left) of the point x, then $\varphi_n(\varphi_{-n})$ is called the central dispersion of the 1st kind with the index n $(-n)$ of (q).

Let n be a positive integer, $q(t) < 0$ for $t \in I$; $x \in I$. Let y_1, y_2 be solutions of (q), $y_1(x) = y_2'(x) = 0$. If $\psi_n(x)$ $(\psi_n(x))$ $[\chi_n(x)(\chi_{-n}(x)), \omega_n(x)(\omega_{-n}(x))]$ is the n-th zero point of $y_2'[y_1', y_2]$ lying to the right (to the left) of the point x, then $\psi_n(\psi_{-n})$ $[\chi_n(\chi_{-n}), \omega_n(\omega_{-n})]$ is called the central dispersion of the 2nd [3rd, 4th] kind with the index n $(-n)$ of (q).

Since (q) is understood to be oscillatory for $t \to \infty$, the functions $\varphi_n, \psi_n, \chi_n, \omega_n$ are defined on I for every positive integer n. The functions $\varphi_{-n}, \psi_{-n}, \chi_{-n}, \omega_{-n}$ are generally defined on an interval of the type $(a_1, \infty) \subset I$ with a_1 depending on the natural number n and on the kind of the central dispersion; especially I° $({}^\circ I)$ will denote the definition interval of $\chi_{-1}(\omega_{-1})$. If $n = 1$, then instead of central dispersions of the k-th kind with the index 1 we shall briefly say basic dispersions of the k-th kind with $k = 1, 2, 3, 4$. In place of $\varphi_1, \psi_1, \chi_1, \omega_1$ we shall write $\varphi, \psi, \chi, \omega$.

In our considerations below we shall very often make use of the following formulas:

(1) $$\chi \circ \omega(t) = \psi(t), \qquad t \in I,$$

$$(2) \qquad \omega \circ \chi(t) = \varphi(t), \qquad t \in I,$$

$$(3) \qquad \begin{aligned} \varphi'(t) &= \frac{q(t_1)}{q(t_3)}, \qquad \psi'(t) = \frac{q(t)}{q \circ \psi(t)} \cdot \frac{q(t_4)}{q(t_2)}, \\ \chi'(t) &= \frac{q(t_1)}{q \circ \chi(t)}, \qquad \omega'(t) = \frac{q(t)}{q(t_2)}, \end{aligned}$$

where t_1, t_2, t_3, t_4 are appropriate numbers for which $t < t_1 < \chi(t) < t_3 < \varphi(t)$, $t < t_2 < \omega(t) < t_4 < \psi(t)$ holds.

The above formulas have been introduced in [1] by Borůvka, p. 116 and 123. Let us note that from the Theorem on page 122 [1] it follows that all the central dispersions of the 1st (2nd, 3rd, 4th) kind have the 3rd (1st) continuous derivative whenever they are defined.

It will be well to recall that in case of y being the solution of (q), $y(t) \neq 0$ for $t \in I_1$ $(\subset I)$ the function $r(t) := \dfrac{y'(t)}{y(t)}$, $t \in I_1$ is the solution of Riccati equation

$$r' + r^2 = q(t)$$

on I_1 (see [2] p. 392).

Lemma 1. *Let x be a number from I and let u, v denote a solution of* (p) *and* (q), *respectively, such that $u(x) = v(x) = 0$. Then*

$$\lim_{t \to x} \left(\frac{u'(t)}{u(t)} - \frac{v'(t)}{v(t)} \right) = 0.$$

Proof: First and foremost $\lim\limits_{t \to x^+} \dfrac{u'(t)}{u(t)} = \infty$, $\lim\limits_{t \to x^+} \dfrac{v'(t)}{v(t)} = \infty$ and therefore with L'Hospital's rule we are led to

$$\lim_{t \to x^+} \left(\frac{u'(t)}{u(t)} - \frac{v'(t)}{v(t)} \right) = \lim_{t \to x^+} \frac{u'(t)v(t) - u(t)v'(t)}{u(t)v(t)} = \lim_{t \to x^+} \frac{(p(t) - q(t))u(t)v(t)}{u'(t)v(t) + u(t)v'(t)} =$$

$$= \lim_{t \to x^+} \frac{p(t) - g(t)}{\dfrac{u'(t)}{u(t)} + \dfrac{v'(t)}{v(t)}} = 0.$$

In a similar fashion we can show that $\lim\limits_{t \to x^-} \left(\dfrac{u'(t)}{u(t)} - \dfrac{v'(t)}{v(t)} \right) = 0$. So we have proved our Lemma.

Lemma 2. *Let* (p), (q) *be oscillatory equations for $t \to \infty$, $p(t) < q(t) < 0$, $t \in I$ and let $\tilde{\varphi}, \tilde{\psi}, \tilde{\chi}, \tilde{\omega}$ denote the basic dispersions of the 1st, 2nd, 3rd, 4th kinds of* (p), *$\tilde{\chi}_{-1}(\tilde{\omega}_{-1})$ the central dispersion of the 3rd (4th) kinds with the index -1 of* (p). *Then*

$$\varphi(t) > \tilde{\varphi}(t), \quad \psi(t) > \tilde{\psi}(t), \quad \chi(t) > \tilde{\chi}(t), \quad \omega(t) > \tilde{\omega}(t), \quad t \in I,$$

$$\chi_{-1}(t) < \tilde{\chi}_{-1}(t), \quad t \in I^\circ,$$

$$\omega_{-1}(t) < \tilde{\omega}_{-1}(t), \quad t \in {}^\circ I.$$

Proof: The validity of the inequalities stated in this Lemma may be proved directly from Picone's identity or, for example, it follows from [3], Theorem 2, page 109.

2. We shall be concerned with equations

(q) $$y'' = q(t)\,y, \quad q \in C_I^0,$$

(p) $$y'' = p(t)\,y, \quad p \in C_I^0,$$

in assuming that

(i) there exist numbers $m, M, 0 < m \leqq M$ such that

(4) $$-\mathrm{M} \leqq \mathrm{q}(t) \leqq -\mathrm{m}, \quad t \in I,$$

(ii) there holds

(5) $$\lim_{t \to \infty} (p(t) - q(t)) = 0.$$

Under these assumptions the equations (p), (q) are oscillatory for $t \to \infty$. Before presenting the main result of this chapter, we will introduce, in keeping with the first part of our work, the following notation:

Let v be an integer, $v \neq 0$. Then $\varphi_v, \psi_v, \chi_v, \omega_v(\tilde{\varphi}_v, \tilde{\psi}_v, \tilde{\chi}_v, \tilde{\omega}_v)$ denote the central dispersions of the 1st, 2nd, 3rd, 4th kinds with the index v of (q), ((p)). In place of $\varphi_1, \psi_1, \chi_1, \omega_1(\tilde{\varphi}_1, \tilde{\psi}_1, \tilde{\chi}_1, \tilde{\omega}_1)$ we write $\varphi, \psi, \chi, \omega(\tilde{\varphi}, \tilde{\psi}, \tilde{\chi}, \tilde{\omega})$. For every number ε, $|\varepsilon| < m$, then $\varphi_v^\varepsilon, \psi_v^\varepsilon, \chi_v^\varepsilon, \omega_v^\varepsilon$ will stand for the central dispersions of the 1st, 2nd, 3rd, 4th kinds with the index v of $(\mathrm{q} + \varepsilon) : y'' = (q(t) + \varepsilon)\,y$. Finally $I^\varepsilon({}^\varepsilon I)$ will be used for the definition interval of $\chi_{-1}^\varepsilon(\omega_{-1}^\varepsilon)$ and $I^\circ({}^\circ I)$ for the definition interval of $\chi_{-1}(\omega_{-1})$.

The main result of this work can be expressed by the following.

Theorem 1. *Let the functions p, q satisfy the assumptions* (i) *and* (ii). *Then*

$$\lim_{t \to \infty} (\varphi(t) - \tilde{\varphi}(t)) = 0, \qquad \lim_{t \to \infty} (\psi(t) - \tilde{\psi}(t)) = 0,$$

$$\lim_{t \to \infty} (\chi(t) - \tilde{\chi}(t)) = 0, \qquad \lim_{t \to \infty} (\omega(t) - \tilde{\omega}(t)) = 0.$$

Remark 1. The result of Theorem 1 has been proved for the central dispersions of the 1st kind in [4] under an additional assumption that all the solutions of (q) are bounded on I.

Before proving Theorem 1 we will state and prove a number of supporting results. It should be noted once more here that the number ε always meets the inequality $|\varepsilon| < m$ (for the $(\mathrm{q} + \varepsilon)$ to be oscillatory for $t \to \infty$) and the functions p, q meet the assumptions (i) and (ii) which we shall explicitly put in the assumptions of the next theorems only.

Lemma 3. *The following inequalities hold:*

$$\frac{\pi}{\sqrt{M}} \leqq \varphi(t) - t \leqq \frac{\pi}{\sqrt{m}}, \qquad \frac{\pi}{\sqrt{M}} \leqq \psi(t) - t \leqq \frac{\pi}{\sqrt{m}},$$

$$\frac{\pi}{2\sqrt{M}} \leqq \chi(t) - t \leqq \frac{\pi}{2\sqrt{m}}, \qquad \frac{\pi}{2\sqrt{M}} \leqq \omega(t) - t \leqq \frac{\pi}{2\sqrt{m}}, \qquad t \in I,$$

$$\frac{\pi}{2\sqrt{M}} \leqq t - \chi_{-1}(t) \leqq \frac{\pi}{2\sqrt{m}}, \qquad t \in I^0,$$

$$\frac{\pi}{2\sqrt{M}} \leqq t - \omega_{-1}(t) \leqq \frac{\pi}{2\sqrt{m}}, \qquad t \in {}^0I.$$

Proof: The equation $(-m)$ $((-M))$ has the 1st and 2nd basic dispersion equal to $t + \frac{\pi}{\sqrt{m}} \left(t + \frac{\pi}{\sqrt{M}}\right)$, the 3rd and 4th basic dispersion equal to $t + \frac{\pi}{2\sqrt{m}}$ $\left(t + \frac{\pi}{2\sqrt{M}}\right)$ and the 3rd and 4th central dispersion with the index -1 equal to $t - \frac{\pi}{2\sqrt{m}} \left(t - \frac{\pi}{2\sqrt{M}}\right)$. The statement of Lemma 3 will now come out immediately by using Lemma 2.

Remark 2. The inequality $\varphi(t) - t \leqq \frac{\pi}{\sqrt{m}}$ has been proved in the proof of Lemma 1 in [4].

Lemma 4. *It holds*

$$|\chi(t) - \chi^{\varepsilon}(t)| \leqq \frac{|\varepsilon|}{m - |\varepsilon|}(\chi(t) - t), \qquad t \in I.$$

Proof: Let $x \in I$ and let u, v be solutions of (q), (q + ε), respectively, $u(x) = v(x) = 0$. Wherever the expressions $\frac{u'(t)}{u(t)}, \frac{v'(t)}{v(t)}$ are meaningful, the functions α, β are defined the formulas

$$\alpha(t) := \frac{u'(t)}{u(t)}, \qquad \beta(t) := \frac{v'(t)}{v(t)}. \tag{6}$$

The functions α, β satisfy the equations

$$\alpha' + \alpha^2 = q(t), \tag{7}$$

$$\beta' + \beta^2 = q(t) + \varepsilon. \tag{8}$$

The functions α, β are decreasing wherever they are defined, $\lim_{t \to x^+} \alpha(t) = \infty$, $\lim_{t \to x^+} \beta(t) = \infty$.

If $\varepsilon = 0$ then the statement of the Lemma is certainly true. We divide the next part of the proof into two parts according to the positivity or negativity of ε.

a) Assume $\operatorname{sign} \varepsilon = 1$. By Lemma 2 we have $\chi(t) < \chi^{\varepsilon}(t)$, $t \in I$. For the sake of simplicity of notation we put $x_1 = \chi(x)$, $x_2 = \chi^{\varepsilon}(x)$; $x < x_1 < x_2$. This gives us $\alpha(x_1) = \beta(x_2) = 0$, $\int_x^{x_1} \alpha(t)\,\mathrm{d}t = \infty$, $\int_x^{x_1} \beta(t)\,\mathrm{d}t = \infty$. With regard to (7) and (8)

$$(9) \qquad \big(\alpha(t) - \beta(t)\big)' + \big(\alpha(t) + \beta(t)\big)\big(\alpha(t) - \beta(t)\big) = -\varepsilon \quad \text{on} \quad (x, x_1\rangle$$

and thus

$$\alpha(t) - \beta(t) = e^{-\int_{x_1}^{t} [(\alpha(s) + \beta(s)] \mathrm{d}s} \left(k - \varepsilon \int_{x_1}^{t} e^{\int_{x_1}^{s} (\alpha(z) + \beta(z)) \mathrm{d}z} \mathrm{d}s\right), \qquad t \in (x, x_1\rangle$$

where $k = -\beta(x_1)$. From the other side

$$k = -\varepsilon \int_x^{x_1} e^{\int_{x_1}^{s} [\alpha(z) + \beta(z)] \mathrm{d}z} \mathrm{d}s,$$

which follows from the equality $\int_x^{x_1} (\alpha(t) + \beta(t))\,\mathrm{d}t = \infty$ and from $\lim_{t \to x^+} (\alpha(t) - \beta(t)) = 0$ proved in Lemma 1. At the same time there holds $\int_{x_1}^{s} (\alpha(t) + \beta(t))\,\mathrm{d}t \leqq 0$ for $s \in (x, x_1\rangle$ thus

$$(10) \qquad k > -\varepsilon(x_1 - x) = -\varepsilon(\chi(x) - x).$$

Integrating (8) from x_1 to x_2 we obtain $(k = -\beta(x_1),\ \beta(x_2) = 0)$

$$k + \int_{x_1}^{x_2} \beta^2(s)\,\mathrm{d}s = \int_{x_1}^{x_2} q(s)\,\mathrm{d}s + \varepsilon(x_2 - x_1).$$

By the mean value theorem there exist numbers ξ_1, ξ_2 on (x_1, x_2) so that

$$k = \big(q(\xi_1) - \beta^2(\xi_2) + \varepsilon\big)(x_2 - x_1)$$

and with regard to (10)

$$\chi^{\varepsilon}(x) - \chi(x) = x_2 - x_1 = \frac{k}{q(\xi_1) - \beta^2(\xi_2) + \varepsilon} < -\frac{\varepsilon}{q(\xi_1) - \beta^2(\xi_2) + \varepsilon}(\chi(x) - x) <$$

$$< \frac{\varepsilon}{-q(\xi_1) - \varepsilon}(\chi(x) - x) \leqq \frac{\varepsilon}{m - \varepsilon}(\chi(x) - x).$$

b) Assume that $\operatorname{sign} \varepsilon = -1$. Then we proceed in an entirely analogous fashion to that in the first part of the proof only that we shall consider $q + \varepsilon$ and q in place of q and $q + \varepsilon$, respectively. Since a similar process will appear once more in the proof of the next Lemma, we shall carry it out at least in this case for the reader's convenience. If we put $x_1 = \chi^\varepsilon(x)$, $x_2 = \chi(x)$ then $x < x_1 < x_2$, which follows from Lemma 2. With respect to (7) and (8) and thus also (9) hold, then $(\beta(x_1) = 0)$

$$\alpha(t) - \beta(t) = e^{-\int_{x_1}^{t} [\alpha(s)+\beta(s)]\,ds} \left(k - \varepsilon \int_{x_1}^{t} e^{\int_{x_1}^{s} [\alpha(z)+\beta(z)]\,dz}\, ds\right), \qquad t \in (x, x_1\rangle,$$

where $k = \alpha(x_1)$. On the other side from the equality $\int_x^{x_1} (\alpha(t) + \beta(t))\,dt = \infty$ and from $\lim_{t \to x^+} (\alpha(t) - \beta(t)) = 0$ proved in Lemma 1 we get

$$k = -\varepsilon \int_x^{x_1} e^{\int_{x_1}^{s} [\alpha(t)+\beta(t)]\,dt}\, ds$$

and there hold the estimates

$$k < -\varepsilon(x_1 - x) < -\varepsilon(x_2 - x) = -\varepsilon(\chi(x) - x).$$

Integrating (7) from x_1 to x_2 we get $(\alpha(x_1) = k,\ \alpha(x_2) = 0)$

$$-k + \int_{x_1}^{x_2} \alpha^2(t)\,dt = \int_{x_1}^{x_2} q(t)\,dt.$$

By the mean value theorem there exist numbers ξ_1, ξ_2 on (x_1, x_2) so that

$$k = (\alpha^2(\xi_1) - q(\xi_2))\,(x_2 - x_1)$$

and therefore

$$\chi(x) - \chi^\varepsilon(x) = x_2 - x_1 = \frac{k}{\alpha^2(\xi_1) - q(\xi_2)} < -\frac{\varepsilon}{\alpha^2(\xi_1) - q(\xi_2)}(\chi(x) - x) <$$

$$< \frac{|\varepsilon|}{m}(\chi(x) - x) < \frac{|\varepsilon|}{m - |\varepsilon|}(\chi(x) - x).$$

Remark 3. From the proof of Lemma 4 we even get the estimates:

a) $0 < \chi^\varepsilon(t) - \chi(t) < \dfrac{\varepsilon}{m - \varepsilon}(\chi(t) - t)$ for $\operatorname{sign} \varepsilon = 1$, $t \in I$,

b) $0 < \chi(t) - \chi^\varepsilon(t) < \dfrac{|\varepsilon|}{m}(\chi^\varepsilon(t) - t)$ for $\operatorname{sign} \varepsilon = -1$, $t \in I$.

Lemma 5. *It holds*

$$|\chi_{-1}(t) - \chi^\varepsilon_{-1}(t)| \leqq \frac{|\varepsilon|}{m - |\varepsilon|}(t - \chi_{-1}(t)), \qquad t \in I^0 \cap I^\varepsilon.$$

Proof: Let $x \in I_0 \cap I^\varepsilon$. Let the functions u, v be solutions of (q), (q + ε) respectively such that $u(x) = v(x) = 0$ and let the functions α, β with the aid of u, v be defined by the formulas (6). Since the Lemma holds for $\varepsilon = 0$ it may be assumed that $\varepsilon \neq 0$. The next part of the proof will be divided again according as the sign of the number ε is $+$ or $-$.

a) Let sign $\varepsilon = 1$. Next let $x_1 = \chi^{\varepsilon}_{-1}(x)$, $x_2 = \chi_{-1}(x)$. Then $\alpha(x_2) = \beta(x_1) = 0$ and by Lemma 2 we get $x_1 < x_2 < x$. The function $\alpha - \beta$ satisfies (9) on $\langle x_2, x)$ and thus

$$\alpha(t) - \beta(t) = e^{-\int_{x_2}^{t} [\alpha(s)+\beta(s)]\,ds} \left(k - \varepsilon \int_{x_2}^{t} e^{\int_{x_2}^{s} (\alpha(z)+\beta(z))\,dz}\, ds\right), \qquad t \in \langle x_2, x),$$

where $k = -\beta(x_2)$. From the properties of the functions α, β it follows $\int_{x_2}^{x} (\alpha(t) + \beta(t))\,dt = -\infty$, by Lemma 1 we get $\lim_{t \to x^-} (\alpha(t) - \beta(t)) = 0$ and consequently

$$k = \varepsilon \int_{x_2}^{x} e^{\int_{x_2}^{s} [\alpha(t)+\beta(t)]\,dt}\, ds.$$

With respect to $\int_{x_2}^{s} (\alpha(t) + \beta(t))\,dt \leqq 0$ for $s \in \langle x_2, x)$ we have

(11) $$k < \varepsilon(x - x_2) = \varepsilon(x - \chi_{-1}(x)).$$

Integrating (8) from x_1 to x_2 we obtain $(\beta(x_1) = 0,\ \beta(x_2) = -k)$

$$-k + \int_{x_1}^{x_2} \beta^2(t)\,dt = \int_{x_1}^{x_2} q(t)\,dt + \varepsilon(x_2 - x_1).$$

By the mean value theorem there exist numbers ξ_1, ξ_2 on (x_1, x_2) for which

$$-k = (q(\xi_1) - \beta^2(\xi_2) + \varepsilon)(x_2 - x_1)$$

holds. Herefrom and from (11) we have

$$\chi_{-1}(x) - \chi^{\varepsilon}_{-1}(x) = x_2 - x_1 = \frac{k}{\beta^2(\xi_2) - q(\xi_1) - \varepsilon} \leqq \frac{\varepsilon}{-q(\xi_1) - \varepsilon}(x - \chi_{-1}(x)) \leqq$$
$$\leqq \frac{\varepsilon}{m - \varepsilon}(x - \chi_{-1}(x)).$$

b) Let sign $\varepsilon = -1$. We may proceed analogously to the proof of Lemma 4 when we consider $q + \varepsilon$ and q in place of q and $q + \varepsilon$. Then there exist numbers ξ_1, ξ_2 on $(\chi_{-1}(x), \chi^{\varepsilon}_{-1}(x))$ so that the estimates

$$\chi^{\varepsilon}_{-1}(x) - \chi_{-1}(x) < -\frac{\varepsilon}{-q(\xi_1) + \alpha^2(\xi_2)}(x - \chi^{\varepsilon}_{-1}(x)) < \frac{|\varepsilon|}{m}(x - \chi^{\varepsilon}_{-1}(x)) <$$
$$< \frac{|\varepsilon|}{m - |\varepsilon|}(x - \chi_{-1}(x))$$

hold true.

Remark 4. From the proof of Lemma 5 we even get the estimates:

a) $0 < \chi_{-1}(t) - \chi^{\varepsilon}_{-1}(t) < \frac{\varepsilon}{m - \varepsilon}(t - \chi_{-1}(t))$ for sign $\varepsilon = 1$, $t \in I^{\varepsilon}$,

b) $0 < \chi^{\varepsilon}_{-1}(t) - \chi_{-1}(t) < \frac{|\varepsilon|}{m}(t - \chi^{\varepsilon}_{-1}(t))$ for sign $\varepsilon = -1$, $t \in I^0$.

Lemma 6. *It holds*

$$|\omega(t) - \omega^{\varepsilon}(t)| \leqq \frac{|\varepsilon| M}{(m - |\varepsilon|)^2}(\omega(t) - t), \qquad t \in I^0 \cap I^{\varepsilon}.$$

Proof: Assume that $x \in I^0 \cap I^{\varepsilon}$. The statement of the Lemma is valid provided that $\varepsilon = 0$. Assume first that sign $\varepsilon = -1$ and let us put $x_1 = \omega^{\varepsilon}(x)$, $x_2 = \omega(x)$, $x_3 = \chi_{-1}(x_1)$. By Lemma 2 we have $x < x_1 < x_2$ and it holds $x_3 < x$, $\chi^{\varepsilon}_{-1}(x_1) = x$, $\omega(x_3) = x_1$. Hence according to remark 4 we see that

$$0 < x - x_3 = \chi^{\varepsilon}_{-1}(x_1) - \chi_{-1}(x_1) < \frac{|\varepsilon|}{m}(x_1 - \chi^{\varepsilon}_{-1}(x_1) = \frac{|\varepsilon|}{m}(x_1 - x)$$

and further—by the mean value theorem—there exists a number ξ and by (3) a number t_1 such that

$$0 < x_2 - x_1 = \omega(x) - \omega^{\varepsilon}(x) = \omega(x) - \omega(x_3) =$$
$$= \omega'(\xi)(x - x_3) = \frac{g(\xi)}{g(t_1)}(x - x_3).$$

Therefore the estimate

$$\omega(x) - \omega^{\varepsilon}(x) < \frac{|\varepsilon|}{m} \cdot \frac{g(\xi)}{g(t_1)}(x_1 - x) < \frac{|\varepsilon| M}{m^2}(x_2 - x) < \frac{|\varepsilon| M}{(m - |\varepsilon|)^2}(\omega(x) - x)$$

holds.

Assume now that sign $\varepsilon = 1$ and let us put $x_1 = \omega(x)$, $x_2 = \omega^{\varepsilon}(x)$, $x_3 = \chi^{\varepsilon}_{-1}(x_1)$. Then $x_3 \in I$, $x_3 < x < x_1 < x_2$, $\omega^{\varepsilon}(x_3) = x_1$ and from remark 4 we get

$$0 < x - x_3 = \chi_{-1}(x_1) - \chi^{\varepsilon}_{-1}(x_1) < \frac{\varepsilon}{m - \varepsilon}(x_1 - \chi_{-1}(x_1)) = \frac{\varepsilon}{m - \varepsilon}(x_1 - x).$$

Next, by the mean value theorem there exists a number ξ and by (3) a number t_1 for which

$$0 < x_2 - x_1 = \omega^{\varepsilon}(x) - \omega(x) = \omega^{\varepsilon}(x) - \omega^{\varepsilon}(x_3) =$$
$$= \omega^{\varepsilon\prime}(\xi)(x - x_3) = \frac{g(\xi) + \varepsilon}{g(t_1) + \varepsilon}(x - x_3).$$

Therefore

$$\omega^{\varepsilon}(x) - \omega(x) < \frac{g(\xi) + \varepsilon}{g(t_1) + \varepsilon} \cdot \frac{\varepsilon}{m - \varepsilon}(x_1 - x) \leqq$$

$$\leqq \frac{\varepsilon(M-\varepsilon)}{(m-\varepsilon)^2}(\omega(x)-x) < \frac{\varepsilon M}{(m-\varepsilon)^2}(\omega(x)-x).$$

Lemma 7. *It holds*

$$|\varphi(t)-\varphi^\varepsilon(t)| \leqq \frac{|\varepsilon|(M+|\varepsilon|)}{(m-|\varepsilon|)^2}(\varphi(t)-t), \qquad t\in I^0\cap I^\varepsilon.$$

Proof: From the formula (2) with some evident modifications we obtain

$$|\varphi(t)-\varphi^\varepsilon(t)| = |\omega\circ\chi(t)-\omega^\varepsilon\circ\chi^\varepsilon(t)| \leqq |\omega\circ\chi(t)-\omega^\varepsilon\circ\chi(t)| + \\ + |\omega^\varepsilon\circ\chi(t)-\omega^\varepsilon\circ\chi^\varepsilon(t)|.$$

By the mean value theorem there exists a number ξ and by (3) a number t_1 such that

$$|\omega^\varepsilon\circ\chi(t)-\omega^\varepsilon\circ\chi^\varepsilon(t)| = \omega^{\varepsilon'}(\xi)|\chi(t)-\chi^\varepsilon(t)| = \frac{g(\xi)+\varepsilon}{g(t_1)+\varepsilon}|\chi(t)-\chi^\varepsilon(t)|$$

and we can see that with regard to Lemmas 4 and 6 the following estimate

$$|\varphi(t)-\varphi^\varepsilon(t)| \leqq |\omega\circ\chi(t)-\omega^\varepsilon\circ\chi(t)| + |\omega^\varepsilon\circ\chi(t)-\omega^\varepsilon\circ\chi^\varepsilon(t)| \leqq$$
$$\leqq \frac{|\varepsilon|M}{(m-|\varepsilon|)^2}(\omega\circ\chi(t)-\chi(t)) + \frac{M+|\varepsilon|}{m-|\varepsilon|}\cdot\frac{|\varepsilon|}{m-|\varepsilon|}(\chi(t)-t) <$$
$$< \frac{|\varepsilon|(M+|\varepsilon|)}{(m-|\varepsilon|)^2}(\varphi(t)-t), \qquad t\in I^0\cap I^\varepsilon,$$

is true.

Lemma 8. *It holds*

$$|\psi(t)-\psi^\varepsilon(t)| \leqq \frac{|\varepsilon|M(M+|\varepsilon|)}{(m-|\varepsilon|)^3}(\varphi(t)-t), \qquad t\in I^0\cap I^\varepsilon.$$

Proof: From the formula (1) with some evident modifications we get

$$|\psi(t)-\psi^\varepsilon(t)| = |\chi\circ\omega(t)-\chi^\varepsilon\circ\omega^\varepsilon(t)| \leqq |\chi\circ\omega(t)-\chi^\varepsilon\circ\omega(t)| + \\ + |\chi^\varepsilon\circ\omega(t)-\chi^\varepsilon\circ\omega^\varepsilon(t)|.$$

By (3) there exist numbers t_1, t_2 and by the mean value theorem a number ξ, such that

$$|\chi^\varepsilon\circ\omega(t)-\chi^\varepsilon\circ\omega^\varepsilon(t)| = \chi^{\varepsilon'}(\xi)|\omega(t)-\omega^\varepsilon(t)| = \frac{g(t_1)+\varepsilon}{g(t_2)+\varepsilon}|\omega(t)-\omega^\varepsilon(t)|$$

Consequently by Lemmas 4 and 6 we have

$$|\psi(t)-\psi^\varepsilon(t)| \leqq |\chi\circ\omega(t)-\chi^\varepsilon\circ\omega(t)| + |\chi^\varepsilon\circ\omega(t)-\chi^\varepsilon\circ\omega^\varepsilon(t)| \leqq$$
$$\leqq \frac{|\varepsilon|}{m-|\varepsilon|}(\chi\circ\omega(t)-\omega(t)) + \frac{|\varepsilon|M(M+|\varepsilon|)}{(m-|\varepsilon|)^3}(\omega(t)-t) \leqq$$
$$\leqq \frac{|\varepsilon|M(M+|\varepsilon|)}{(m-|\varepsilon|)^3}(\varphi(t)-t), \qquad t\in I^0\cap I^\varepsilon.$$

Theorem 2. *Let the functions p, q satisfy the assumptions* (i) *and* (ii). *Then the following estimates hold for* $t \in I^0 \cap I^\varepsilon$:

$$|\varphi(t) - \varphi^\varepsilon(t)| \leqq \frac{|\varepsilon|(M + |\varepsilon|)}{(m - |\varepsilon|)^2} \cdot \frac{\pi}{\sqrt{m}},$$

$$|\psi(t) - \psi^\varepsilon(t)| \leqq \frac{|\varepsilon| M(M + |\varepsilon|)}{(m - |\varepsilon|)^3} \cdot \frac{\pi}{\sqrt{m}},$$

$$|\chi(t) - \chi^\varepsilon(t)| \leqq \frac{|\varepsilon|}{m - |\varepsilon|} \cdot \frac{\pi}{2\sqrt{m}},$$

$$|\omega(t) - \omega^\varepsilon(t)| \leqq \frac{|\varepsilon| M}{(m - |\varepsilon|)^2} \cdot \frac{\pi}{2\sqrt{m}},$$

$$|\chi_{-1}(t) - \chi^\varepsilon_{-1}(t)| \leqq \frac{|\varepsilon|}{m - |\varepsilon|} \cdot \frac{\pi}{2\sqrt{m}}.$$

Proof: The statement of the Theorem immediately follows from Lemmas 3–8.

Corollary 1. *Let* $\{q_n(t)\}$ *be a sequence of continuous functions on I cornverging uniformly on* $I^0(= (b, \infty))$ *towards the function* $q(t)$ *and let* ${}^n\varphi$, ${}^n\psi$, ${}^n\chi$, ${}^n\omega$ *be basic dispersions of the 1st, 2nd, 3rd, 4th kinds of* (q_n), $c > 0$ *a number. Then the sequences of functions* $\{{}^n\varphi(t)\}$, $\{{}^n\psi(t)\}$, $\{{}^n\chi(t)\}$, $\{{}^n\omega(t)\}$ *on the interval* $\langle b + c, \infty)$ *converge uniformly towards* $\varphi(t)$, $\psi(t)$, $\chi(t)$, $\omega(t)$, *respectively.*

Proof: Let $\varepsilon_1 > 0$ and let us choose $\varepsilon > 0$ so that $\dfrac{\varepsilon M(M + \varepsilon)}{(m - \varepsilon)^3} \cdot \dfrac{\pi}{\sqrt{m}} \leqq \varepsilon_1$, $I^{\varepsilon_1} \supset \langle b + c, \infty)$. Let now N be such an index that for all $n > N$ we have $|q(t) - q_n(t)| < \varepsilon$ on I^0. Lemma 2 yields the inequalities $\varphi^{-\varepsilon}(t) < {}^n\varphi(t) < \varphi^\varepsilon(t)$, $\psi^{-\varepsilon}(t) < {}^n\psi(t) < \psi^\varepsilon(t)$, $\chi^{-\varepsilon}(t) < {}^n\chi(t) < \chi^\varepsilon(t)$, $\omega^{-\varepsilon}(t) < {}^n\omega(t) < \omega^\varepsilon(t)$ for $t \in I^0$ and every $n > N$. Hence $|{}^n\varphi(t) - \varphi(t)| \leqq \varepsilon_1$, $|{}^n\psi(t) - \psi(t)| \leqq \varepsilon_1$, $|{}^n\chi(t) - \chi(t)| \leqq \varepsilon_1$, $|{}^n\omega(t) - \omega(t)| \leqq \varepsilon_1$ for $t \in \langle b + c, \infty)$, which follows from Theorem 2. By this we have proved the foregoing Corollary.

We proceed now to the proof of Theorem 1.

Proof of Theorem 1: Let $\varepsilon_1 > 0$ be an arbitrary number. It is sufficient for this proof to show the existence of a number $t_1 \in I$ that for $t \geqq t_1$ it is $|\varphi(t) - \tilde{\varphi}(t)| \leqq \varepsilon_1$, $|\psi(t) - \tilde{\psi}(t)| \leqq \varepsilon_1$, $|\chi(t) - \tilde{\chi}(t)| \leqq \varepsilon_1$, $|\omega(t) - \tilde{\omega}(t)| \leqq \varepsilon_1$. Let $\varepsilon > 0$ be a number for which the inequality $\dfrac{\varepsilon M(M + \varepsilon)}{(m - \varepsilon)^3} \cdot \dfrac{\pi}{\sqrt{m}} \leqq \varepsilon_1$ is true and $t_1 \in I^0 \cap I^\varepsilon$ such a number where $q(t) - \varepsilon_1 < p(t) < q(t) + \varepsilon_1$ for $t \in \langle t_1, \infty)$. The existence of such a number t_1 follows from the assumption (5). Thus for $t \geqq t_1$ we have $\varphi^{-\varepsilon_1}(t) < \tilde{\varphi}(t) < \varphi^{\varepsilon_1}(t)$, $\psi^{-\varepsilon_1}(t) < \tilde{\psi}(t) < \psi^{\varepsilon_1}(t)$, $\chi^{-\varepsilon_1}(t) < \tilde{\chi}(t) < \chi^{\varepsilon_1}(t)$, $\omega^{-\varepsilon_1}(t) < \tilde{\omega}(t) < \omega^{\varepsilon_1}(t)$, which follow from Lemma 2. Theorem 2 yields then $|\varphi(t) - \tilde{\varphi}(t)| \leqq \varepsilon_1$, $|\psi(t) - \tilde{\psi}(t)| \leqq \varepsilon_1$, $|\chi(t) - \tilde{\chi}(t)| \leqq \varepsilon_1$, $|\omega(t) - \tilde{\omega}(t)| \leqq \varepsilon_1$ for $t > t_1$.

Corollary 2. *For every positive integer n*

$$\lim_{t\to\infty} (\varphi_n(t) - \tilde{\varphi}_n(t)) = 0, \qquad \lim_{t\to\infty} (\psi_n(t) - \tilde{\psi}_n(t)) = 0,$$
$$\lim_{t\to\infty} (\chi_n(t) - \tilde{\chi}_n(t)) = 0, \qquad \lim_{t\to\infty} (\omega_n(t) - \tilde{\omega}_n(t)) = 0.$$

Proof: Since the proof of all four equalities above are similar to each other, we shall give here just one of them. So, let us prove $\lim_{t\to\infty} (\varphi_n(t) - \tilde{\varphi}_n(t)) = 0$, $n = 1, 2, 3, \ldots$ This can be done by mathematical induction. The instance $n = 1$ has been proved in Theorem 1. Thus, let $\lim_{t\to\infty} (\varphi_k(t) - \tilde{\varphi}_k(t)) = 0$ for $k \geqq 1$. From the formulas (see [1] p. 115) $\varphi_{k+1}(t) = \varphi \circ \varphi_k(t)$, $\tilde{\varphi}_{k+1}(t) = \tilde{\varphi} \circ \tilde{\varphi}_k(t)$ we obtain $\varphi_{k+1}(t) - \tilde{\varphi}_{k+1}(t) = (\varphi \circ \varphi_k(t) - \varphi \circ \tilde{\varphi}_k(t)) + (\varphi \circ \tilde{\varphi}_k(t) - \tilde{\varphi} \circ \tilde{\varphi}_k(t)) = \varphi'(\eta) (\varphi_k(t) - \tilde{\varphi}_k(t)) + (\varphi \circ \tilde{\varphi}_k(t) - \tilde{\varphi} \circ \tilde{\varphi}_k(t))$, where η is a appropriate number. From the assumption $\lim_{\to\infty} (\varphi_k(t) - \tilde{\varphi}_k(t)) = 0$ and from the boundedness of the function φ' on I following rom the assumption (i) and (3) we come to $\lim_{t\to\infty} (\varphi_{k+1}(t) - \tilde{\varphi}_{k+1}(t)) = 0$.

Remark 5. Eventually we can prove that the statement of Corollary 2 applies to every integer n, $n \neq 0$.

From the next two examples it is apparent that Theorem 1 does not hold when m, in the assumption (i), is not a positive number or when $m = 0$ and at the some time $M = -\infty$.

Example 1. Let $I = (1, \infty)$ and let $p(t) := -\frac{1}{t^2}$, $q(t) := -\frac{1}{4t^2}$ for $t \in I$. Then the equation (q) is nonoscillatory and the equation (p) is oscillatory for $t \to \infty$ ([2] p. 427). Thereby $\lim_{t\to\infty} (p(t) - q(t)) = 0$ and we have $m = 0$, $M = \frac{1}{4}$ in (4). Since the functions φ, ψ, χ, ω are not defined on I, Theorem 1 is meaningless.

Example 2. Let $I = (0, \infty)$, $q \in C_I^0$, $q(t) < 0$ for $t \in I$. Let next q be lower unbounded and such that $\int_0^\infty q(t)\,dt$ converges and $\lim_{t\to\infty} t \int_0^t q(s)\,ds = k > -\frac{1}{4}$. By Theorem 2.1 in [5], page 45 (q) is nonoscillatory for $t \to \infty$. Let Δ be such a function, where $\Delta \in C_I^0$, $\int_0^\infty \Delta(t)\,dt = -\infty$, $\lim_{t\to\infty} \Delta(t) = 0$. Putting $p(t) := q(t) + \Delta(t)$ for $t \in I$ then we have $\lim_{t\to\infty} (p(t) - q(t)) = \lim_{t\to\infty} \Delta(t) = 0$ and $\int_0^\infty p(t)\,dt = -\infty$. The equation (p) is oscillatory for $t \to \infty$ (see [5], p. 70, Theorem 2.24). The equality (4) is fulfilled for $m = 0$ and $M = -\infty$. In that case Theorem 1 is anew meaningless because of the functions φ, ψ, χ, ω not being defined on I.

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