Archivum Mathematicum

Miroslav Bartušek

On asymptotic properties and distribution of zeros of solutions of $y'' + f(t, y, y') = 0$

Archivum Mathematicum, Vol. 14 (1978), No. 1, 1--12

Persistent URL: http://dml.cz/dmlcz/106986

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIV: 1—12, 1978

ON ASYMPTOTIC PROPERTIES AND DISTRIBUTION OF ZEROS OF SOLUTIONS OF $y'' + f(t, y, y') = 0$

MIROSLAV BARTUŠEK, Brno

(Received May 31, 1976)

1. Consider a differential equation

$$(1)\quad \begin{cases} y'' + f(t, y, y') = 0\,, \\ \text{where the function } f \text{ is continuous in } D = \{(t, y, v): \\ t \in [t_0, \infty), y \in R, v \in R\},\ f(t, y, v)\, y > 0 \text{ for } y \neq 0\,. \end{cases}$$

It is evident that Cauchy initial problem for (1) has a solution but we do not suppose its uniqueness. In the present paper we shall omit the trivial solution $y \equiv 0$ from our considerations.

A solution y of (1) is called oscillatory if there exists a sequence of numbers $\{t_k\}_1^\infty$ such that $t_0 \leqq t_k < t_{k+1}$, $y(t_k) = 0$, $y(t) \neq 0$ on (t_k, t_{k+1}), $k = 1, 2, 3, \ldots$

In this paper we shall deal only with oscillatory solutions of (1) that exist on the whole interval $[t_0, \infty)$, i.e. $\lim_{k\to\infty} t_k = \infty$. We shall study some asymptotic properties of them and the distribution of their zeros.

If y is an oscillatory solution of (1), then the distribution of its zeros is characterized by the sequence $\{\Delta_k\}_1^\infty$. $\Delta_k = t_{k+1} - t_k$ where $\{t_k\}_1^\infty$ is the sequence of all zeros of y. There exists exactly one sequence $\{\tau_k\}_1^\infty$ called the sequence of extremants of y, with the property $t_k < \tau_k < t_{k+1}$, $y'(\tau_k) = 0$ (see [2]). The symbols t_k, τ_k, Δ_k have the above mentioned meaning in the present paper.

We shall need the simple lemma the proof of which can be found in [2], [3].

Lemma 1. *Let y be an arbitrary non-trivial solution of (1) and $t_1 < t_2$ its consecutive zeros ($y(t) \neq 0$ on (t_1, t_2)). Then there exists exactly one number τ, $t_1 < \tau < t_2$ such that $y'(\tau) = 0$ holds, the function y' sgn y is decreasing on (t_1, t_2) and the inequalities*

$$|y'(t_1)|(\tau - t_1) > |y(\tau)|, \qquad |y'(t_2)|(t_2 - \tau) > |y(\tau)|$$

are valid.

2. Theorem 1. *Let y be an oscillatory solution of (1) and let there exist a constant $M > 0$ such that for an arbitrary number $M_1 > 0$ the following relation holds:*

$$\lim_{t\to\infty} f(t, y, v) = 0, \qquad \textit{uniformly for} \quad |y| \leqq M_1, |v| \leqq M.$$

Then at least one of the following assertions is valid

(i) $\lim_{t\to\infty} y'(t) = 0$,

(ii) *y is unbounded on* $[t_0, \infty)$.

If, in addition, constants M_2, M_3 exist such that

$$0 < M_2 \leqq |y(\tau_k)| \leqq M_3, \qquad k = 1, 2, 3, \ldots,$$

then $\lim_{k\to\infty} \Delta_k = \infty$.

Proof. Assume that y is bounded on $[t_0, \infty)$. We shall prove at first the relation $\lim_{t\to\infty} y'(t) = 0$. Let

$$|y(t)| \leqq N = \text{const.} < \infty, \qquad t \in [t_0, \infty).$$

Put

$$H_k(v) = \max |f(t, y, v)| > 0 \qquad \text{for } v \in R,$$
$$|y| \leqq N, \qquad t_k \leqq t \leqq t_{k+1}.$$

It follows from the assumptions of the theorem that

(2) $$\lim_{k\to\infty} H_k(v) = 0 \qquad \text{uniformly for } |v| \leqq M.$$

By multiplicating the equation (1) by $-\dfrac{y'}{H_k(y')}$ and by integration we obtain

(3) $$\int_0^{|y'(t_k)|} \frac{t\,dt}{H_k(t \operatorname{sgn} y'(t_k))} = -\int_{t_k}^{\tau_k} \frac{y''(t)\,y'(t)}{H_k(y'(t))}\,dt =$$

$$= \int_{t_k}^{\tau_k} \frac{|f(t, y(t), y'(t))|\,|y'(t)|}{H_k(y'(t))}\,dt \leqq \int_{t_k}^{\tau_k} |y'(t)|\,dt = |y(\tau_k)| \leqq N, \qquad k = 1, 2, \ldots$$

Suppose that y' does not converge to zero for $t \to \infty$. Then there exists a sequence of integers $\{k_i\}_1^\infty$ such that $|y'(t_{k_i})| \geqq \varepsilon > 0$, $i = 1, 2, 3, \ldots$ holds in case ε, $\varepsilon \leqq M$ is a suitable number. According to (2)

$$\lim_{i\to\infty} \int_0^{|y'(t_{k_i})|} \frac{t\,dt}{H_{k_i}(t \operatorname{sgn} y'(t_{k_i}))} \geqq \lim_{i\to\infty} \int_0^{\varepsilon} \frac{t\,dt}{H_{k_i}(t \operatorname{sgn} y'(t_{k_i}))} = \infty$$

and we get a contradiction to the inequality (3). Thus the first part of the statement is proved.

Let M_2, M_3 be constants such that $0 < M_2 \leqq |y(\tau_k)| \leqq M_3$, $k = 1, 2, 3, \ldots$ Thus according to the proved part of the theorem $\lim_{t\to\infty} y'(t) = 0$ holds and the rest of the statement follows from Lemma 1:

$$\tau_k - t_k > \left| \frac{y(\tau_k)}{y'(t_k)} \right| \geqq \frac{M_2}{|y'(t_k)|} \underset{k\to\infty}{\to} \infty,$$

$$t_{k+1} - \tau_k > \frac{|y(\tau_k)|}{|y'(t_{k+1})|} \geqq \frac{M_2}{|y'(t_{k+1})|} \underset{k\to\infty}{\to} \infty.$$

The theorem is proved.

Theorem 2. *Let y be an oscillatory solution of (1) and let a constant M, $0 < M$ exist such that for arbitrary numbers M_1, M_2, $0 < M_2 \leqq M$, $0 < M_1$ the following relation holds*

$$\lim_{t\to\infty} |f(t, y, v)| = \infty \qquad \textit{uniformly for } M_2 \leqq |y| \leqq M, |v| \leqq M_1.$$

Then at least one of the following assertions is valid

(i) $\lim_{t\to\infty} y(t) = 0$,

(ii) *y' is unbounded on* $[t_0, \infty)$.

Proof. Suppose that (i) is not valid. Then there exists a sequence of integers $\{k_i\}_1^\infty$ such that $|y(\tau_{k_i})| \geqq \varepsilon > 0$, $i = 1, 2, 3, \ldots$ where ε, $\varepsilon \leqq M$ is a suitable number. Put

$$H_i(v) = \min |f(t, y, v)| > 0, \quad \text{for}$$
$$\varepsilon/2 \leqq |y| \leqq \varepsilon, \quad t_{k_i} \leqq t \leqq t_{k_i+1}, \quad v \in R; \quad i = 1, 2, \ldots$$

With respect to the assumptions of the theorem we have

$$\lim_{i\to\infty} H_i(v) = \infty \qquad \text{uniformly for } |v| \leqq \text{const.} \tag{4}$$

By multiplying (1) by $-\dfrac{y'}{H_i(y')}$ and by integrating we obtain

$$\int_0^{|y'(t_{k_i})|} \frac{t\,\mathrm{d}t}{H_i(t \operatorname{sgn} y'(t_{k_i}))} = \int_{t_{k_i}}^{\tau_{k_i}} \frac{|f(t, y(t), y'(t))|\,|y'(t)|}{H_i(y'(t))}\,\mathrm{d}t \geqq \tag{5}$$
$$\geqq \int_{t_i^1}^{t_i^2} \frac{|f(t, y(t), y'(t))|\,|y'(t)|\,\mathrm{d}t}{H_i(y'(t))} \geqq \int_{t_i^1}^{t_i^2} |y'(t)|\,\mathrm{d}t = |y(t_i^2)| - |y(t_i^1)| = \frac{\varepsilon}{2},$$

where t_i^1, t_i^2 are such numbers that $t_{k_i} < t_i^1 < t_i^2 \leqq \tau_{k_i}$, $|y(t_i^1)| = \varepsilon/2$, $|y(t_i^2)| = \varepsilon$, $i = 1, 2, 3, \ldots$

Let y' be bounded on $[t_0, \infty]$. Thus there exists a constant $N > 0$ such that $|y'(t)| \leqq N$, $t \in [t_0, \infty)$. Then according to (4) we can conclude

$$\int_0^{|y'(t_{ki})|} \frac{t\,\mathrm{d}t}{H_i(t \operatorname{sgn} y'(t_{ki}))} \leqq N \int_0^N \frac{\mathrm{d}t}{H_i(t \operatorname{sgn} y'(t_{ki}))} \underset{i\to\infty}{\longrightarrow} 0.$$

But this fact is in contradiction with the inequality (5) and the function y' is unbounded on $[t_0, \infty)$. The theorem is proved.

Remark 1. It is evident from (3) and (5) that under the assumptions of Theorem 1

$$\liminf_{k\to\infty} \{|y(\tau_k)|\} < \infty \Rightarrow \liminf_{k\to\infty} \{|y'(t_k)|\} = 0$$

holds, too. Similarly, if the assumptions of Theorem 2 are valid, then

$$\liminf_{k\to\infty} \{|y(\tau_k)|\} > 0 \Rightarrow \lim_{k\to\infty} \{|y'(t_k)|\} = \infty$$

holds.

Remark 2. Let the assumptions of Theorem 1 be valid and let y be bounded on $[t_0, \infty)$. Then

$$\lim_{t\to\infty} y'(t) = 0, \qquad \lim_{t\to\infty} y''(t) = 0.$$

The last result follows from the previous one and from the relation

$$\lim_{t\to\infty} y''(t) = -\lim_{t\to\infty} f(t, y(t), y'(t)) = 0,$$

Similarly, let y be an oscillatory solution of (1) and let for an arbitrary constant $M_2 > 0$ the following relation hold

$$\lim_{i\to\infty} |f(t, y, v)| = \infty \qquad \text{uniformly for } |v| < \infty, M_2 \leqq |y| < \infty.$$

Let $\limsup_{k\to\infty} \{|y(\tau_k)|\} > 0$. Then the functions y', y'' are unbounded on $[t_0, \infty)$.

If, in addition, $\liminf_{k\to\infty} \{|y(\tau_k)|\} > 0$, then

$$\lim_{k\to\infty} |y'(t_k)| = \lim_{k\to\infty} \sigma_k = \infty \qquad \text{where} \qquad \sigma_k = \max_{t\in[t_k, t_{k+1}]} |y''(t)|.$$

Remark 3. The results of Theorems 1 and 2 were studied in [2], [3] for the differential equation

$$(p(t)\, y')' + f(t, y, y') = 0.$$

The statements of Theorems 1 and 2 generalize some conclusions of the above mentioned papers for (1).

Corollary 1. Let y be an oscillatory solution of a differential equation

$$(6) \qquad \begin{cases} y'' + a(t) f(y, y') = 0, \\ \text{where } a(t), f(y, v) \text{ are continuous functions} \\ \text{for } t \in [t_0, \infty),\ y \in R,\ v \in R, a > 0, \\ f(y, v)\, y > 0 \text{ for } y \neq 0. \end{cases}$$

(i) Let $\lim_{t\to\infty} a(t) = 0$. If y is bounded on $[t_0, \infty)$, then

$$\lim_{t\to\infty} y'(t) = 0, \qquad \lim_{t\to\infty} y''(t) = 0.$$

If, in addition, $\liminf_{t\to\infty}\{|y(\tau_k)|\} > 0$, then $\lim_{k\to\infty} \Delta_k = \infty$ holds.

(ii) Let $\lim_{t\to\infty} a(t) = \infty$. If y' is bounded on $[t_0, \infty)$, then

$$\lim_{t\to\infty} y(t) = 0.$$

(iii) Let $\lim_{t\to\infty} a(t) = \infty$. If there exist constants M, M_1 such that $0 < M \leqq \leqq |y(\tau_k)| \leqq M_1 < \infty$ holds, then $\lim_{k\to\infty} |y'(t_k)| = \infty$.

If, in addition, for an arbitrary constant $N > 0$ there exists a number $M_2 > 0$ such that

$$|f(y, v)| \geqq M_2 > 0, \qquad |y| \geqq N, |v| < \infty$$

holds, then $\lim_{k\to\infty} \sigma_k = \infty$, $\sigma_k = \max_{t\in[t_k, t_{k+1}]} |y''(t)|$.

3. In this paragraph some well-known results for the linear differential equation (see [15], [1], [9])

$$y'' + q(t)y = 0, \qquad t \in [t_0, \infty)$$

q continuous, $0 < M = \text{const.} \leqq q(t) \leqq M_1 = \text{const.} < \infty$, $t \in [t_0, \infty)$ are extended to the equation (1).

Theorem 3. *Let y be an oscillatory solution of (1). Let constants M, M_1, M_3, $0 < M$, $0 < M_1$, $0 < M_3$ exist such that for an arbitrary number M_2, $0 < M_2 \leqq M$ there hold*

$$|f(t, y, v)| \leqq M_3, \qquad t \in [t_0, \infty), \qquad |y| \leqq M, \qquad |v| \leqq M_1,$$
$$0 < M_4 \leqq |f(t, y, v)|, \qquad t \in [t_0, \infty), \qquad M_2 \leqq |y| \leqq M, \qquad |v| \leqq M_1,$$

where M_4 *is a constant (depending on M_2).*

Then

$$\lim_{t\to\infty} y(t) = 0 \qquad \text{if, and only if} \qquad \lim_{t\to\infty} y'(t) = 0.$$

Proof. Let $\lim_{t\to\infty} y(t) = 0$. Then there exists a number $\bar{t} \in [t_0, \infty)$ and an integer $\bar{k}$ such that $|y(t)| \leqq M$, $t \in [\bar{t}, \infty)$, $\tau_{\bar{k}} \geqq t$. We can define the function $H_k(v)$ in the same way as in Theorem 1 ($N = M$), the estimation (3) holds, thus especially

$$\int_0^{|y'(t_k)|} \frac{t\,dt}{H_k(t \operatorname{sgn} y'(t_k))} \leqq |y(\tau_k)|, \qquad k = \bar{k}, \bar{k}+1, \bar{k}+2, \ldots \tag{7}$$

If $\lim_{k\to\infty} y(\tau_k) = 0$, then according to the estimation

$$H_k(v) \leqq M_3 = \text{const.} < \infty, \qquad |v| \leqq M_1, \qquad k = \bar{k}, \bar{k}+1, \bar{k}+2, \ldots$$

(the validity of which follows from the assumptions of the theorem) we can conclude $\lim_{k\to\infty} y'(t_k) = 0$.

On the contrary, let $\lim_{t\to\infty} y'(t) = 0$. We shall use the indirect proof for proving the relation $\lim_{t\to\infty} y(t) = 0$. If this relation is not valid, then there exists a sequence $\{k_i\}_1^\infty$ and a number ε, $0 < \varepsilon$ such that $|y(\tau_{k_i})| \geqq \varepsilon$, $i = 1, 2, \ldots$ and (5) hold. Thus

$$\int_0^{|y'(t_{k_i})|} \frac{t\,\mathrm{d}t}{H_i(t \operatorname{sgn} y'(t_{k_i}))} \geqq \frac{\varepsilon}{2} > 0.$$

From this and according to $\lim_{t\to\infty} y'(t) = 0$ we obtain a contradiction because from the assumptions of the theorem there follows

$$0 < N_2 = \text{const.} \leqq H_i(v), \qquad i = 1, 2, \ldots$$

uniformly in some neighbourhood of $v = 0$. The theorem is proved.

Remark 4. It is evident from the proof of Theorem 3 that the following assertion is valid, too

$$\liminf_{k\to\infty} |y(\tau_k)| = 0 \Leftrightarrow \liminf_{k\to\infty} |y'(t_k)| = 0.$$

Corollary 2. Let y be an oscillatory solution of (6). Let constants M, M_1 exist such that

$$0 < M \leqq a(t) \leqq M_1, \qquad t \in [t_0, \infty)$$

holds. Then

$$\lim_{t\to\infty} y(t) = 0 \qquad \text{if and only if} \qquad \lim_{t\to\infty} y'(t) = 0.$$

Theorem 4. *Let y be an oscillatory solution of* (1). *Let a continuous function $g(v)$, $v \in R$, $g > 0$, $\int_0^\infty \frac{t}{g(\pm t)} \mathrm{d}t = \infty$ exist such that for arbitrary constants M, M_1, M_2, $0 < M_1 \leqq M < \infty$ there hold $|f(t, y, v)| \leqq M_3 g(v)$, $t \in [t_0, \infty)$, $|y| \leqq M$, $v \in R$ and*

(8) $$0 < M_4 \leqq |f(t, y, v)|, \quad t \in [t_0, \infty), \quad M_1 \leqq |y|, \quad |v| \leqq M_2.$$

Here M_3 (M_4) is a suitable number that depends on M (on M_1, M_2). Then

(i) *y is bounded on $[t_0, \infty)$ if and only if y' is bounded on $[t_0, \infty)$*

(ii) $\lim_{k\to\infty} |y(\tau_k)| = \infty$ *if and only if* $\lim_{k\to\infty} |y'(t_k)| = \infty$

(iii) *If* $0 < M_5 = \text{const.} \leqq |y(\tau_k)| \leqq M_6 = \text{const.}$, $k = 1, 2, 3, \ldots$, *then there exist numbers C, C_1 such that*

(9) $$0 < C \leqq \Delta_k \leqq C_1, \qquad k = 1, 2, 3, \ldots$$

holds.

Proof. (i) Let y be bounded $|y(t)| \leqq N$, $t \in [t_0, \infty)$. Then we can define the function $H_k(v)$ in the same way as in Theorem 1 and (3) holds. Thus, especially

$$\int_0^{|y'(t_k)|} \frac{t\,dt}{H_k(t \operatorname{sgn} y'(t_k))} \leqq |y(\tau_k)| \leqq N, \qquad k = 1, 2, 3, \ldots$$

According to the assumptions of the theorem a constant $N_1 < \infty$ exists such that $H_k(v) \leqq N_1 g(v)$, $k = 1, 2, \ldots$ From this

$$\frac{1}{N_1} \int_0^{|y'(t_k)|} \frac{t\,dt}{g(t \operatorname{sgn} y'(t_k))} \leqq \int_0^{|y'(t_k)|} \frac{t\,dt}{H_k(t \operatorname{sgn} y'(t_k))} \leqq N.$$

Thus with respect to the assumptions of the function g we obtain that y' must be bounded on $[t_0, \infty)$.

On the contrary let y' be bounded $|y'(t)| \leqq N_2$, $t \in [t_0, \infty)$. The statement will be proved by the indirect proof. Thus suppose that there exists a sequence of integers $\{k_i\}_1^\infty$ such that $\lim_{i \to \infty} |y(\tau_{k_i})| = \infty$. Let $\varepsilon > 0$, σ_i, $i = 1, 2, \ldots$ be such numbers that

$$|y(\tau_{k_i})| > \varepsilon, \qquad |y(\sigma_i)| = \varepsilon, \qquad t_{k_i} < \sigma_i < \tau_{k_i}, \qquad i = 1, 2, \ldots$$

hold. Put

$$A_i(v) = \min |\,\mathrm{f}(t, y, v) > 0 \qquad \text{for } \varepsilon \leqq y \leqq |y(\tau_{k_i})|, \quad t_k \leqq t \leqq t_{k+1}.$$

There exists a constant N_3 such that

(10) $$A_i(v) \geqq N_3, \qquad |v| \leqq N_2, \qquad i = 1, 2, \ldots$$

and thus (according to (1))

$$\infty > \int_0^{N_2} \frac{t\,dt}{N_3} \leqq \int_0^{|y'(t_{k_i})|} \frac{t\,dt}{A_i(t \operatorname{sgn} y'(t_{k_i}))} = \int_{t_{k_i}}^{\tau_{k_i}} \frac{|f(t, y(t), y'(t))|\;|y'(t)|}{A_i(y'(t))}\,dt \geqq$$

$$\geqq \int_{\sigma_i}^{\tau_{k_i}} |y'(t)|\,dt = |y(\tau_{k_i})| - \varepsilon \xrightarrow[i \to \infty]{} \infty.$$

But this is a contradiction.

(ii) The result follows from the proved part of the theorem and from the following conclusion which can be proved in the same way as (i):

Let $\{k_i\}_1^\infty$ be a sequence of integers. The sequence $\{|y(\tau_{k_i})|\}_1^\infty$ is bounded on $[t_0, \infty$ iff $\{|y'(t_{k_i})|\}_1^\infty$ is bounded on this interval.

(iii) It follows from the proved part of the theorem that

(11) $$|y'(t_k)| \leqq N_5 = \text{const.}, \qquad k = 1, 2, \ldots$$

holds. Denote by σ_k, $\bar{\sigma}_k$ such numbers that

$$t_k < \sigma_k < \bar{\sigma}_k \leqq \tau_k, \qquad |y(\sigma_k)| = \frac{M_5}{2}, \qquad |y(\bar{\sigma}_k)| = M_5.$$

As the function $y'(t) \operatorname{sgn} y(t)$ is decreasing in the interval (t_k, τ_k) (see Lemma 1), the following inequalities are valid

$$0 < \sigma_k - t_k < \bar{\sigma}_k - \sigma_k \leqq \tau_k - \sigma_k.$$

Put

$$A_k(v) = \min |f(t, y, v)| > 0$$

for $\quad M_5/2 \leqq |y| \leqq M_6, \quad t_k \leqq t \leqq \tau_k, \quad v \in R; \quad k = 1, 2, \dots$

Then $A_k(v) \geqq N_6 = \text{const.} > 0$ for $|v| \leqq N_5$. By multiplicating (1) by $-A_k^{-1}(y'(t))$ and by integration we can conclude

$$\infty > \frac{N_5}{N_6} \geqq \int_0^{|y'(t_k)|} \frac{\mathrm{d}t}{A_k(t \operatorname{sgn} y'(t_k))} = \int_{t_k}^{\tau_k} \frac{|f(t, y(t), y'(t))|}{A_k(y'(t))} \mathrm{d}t \leqq$$

$$\geqq \int_{\sigma_k}^{\tau_k} \frac{|f(t, y(t), y'(t))|}{A_k(y'(t))} \mathrm{d}t \geqq \int_{\sigma_k}^{\tau_k} \mathrm{d}t = \tau_k - \sigma_k .$$

Thus $\tau_k - t_k = (\tau_k - \sigma_k) + (\sigma_k - t_k)$ is bounded above for $k = 1, 2, 3, \dots$ It can be proved similarly that $\{t_{k+1} - \tau_k\}_1^\infty$ is bounded.

The boundedness of $\{\Delta_k\}_1^\infty$ from below by a positive constant is a consequence of Lemma 1 and (11). The theorem is proved.

Remark 5. It is evident from the proof that under the assumptions of Theorem 4 the following assertions are valid

(i) $\quad \liminf_{k \to \infty} |y(\tau_k)| < \infty \Leftrightarrow \liminf_{k \to \infty} |y'(t_k)| < \infty$

(ii) $\quad \limsup_{t \to \infty} |y(t)| = \infty \Leftrightarrow \limsup_{t \to \infty} |y'(t)| = \infty.$

Remark 6. It can be easily seen from the proof of Theorem 4 that the conslusion (iii) holds even if we suppose

$$0 < M \leqq |f(t, y, v)|, \qquad t \in [t_0, \infty), \qquad M_1 \leqq |y| \leqq M, \qquad |v| \leqq M_2$$

instead of (8).

Theorem 5. *Let y be an oscillatory solution of* (1) *and let a continuous function g exist, $g(t) > 0$, $t \in [t_0, \infty)$, $\int_0^\infty \frac{t\,\mathrm{d}t}{g(\pm t)} < \infty$ such that for an arbitrary constant M_1, $0 < M_1$ it holds*

$$0 < M_2 g(v) \leqq |f(t, y, v)|, \qquad t \in [t_0, \infty), \qquad M_1 \leqq |y|, \qquad v \in R,$$

where $M_2 > 0$ is a suitable number (depending on M_1). Then y is bounded on $[t_0, \infty)$. If, in addition, $0 < M_3 =$ const. $\leqq |y(\tau_k)|$, $k = 1, 2, 3, \dots$, then $\{\Delta_k\}_1^\infty$ is bounded.

Proof. The first part of the theorem can be proved similarly to the second part of the statement (i) in Theorem 4 (i.e. y' is bounded $\Rightarrow$ y is bounded). We must use the estimation

$$\mathrm{A}_i(v) \geqq N_3 g(v), \qquad v \in R, \qquad i = 1, 2, 3, \dots$$

instead of (10). The boundedness of Δ_k can be proved similar to the same result in Theorem 4.

4. Consider a differential equation

$$y'' + f(t, y)\, g(y') = 0, \tag{12}$$

where $f(t, y)$, $g(v)$ are continuous functions in $D = \{(t, y) : t \in [t_0, \infty), y \in R\}$, $v \in R$, $g > 0$, $f(t, y)\, y > 0$ for $y \neq 0$.

In our further considerations we must suppose very often the validity of the conditions

$$f(t, y) = -f(t, -y), \qquad g(v) = g(-v). \tag{13}$$

This paragraph contains especially some consequences of the previous paragraphs and of some results of [5]. At first we mention here necessary results of [5].

Theorem. *Let y be an oscillatory solution of* (12). *Let the function $|f(t, y)|$ is non-increasing (non-decreasing) with respect to t in D.*

(i) *If $g(v) = g(-v)$ for $v \in (-\infty, \infty)$, then the sequence $\{|y'(t_k)|\}_1^\infty$ is non-increasing (non-decreasing) and $\tau_k - t_k \leqq t_{k+1} - \tau_k$ ($\tau_k - t_k \geqq t_{k+1} - \tau_k$) holds.*

(ii) *If $f(t, y) = -f(t, -y)$ in D, then the sequence $\{|y(\tau_k)|\}_1^\infty$ is non-decreasing (non-increasing).*

Theorem 6. *Let y be an oscillatory solution of* (12) *and let* (13) *hold. Let*

(i) (i) *$|f(t, y)|$ be non-decreasing with respect to t in D*

(ii) *a constant $M > 0$ exist such that $f(t, y)$ is non-decreasing with respect to y in $D_1 = \{(t, y) : t \in [t_0, \infty), |y| \leqq M\}$.*

If $\lim_{t\to\infty} y(t) = 0$, *then* $\lim_{k\to\infty} \Delta_k = 0$.

Proof. It follows from Theorem that $\{|y(\tau_k)|\}_1^\infty$ is non-increasing and $\{|y'(t_k)|\}_1^\infty$ is non-decreasing. Thus

$$|y'(t_k)| \geqq |y'(t_1)| = N > 0, \qquad k = 1, 2, 3, \ldots$$

Let $\varepsilon > 0$ be an arbitrary number, $\varepsilon \leqq \dfrac{\min_{|t|\leqq N} g(t)}{\max_{|t|\leqq N} g(t)} \dfrac{N}{2}$. Denote by σ_k, $k = 1, 2, 3, \ldots$ numbers such that $t_k < \sigma_k < \tau_k$, $|y'(\sigma_k)| = \varepsilon$. First we prove that $\lim_{k\to\infty} \sigma_k - t_k = 0$ holds. According to the Rolle theorem there exists a number $\xi_k \in (t_k, \sigma_k)$ such that $y'(\xi_k) = \dfrac{y(\sigma_k) - y(t_k)}{\sigma_k - t_k}$. From this

$$\sigma_k - t_k = \frac{|y(\sigma_k)|}{|y'(\xi_k)|} \leqq \frac{|y(\sigma_k)|}{\varepsilon} \xrightarrow[k\to\infty]{} 0$$

and thus $\lim_{k\to\infty} \sigma_k - t_k = 0$.

According to $\lim_{t\to\infty} y(t) = 0$ there exists an integer n such that $|y(\tau_k)| \leqq M$, $k \geqq n$.

Consider the function

$$F(t) = \int_t^{\tau_k} \frac{y''\,\mathrm{d}t}{g(y')} = -\int_0^{y'(t)} \frac{\mathrm{d}t}{g(t)} = -\int_t^{\tau_k} f(t, y(t))\,\mathrm{d}t, \qquad t \in [t_k, \tau_k].$$

If $\sigma_k \leqq t_1 \leqq t_2 \leqq \tau_k$, then $|y(t_1)| \leqq |y(t_2)|$ and

$$F'(t_2) - F'(t_1) = f(t_2, y(t_2)) - f(t_1, y(t_1)) =$$
$$= [f(t_2, y(t_2)) - f(t_2, y(t_1))] + [f(t_2, y(t_1)) - f(t_1, y(t_1))] \geqq$$
$$\geqq 0 \qquad \text{for } y(t_1) > 0$$
$$\leqq 0 \qquad \text{for } y(t_1) < 0\,.$$

As $F'(t) > 0$ (< 0) if $y > 0$ ($y < 0$), the function $|F'|$ is non-decreasing on (t_k, τ_k). Denote by $\bar{\sigma}_k$ such number that $t_k \leqq \bar{\sigma}_k < \sigma_k$, $|F(\bar{\sigma}_k)| = 2|F(\sigma_k)|$. $\bar{\sigma}_k$ really exists because $|F|$ is non-increasing and

$$|F(\sigma_k)| = \int_0^s \frac{\mathrm{d}t}{g(t)} \leqq \frac{\varepsilon}{\min\limits_{|t|\leqq N} g(t)} \leqq \frac{N}{2}\,\frac{1}{\max\limits_{|t|\leqq N} g(t)},$$

$$|F(t_k)| = \int_0^{|y'(t_k)|} \frac{\mathrm{d}t}{g(t)} \leqq \frac{N}{\max\limits_{|t|\leqq N} g(t)}.$$

According to the mean value theorem we have

$$|F(\sigma_k)| = |F(\tau_k) - F(\tau_k)| = |F'(\xi_1)|\,(\tau_k - \sigma_k), \qquad \xi_1 \in (\sigma_k, \tau_k),$$
$$|F(\sigma_k)| = |F(\sigma_k) - F(\bar{\sigma}_k)| = |F'(\xi_2)|\,(\sigma_k - \bar{\sigma}_k), \qquad \xi_2 \in (\bar{\sigma}_k, \sigma_k).$$

From this and with respect to $|F'|$ being non-decreasing, we can conclude

$$\tau_k - \sigma_k \leqq \sigma_k - \bar{\sigma}_k \leqq \sigma_k - t_k \xrightarrow[k\to\infty]{} 0.$$

Thus finally $\lim_{k\to\infty} \tau_k - t_k = 0$.

By Theorem there holds $t_{k+1} - \tau_k \leqq \tau_k - t_k$ and the theorem is proved.

Remark 7. Katranov [11], [12] deals with the problem of Theorem 6. He proved the statement of this theorem but under the more restrictive assumptions. He must, in addition, suppose that

1° there exists $\frac{\partial}{\partial t} f(t, y)$ and it is continuous

2° for an arbitrary $M \neq 0$ there holds $\lim_{t\to\infty} |f(t, M)| = \infty$

3° there exists a constant $g_1 > 0$ such that $g(v) > g_1$, $v \in R$

4° the uniqueness of the Cauchy initial problem holds.

Theorem 7. *Let y be an oscillatory solution of* (12), (13) *and let* $|f(t, y)|$ *be non-increasing with respect to t in D. Further, let for an arbitrary constant* $0 < M$ *there holds* $\lim_{t\to\infty} f(t, y) = 0$ *uniformly for* $|y| \leqq M$. *Then*

$$\lim_{k\to\infty} \Delta_k = \infty.$$

Proof. It is evident that the assumptions of Theorem 1 are fulfilled. Thus either $\lim_{t\to\infty} y'(t) = 0$ or y is unbounded. From this and according to Lemma 1 and Theorem we have

$$\Delta_k > \frac{|y(\tau_k)|}{|y'(t_k)|} \xrightarrow[k\to\infty]{} \infty.$$

The theorem is proved.

The following Corollary is a consequence of Theorem 4 and Theorem.

Corollary 3. Let y be an oscillatory solution of (12), (13). Let $|f(t, y)|$ be non-increasing (non-decreasing) with respect to t in D and let $\int_0^\infty \frac{t}{g(t)} \mathrm{d}t = \infty$. Further suppose that for an arbitrary constant M, $0 < M$ there exists a number M_1 such that

$$0 < M_1 \leqq \lim |f(t, y)|, \qquad M \leqq |y|,$$
$$(\lim_{t\to\infty} |f(t, y)| \leqq M_1 \qquad \text{for } |y| \leqq M)$$

holds. Then the sequences $\{|y(\tau_k)|\}_1^\infty$, $\{|y'(t_k)|\}_1^\infty$, $\{\Delta_k\}_1^\infty$ are bounded above and are bounded away from zero.

Remark 8. The problems concerning the boundedness of y and y' are studied in [6], [7], [14], [18], [19] (these papers deal with the differential equation (1), $f(t, y, v) = a(t) r(y) h(y')$) and in [8], [10], [13], [16], [17] (for $f(t, y, v) = a(t) r(y)$), but mostly without the assumptions (13) and $\int_0^\infty \frac{t}{g(t)} = \infty$. The other assumptions of Corollary 3 are supposed, too.

REFERENCES

[1] Bartušek M.: *On Asymptotic Properties and Distribution of Zeros of Solutions of* $y'' = q(t) y$. Acta F.R.N. Univ. Comenianae, Math., XXXII, 1975, 69—86.

[2] Бартушек М.: *О нулях колеблющихся решений уравнения* $[p(t) x']' + f(t, x, x') = 0$. Дудар. урав. XII, 1976, 621—625.

[3] Bartušek M.: *On Zeros of Solutions of the Differential Equation* $[p(t) y']' + f(t, y, y') = 0$. Arch. Math. XI, No 4, 1975, 187—192.

[4] Bartušek M.: *On Distribution of Zeros of Solutions of Differential Equations* $y'' + f(t, y, y') = 0$ Arch. Math. XIII, No 2, 1977, 69—74.

[5] Bartušek M.: *Monotonicity Theorems Concerning Differential Equations* $y'' + f(t, y, y') = 0$. Arch. Math. XII, No 4, 1976, 169–178.

[6] Bihari I.: *Oscillation and Monotonity Theorems Concerning Non-linear Differential Equations of the Second Order.* Acta Math. Acad. Sci. Hung., IX, No 1—2, 1958, 83—104.

[7] Burton T., Grimmer R.: *On the Asymptotic Behaviour of Solutions of* $x'' + a(t) f(x) = 0$. Proc. Camb. Phil. Soc., 70, 1971, 77—88.

[8] Dlotko T.: *Sur làllure asymptotique des solutions de l'equation différentielle ordinaire du second ordre.* Ann. polon. math., 11, 1961, No 3, 261—273.

[9] Ендовицкий И. И.: *Некоторые условия неустойчивости решений уравнения* $u'' + -p(x)u = 0$. Изв. вузов, мат., 140, Но 1, 1974, 47—51.

[10] Изюмова Д. В., Кичурадзе И. Т.: *Некоторые замечания о решениях уравнения* $u'' + a(t)f(u) = 0$. Дифф. урав., IV, Но 4, 1968, 589—605.

[11] Катранов А. Г.: *О нулях колеблющихся решений уравнения* $x'' + a(t)f(x) = 0$. Дифф. урав., VII, Но 5, 1971, 930—933.

[12] Катранов А. Г.: *Об асимптотическом поведении колеблющихся решений уравнения* $\ddot{x} + f(t, x) g(\dot{x}) = 0$. Дифф. урав., VIII, Но 6, 1972, 1111—1115.

[13] Kroopnik A.: *Properties of Solutions of Differential Equations of the Form* $y'' + a(t) b(y) = 0$. Proc. Amer. Math. Soc., 34, No 1, 1972, 319—320.

[14] Lalli B. S.: *On Boundedness of Solutions of Certain Second Order Differential Equations.* J. Math. Anal. Appl., 25, 1969, 182—188.

[15] Сансоне Дж.: *Обыкновенные дифференциальные уравнения* II. Moscow 1954.

[16] Utz: *Properties of Solutions of* $u'' + q(t) u^{2n-1} = 0$. II. Monatsh. Math., 69, No 4, 1965, 353—361.

[17] Waltman P.: *Some Properties of Solutions of* $u'' + a(t) f(u) = 0$. Monatsh. Math., 67, 1963, No 1, 50—54.

[18] Wong J. S. W.: *Boundedness Theorems for Solutions of* $u''(t) + a(t) f(u) g(u') = 0$. IV. L'enseignement Math., VII, 1961, 157—165.

[19] Wong J. S. W., Burton T.: *Some Properties of Solutions of* $u''(t) + a(t) f(u) g(u') = 0$. II. Monatsh. Math., 69, No 4, 1965, 368—374.

M. Bartušek
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia