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AN EMBEDDING PROBLEM AND ITS APPLICATION IN LINGUISTICS

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0. INTRODUCTION

We say that an ordered pair (V, L) is a *language* if V is a finite set (the *alphabet*) and L is a subset of the free monoid V^* over V.

Let (V, L) be a language. We define for $a \in V$

$$\sigma_L(a) = \{(u, v); (u, v) \in V^* \times V^* \text{ and } uav \in L\}.$$

We call $\sigma_L(a)$ the *set of all contexts accepted by a in* (V, L). We

$$\text{put} \begin{Bmatrix} \mathbf{H}(V,L) \\ \mathbf{R}(V,L) \\ \mathbf{P}(V,L) \\ \mathbf{I}(V,L) \\ \mathbf{N}(V,L) \\ \mathbf{F}(V,L) \\ \mathbf{C}(V,L) \end{Bmatrix} = \{\sigma_L(a); a \text{ is a} \begin{Bmatrix} \text{pure homonym} \\ \text{root} \\ \text{partial homonym} \\ \text{initial word-form} \\ \text{nonhomonym} \\ \text{free homonym} \\ \text{complete element} \end{Bmatrix} \text{in } (V, L)\}.$$

The definitions of the above mentioned special types of elements of the alphabet can be found in [6] or in [4]. Let us denote

$$\mathfrak{A}(V, L) = \{\sigma_L(a); a \in V\}.$$

We say that a language (V, L) *contains no parasitary elements* whenever the empty set $\emptyset$ is not in $\mathfrak{A}(V, L)$. The set $\mathfrak{A}(V, L)$, ordered by inclusion, is a finite poset for each language (V, L).

Let G be a poset. If (V, L) is a language and $r: G \to \mathfrak{A}(V, L)$ an isomorphism then we call the ordered pair $(r, (V, L))$ a *p-representation* of G.

In the Main theorem we characterize, for a given finite poset G, all ordered seventuples (H, R, P, I, N, F, C) of elements from 2^G (the set of all subsets of G) such that there exists a p-representation $(r, (V, L))$ of G with the following properties. (V, L) contains no parasitary elements and $\{r(a); a \in M\} = M(V, L)$ for $M = H, R, P, I, N, F, C$.

1. FORMULATION OF THE PROBLEM

Let $n > 0$ be an integer. We denote by $\{a_1, a_2, \ldots, a_n\}$ the finite set containing just the elements $a_1, a_2, \ldots, a_n$. In case $n = 0$ we define $\{a_1, a_2, \ldots, a_n\} = \emptyset$. Further, we put $\bigcup \emptyset = \emptyset$.

Let A, D be sets and $e: A \to D$ a map. We denote $e[B] = \{e(b); b \in B\}$ for each $B \subseteq A$. If $e(a) = a$ for all $a \in A$ then we call e an *identity map*. If, moreover, $D = A$ then we put $e = 1_A$. We say that e is an *embedding* (of A into D) if A, D are posets and if $a \leqq b \Leftrightarrow e(a) \leqq e(b)$ for all $a, b \in A$. Obviously, any embedding is an injection.

The definitions of the *minimal condition*, the *closure operator*, and the *Galois connection* can be found in [3].

Let G be a poset. We put $\omega_G(a) = \{b; b \in G \text{ and } b \leqq a\}$ for each $a \in G$. Each subset of G is considered partially ordered by the restriction of the ordering on G.

We denote by $\bigvee_G A$ the l.u. bound of A in G for each $A \subseteq G$ and write $a \vee b$ instead of $\bigvee_G \{a, b\}$. We define $\bigvee_G \emptyset$ iff there exists the smallest element o in G; then we put $\bigvee_G \emptyset = o$.

1.1. Lemma. *Let G be a poset, $a \in G$, $B \subseteq G$, $C(b) \subseteq G$ for each $b \in B$. If $a = \bigvee_G B$ and $b = \bigvee_G C(b)$ for each $b \in B$ then $a = \bigvee_G \bigcup_{b \in B} C(b)$.*

1.2. Definition. Let G be a poset. We call $a \in G$ (*completely additively*) *irreducible* (in G) if $a = \bigvee_G A \Rightarrow a \in A$ for each $A \subseteq G$.

We denote by $\mathbf{IR}_G$ the set of all irreducible elements in G. Further, we put

$$IR_G = \begin{cases} \mathbf{IR}_G \cup \{o\} & \text{if } o \text{ is the smallest element in } G, \\ \mathbf{IR}_G & \text{if there is not a smallest element in } G. \end{cases}$$

1.3. Definition. Let H be a poset and $G \subseteq H$. We call G a *σ-dense subset* (in H) if there exists $A(a) \subseteq G$ such that $a = \bigvee_H A(a)$ for each $a \in H$.

1.4. Lemma. *Let H be a poset and G a σ-dense subset in H. Then $\mathbf{IR}_H \subseteq G$.*

We now give a sufficient condition under which the converse of 1.4 is true.

1.5. Lemma. *Let H be a poset and let there exist a σ-dense subset G, satisfying the minimal condition, in H. Then $\mathbf{IR}_H$ is a σ-dense subset in H.*

Proof. Let us put $O = \{a; a \in G \text{ and } a \neq \bigvee_H A \text{ for each } A \subseteq \mathbf{IR}_H\}$. If $O \neq \emptyset$ then there exists m minimal in O. Since $m \notin \mathbf{IR}_H$, we can find $A \subseteq H$ such that $m \notin A$, $m = \bigvee_H A$. As G is σ-dense in H, there exists $B(a) \subseteq G$ with the property

$a = \bigvee_H B(a)$ for each $a \in A$. If we put $B = \bigcup_{a \in A} B(a)$ then $B \subseteq G$ and $m = \bigvee_H B$ by 1.1. The obvious fact $B \cap O = \emptyset$ implies the existence of $C(b) \subseteq \mathbf{IR}_H$ such that $b = \bigvee_H C(b)$ for each $b \in B$. By this and by 1.1, it follows that $m = \bigvee_H C$ for $C = \bigcup_{b \in B} C(b)$. Since $C \subseteq \mathbf{IR}_H$, we have a contradiction with $m \in O$. Thus, $O = \emptyset$ and the statement follows by 1.1.

1.6. Definition. Let G be a poset and S a complete lattice with the smallest element o. We call the map e: $G \to S$ a *σ_0-dense embedding* (of G into S) if e is an embedding, $e[G]$ is a σ-dense subset in S, $o \notin e[G]$.

1.7. Remark. (i) If we omit the requirement $o \notin e[G]$ in 1.6 then we obtain the concept of a σ-dense embedding which was studied in [1], [5], [7] and in many other works.

(ii) Let S be a lattice. Then S is finite and nonempty whenever there exists a σ_0-dense embedding of a finite poset into S.

1.8. Definition. Let S be a lattice.

(i) We call $a \in S$ *strong* (in S) if $b < c$, $a \parallel c \Rightarrow a \vee b < a \vee c$ for all $b, c \in S$.

(ii) We call $a \in S$ (*completely additively*) *primitive* (in S) if $a \leqq \bigvee_S A \Rightarrow$ there exists $b \in A$ such that $a \leqq b$ for each $A \subseteq S$.

We denote by $\mathbf{S}_S$, $\mathbf{P}_S$ the set of all strong, primitive elements in S, respectively.

1.9. Definition. Let S be a lattice and o the smallest element in S.

(i) We call $a \in S$ an *atom* (in S) if $b < a \Rightarrow b = o$ for each $b \in S$.

We denote by $\mathbf{A}_S$ the set of all atoms in S.

(ii) We call $N \subseteq S$ a *nonhomonymous set* (in S) if N is finite, $N \subseteq \mathbf{S}_S \cap \mathbf{P}_S \cap \mathbf{A}_S$ and if no element from $\mathbf{A}_S$ is the smallest one in $S - \omega_S(\bigvee_S N)$.

We denote by $\mathfrak{N}_S$ the set of all nonhomonymous sets in S.

1.10. Lemma. *Let S be a lattice with a smallest element and $N \subseteq \mathbf{S}_S \cap \mathbf{P}_S \cap \mathbf{A}_S$ a finite set. Then $\mathbf{IR}_S \cap \omega_S(\bigvee_S N) = N$.*

Proof. Clearly, $N \subseteq \mathbf{IR}_S \cap \omega_S(\bigvee_S N)$. If $a \in \omega_S(\bigvee_S N)$ then $a = \bigvee_S A$ for $A = \{b; b \in N \text{ and } b \leqq a\}$ by [4] II, 1.16. If, moreover, $a \in \mathbf{IR}_S$ then $a \in A \subseteq N$. Thus, $\mathbf{IR}_S \cap \omega_S(\bigvee_S N) \subseteq N$.

We shall see that our Main theorem is an easy consequence of the main results from [4] II and of the solution of the following

1.11. Problem. *Let G be a poset satisfying the minimal condition. What are the necessary and sufficient conditions imposed on an ordered fourtuple (I, R, N, C) of subsets of G for the existence of a complete lattice S and a σ_0-dense embedding e of G into S such that $e[I] = \mathbf{A}_S$, $e[R] = \mathbf{IR}_S$, $e[N] \in \mathfrak{N}_S$, $e[C] = \mathbf{P}_S$.*

2. 0-GENERATING SYSTEMS AND 0-EMBEDDING OPERATORS

2.1. Definition. Let G be a poset. We call $A \subseteq G$ an *initial segment* (in G) if $\omega_G(a) \subseteq A$ for each $a \in A$.

We denote by Ω_G the set of all initial segments in G.

2.2. Definition. Let G be a poset and $\mathfrak{G} \subseteq 2^G$. We call $\mathfrak{G}$ a *0-generating system* (on G) if

(i) $\mathfrak{G} \subseteq \Omega_G$,

(ii) $\bigcap \mathfrak{A} \in \mathfrak{G}$ for each $\mathfrak{A} \subseteq \mathfrak{G}$, $\mathfrak{A} \neq \emptyset$,

(iii) $\omega_G[G] \subseteq \mathfrak{G}$,

(iv) $\{\emptyset, G\} \subseteq \mathfrak{G}$.

We denote by $\mathrm{Gs}(G)$ the set of all 0-generating systems on G.

2.3. Lemma. *Let G be a poset, $\mathfrak{G} \in \mathrm{Gs}(G)$, $\mathfrak{A} \subseteq \mathfrak{G}$, $\mathfrak{H} = \mathfrak{G} - \mathfrak{A}$. If*

(i) $\mathfrak{A} \cap (\omega_G[G] \cup \{\emptyset, G\}) = \emptyset$ *and*

(ii) *for each $A \in \mathfrak{A}$ there exists $a \in G - A$ such that*

$$A \subseteq B, a \notin B \Rightarrow B \in \mathfrak{A} \quad \textit{for all } B \in \mathfrak{G}$$

then $\mathfrak{H} \in \mathrm{Gs}(G)$.

Proof. Clearly, 2.2 (i), (iii), (iv) hold for $\mathfrak{H}$. For an arbitrary $\mathfrak{B} \subseteq \mathfrak{H}$, $\mathfrak{B} \neq \emptyset$ we have $\bigcap \mathfrak{B} \in \mathfrak{G}$ and we can find $A(a) \in \mathfrak{B}$ with the properties $\bigcap \mathfrak{B} \subseteq A(a)$, $a \notin A(a)$ for each $a \in G - \bigcap \mathfrak{B}$. This and $\mathfrak{A} \cap \mathfrak{B} = \emptyset$ give $\bigcap \mathfrak{B} \notin \mathfrak{A}$. It follows that $\bigcap \mathfrak{B} \in \mathfrak{H}$ and 2.2 (ii) is true.

2.4. Lemma. *Let G be a poset and φ a closure operator on 2^G. Then the assertions* (i) *and* (ii) *hold for all $A, B \subseteq G$.*

(i) $B \subseteq A \subseteq \varphi(B) \Rightarrow \varphi(A) = \varphi(B)$.

(ii) $\varphi(\varphi(A) \cup B) = \varphi(A \cup B)$.

2.5. Definition. Let G be a poset and φ a closure operator on 2^G. We call φ a 0-embedding operator (on 2^G) if $\varphi(\{a\}) = \omega_G(a)$ for each $a \in G$ and $\varphi(\emptyset) = \emptyset$.

We denote by $\mathrm{Op}(G)$ the set of all 0-embedding operators on 2^G.

2.6. Remark. If we consider $\mathrm{Gs}(G)$, $\mathrm{Op}(G)$ partially ordered then the ordering on $\mathrm{Gs}(G)$ is the inclusion and that on $\mathrm{Op}(G)$ is the following. For arbitrary $\varphi, \psi \in \mathrm{Op}(G)$ we have $\varphi \leqq \psi$ whenever $\varphi(A) \subseteq \psi(A)$ for each $A \subseteq G$.

2.7. Definition. Let G be a poset. We associate a map $\xi_G\mathfrak{G} : 2^G \to 2^G$, defined by $\xi_G\mathfrak{G}(A) = \bigcap_{A \subseteq B \in \mathfrak{G}} B$ for every $A \subseteq G$, with each $\mathfrak{G} \in \mathrm{Gs}(G)$ and a set $\mathfrak{C}_G\varphi = \varphi[2^G]$ with each $\varphi \in \mathrm{Op}(G)$.

2.8. Theorem. *Let G be a poset. The pair ξ_G, $\mathfrak{C}_G$ forms a Galois connection between the posets* $\mathrm{Gs}(G)$, $\mathrm{Op}(G)$ *and it holds*

$$\mathfrak{C}_G\xi_G = 1_{\mathrm{Gs}(G)}, \qquad \xi_G\mathfrak{C}_G = 1_{\mathrm{Op}(G)}.$$

2.9. Lemma. *Let G be a poset and $\mathfrak{G} \in \mathrm{Gs}(G)$. Then $\mathfrak{G}$, ordered by inclusion, is a complete lattice in which meets coincide with intersections and $\bigvee_{\mathfrak{G}}\mathfrak{A} = \xi_G\mathfrak{G}(\bigcup\mathfrak{A})$ for each $\mathfrak{A} \subseteq \mathfrak{G}$.*

Proof. The first part of the statement follows by 2.2 (ii), (iv) and by theorem 10 from [3]. By 2.8, $\xi_G\mathfrak{G}$ is a closure operator on the complete lattice 2^G. The second part of the statement is now a consequence of theorem 15 from [3].

The connection between the concept of a 0-generating system and that of a σ_0-dense embedding is formulated in the following fundamental theorem which was proved in [5] for the case of σ-dense embeddings.

2.10. Theorem. *Let G be a poset. Then*

(i) *For each $\mathfrak{G} \in \mathrm{Gs}(G)$, $\omega_G : G \to \mathfrak{G}$ is a σ_0-dense embedding.*

(ii) *For each σ_0-dense embedding e of G into a complete lattice S there exist $\mathfrak{G} \in \mathrm{Gs}(G)$ and an isomorphism $\iota : S \to \mathfrak{G}$ such that $\iota e = \omega_G$.*

2.11. Corollary. *Let G be a poset and $\mathfrak{G} \in \mathrm{Gs}(G)$. Then $\mathbf{IR}_{\mathfrak{G}}$, $\mathbf{P}_{\mathfrak{G}}$, $\mathbf{A}_{\mathfrak{G}}$, and all $\mathbf{N} \in \mathfrak{N}_{\mathfrak{G}}$ are subsets of $\omega_G[G]$.*

Proof. This assertion is a consequence of 2.10 (i), 1.4, and of the inclusions $\mathbf{P}_{\mathfrak{G}} \subseteq \mathbf{IR}_{\mathfrak{G}}$, $\mathbf{A}_{\mathfrak{G}} \subseteq \mathbf{IR}_{\mathfrak{G}}$, $\mathbf{N} \subseteq \mathbf{IR}_{\mathfrak{G}}$ for each $\mathbf{N} \in \mathfrak{N}_{\mathfrak{G}}$.

3. SPECIAL PROPERTIES OF ELEMENTS IN 0-GENERATING SYSTEMS AND THE 0-EMBEDDING OPERATOR φ_G^R

3.1. Definition. Let G be a poset and $a \in G$. We put

$$\omega_G^-(a) = \omega_G(a) - \{a\}, \qquad \varepsilon_G(a) = \{b;\, b \in G \text{ and } a \leqq b\},$$
$$\bar{\varepsilon}_G(a) = G - \varepsilon_G(a).$$

3.2. Lemma. *Let G be a poset. Then*

(i) *$\bar{\varepsilon}_G(a) \subseteq \bar{\varepsilon}_G(b) \Leftrightarrow a \leqq b$ for all $a, b \in G$.*

(ii) *$\omega_G^-(a) \subseteq \bar{\varepsilon}_G(b)$, $a \in \varepsilon_G(b) \Rightarrow a = b$ for all $a, b \in G$.*

(iii) *$A \nsubseteq \bar{\varepsilon}_G(a) \Leftrightarrow a \in A$ for all $a \in G$, $A \in \Omega_G$.*

(iv) *$\omega_G^-(a) \subseteq A \Leftrightarrow A \cup \{a\} \in \Omega_G$ for all $a \in G$, $A \in \Omega_G$.*

3.3. Lemma. *Let G be a poset and $a \in G$. Then $a \notin \mathbf{IR}_G$ if and only if $a = \bigvee_G\omega_G^-(a)$.*

3.4. Lemma. *Let G be a poset, $\mathfrak{G} \in \mathrm{Gs}(G)$, $a \in G$. Then the assertions* (i) *and* (ii) *are equivalent.*

(i) $\omega_G(a) \in \mathbf{IR}_{\mathfrak{G}}$.
(ii) $\omega_G^-(a) \in \mathfrak{G}$.

Proof. Let us put $\mathfrak{A} = \omega_{\mathfrak{G}}^-(\omega_G(a))$. If $b \in \omega_G^-(a)$ then $b \in \omega_G(b) \in \mathfrak{A}$ by 2.10 (i) and we have $\omega_G^-(a) \subseteq \bigcup\mathfrak{A}$. This inclusion and the obvious validity of its converse imply $\omega_G^-(a) = \bigcup\mathfrak{A}$. By 2.9, it follows that $\bigvee_{\mathfrak{G}}\mathfrak{A} = \xi_G\mathfrak{G}(\bigcup\mathfrak{A}) = \xi_G\mathfrak{G}(\omega_G^-(a))$. Since either $\xi_G\mathfrak{G}(\omega_G^-(a)) = \omega_G^-(a)$ or $\xi_G\mathfrak{G}(\omega_G^-(a)) = \omega_G(a)$, it holds $\bigvee_{\mathfrak{G}}\mathfrak{A} = \omega_G(a)$ iff $\omega_G^-(a) \notin \mathfrak{G}$. Now, the statement follows by 3.3.

3.5. Lemma. *Let G be a poset, $\mathfrak{G} \in \mathrm{Gs}(G)$, $a \in G$. Then the assertions* (i), (ii), (iii) *are equivalent.*

(i) $\omega_G(a) \in \mathbf{P}_{\mathfrak{G}}$.
(ii) $\varepsilon_G(a) \in \mathfrak{G}$.
(iii) $a \notin \xi_G\mathfrak{G}(A)$ *for each $A \in \Omega_G$ such that $a \notin A$.*

Proof. Let us assume $\omega_G(a) \in \mathbf{P}_{\mathfrak{G}}$. Then $\omega_G(a) \nsubseteq \bigvee_{\mathfrak{G}}\mathfrak{A}$ for each $\mathfrak{A} \subseteq \mathfrak{G}$ such that $\omega_G(a) \nsubseteq \bigcup\mathfrak{A}$. We have $\bigcup\mathfrak{B} = \varepsilon_G(a)$ for $\mathfrak{B} = \omega_G[\varepsilon_G(a)]$. By $\omega_G(a) \nsubseteq \bigcup\mathfrak{B}$ and by 2.9, it follows that $\omega_G(a) \nsubseteq \bigvee_{\mathfrak{G}}\mathfrak{B} = \xi_G\mathfrak{G}(\varepsilon_G(a))$. Hence, $a \notin \xi_G\mathfrak{G}(\varepsilon_G(a)) \in \mathfrak{G}$ and, consequently, $\xi_G\mathfrak{G}(\varepsilon_G(a)) \subseteq \varepsilon_G(a)$ by 3.2 (iii). Since the inverse inclusion is obvious, we obtain $\xi_G\mathfrak{G}(\varepsilon_G(a)) = \varepsilon_G(a)$ and, consequently, $\varepsilon_G(a) \in \mathfrak{G}$. We have proved (i) $\Rightarrow$ (ii).

If $\varepsilon_G(a) \in \mathfrak{G}$ then for $A \in \Omega_G$ satisfying $a \notin A$ it holds $A \subseteq \varepsilon_G(a)$, $\xi_G\mathfrak{G}(A) \subseteq \xi_G\mathfrak{G}(\varepsilon_G(a)) = \varepsilon_G(a)$ and $a \notin \xi_G\mathfrak{G}(A)$. Thus, (ii) $\Rightarrow$ (iii).

Suppose $a \notin \xi_G\mathfrak{G}(A)$ for each $A \in \Omega_G$ with the property $a \notin A$. If $\omega_G(a) \subseteq \bigvee_{\mathfrak{G}}\mathfrak{A}$ for $\mathfrak{A} \subseteq \mathfrak{G}$ then $a \in \xi_G\mathfrak{G}(\bigcup\mathfrak{A})$ by 2.9. This gives $a \in \bigcup\mathfrak{A}$ and there exists $A \in \mathfrak{A}$ such that $a \in A$; we have $\omega_G(a) \subseteq A$ and, therefore, $\omega_G(a) \in \mathrm{P}_{\mathfrak{G}}$. Hence (iii) $\Rightarrow$ (i).

3.6. Definition. Let G be a poset, $\mathfrak{G} \in \mathrm{Gs}(G)$, $a \in G$. We denote by $\mathscr{V}(\mathfrak{G}, a)$ the following assertion.

$$A \vee \omega_G(a) \subseteq A \cup \omega_G(a) \cup \varepsilon_G(a) \quad \text{for each } A \in \mathfrak{G}.$$

3.7. Lemma. *Let G be a poset, $\mathfrak{G} \in \mathrm{Gs}(G)$, $a \in G$. Then*

(i) $\omega_G(a) \in \mathbf{S}_{\mathfrak{G}} \cap \mathbf{P}_{\mathfrak{G}} \Rightarrow \mathscr{V}(\mathfrak{G}, a)$.
(ii) $\omega_G(a) \in \mathbf{A}_{\mathfrak{G}}, \mathscr{V}(\mathfrak{G}, a) \Rightarrow \omega_G(a) \in \mathbf{S}_{\mathfrak{G}}$.

Proof. Suppose $\omega_G(a) \in \mathbf{S}_{\mathfrak{G}} \cap \mathbf{P}_{\mathfrak{G}}$. Let us admit that there exists $A \in \mathfrak{G}$ such that $A \vee \omega_G(a) \nsubseteq A \cup \omega_G(a) \cup \varepsilon_G(a)$. Then, clearly, $A \parallel \omega_G(a)$ and we can find $b \in A \vee \omega_G(a)$ with the properties $\omega_G(b) \nsubseteq A$, $\omega_G(b) \parallel \omega_G(a)$. If we put $B = A \vee \omega_G(b)$ then $A \subset B$. Further, $B \subseteq \omega_G(a)$ would imply $A \subseteq \omega_G(a)$ which is a contradiction. Similarly, $\omega_G(a) \subseteq B = A \vee \omega_G(b)$ would imply either $\omega_G(a) \subseteq A$ or $\omega_G(a) \subseteq \omega_G(b)$ because $\omega_G(a) \in \mathbf{P}_{\mathfrak{G}}$; both cases are impossible. Thus, $B \parallel \omega_G(a)$. Simultaneously,

$A \vee \omega_G(a) = (A \vee \omega_G(a)) \vee \omega_G(b) = B \vee \omega_G(a)$ which contradicts $\omega_G(a) \in \mathbf{S}_{\mathfrak{G}}$. Thus, $\mathscr{V}(\mathfrak{G}, a)$ is true.

Suppose $\omega_G(a) \in \mathbf{A}_{\mathfrak{G}}$, $\mathscr{V}(\mathfrak{G}, a)$. Let us take $A, B \in \mathfrak{G}$ such that $A \subset B$, $B \parallel \omega_G(a)$. Then $a \notin B$, $a \notin A$ and there exists $b \in B - A$. The facts $\omega_G(a) \in \mathbf{A}_{\mathfrak{G}}$, $B \cap \omega_G(a) \subset \subset \omega_G(a)$ give $B \cap \omega_G(a) = \emptyset$. For this reason $b \nleq a$. As, at the same time, $a \notin B$, we obtain $a \nleq b$. Hence, $b \parallel a$ and it follows that $b \notin A \vee \omega_G(a)$ according to $\mathscr{V}(\mathfrak{G}, a)$. Since $A \subseteq B$ and $b \in B \vee \omega_G(a)$, it holds $A \vee \omega_G(a) \subset B \vee \omega_G(a)$. We have proved $\omega_G(a) \in \mathbf{S}_{\mathfrak{G}}$.

3.8. Corollary. *Let G be a poset. Let us take $\mathfrak{G} \in \mathrm{Gs}(G)$ and $a \in G$ in such a way that $\omega_G(a) \in \mathbf{P}_{\mathfrak{G}} \cap \mathbf{A}_{\mathfrak{G}}$. Then the assertions* (i) *and* (ii) *are equivalent.*

(i) $\omega_G(a) \in \mathbf{S}_{\mathfrak{G}}$.

(ii) $\mathscr{V}(\mathfrak{G}, a)$.

We shall now deal with a 0-embedding operator of a special kind which will often appear in our considerations.

3.9. Definition. Let G be a poset. We call R an *irreducible set* (in G) if $IR_G \subseteq R \subseteq G$.

3.10. Definition. Let G be a poset and R an irreducible set in G. We put

$$\mathfrak{H}_G^R = \{A;\, A \in \Omega_G \text{ and } \omega_G^-(a) \subseteq A \Rightarrow a \in A \text{ for each } a \in G - R\}.$$

3.11. Lemma. *Let G be a poset and R an irreducible set in G. Then $\mathfrak{H}_G^R \in \mathrm{Gs}(G)$.*

Proof. The condition 2.2 (i) is satisfied trivially. Let $\mathfrak{A} \subseteq \mathfrak{H}_G^R$ be nonempty. Then, clearly, $\bigcap\mathfrak{A} \in \Omega_G$. If $\omega_G^-(a) \subseteq \bigcap\mathfrak{A}$ for $a \in G - R$ then $\omega_G^-(a) \subseteq A$ and $a \in A$ for each $A \in \mathfrak{A}$. It follows that $a \in \bigcap\mathfrak{A}$. We have $\bigcap\mathfrak{A} \in \mathfrak{H}_G^R$ which proves 2.2 (ii). If $\omega_G^-(a) \subseteq \omega_G(b)$ for $a \in G - R$, $b \in G$ then b is an upper bound of $\omega_G^-(a)$. Since $a = \bigvee_G \omega_G^-(a)$ by 3.3, we obtain $a \in \omega_G(b)$. For this reason $\omega_G(b) \in \mathfrak{H}_G^R$ and 2.2 (iii) is true. If $\omega_G^-(a) \subseteq \emptyset$ for $a \in G$ then a is minimal in G. Clearly, $a \in IR_G$ and, consequently, $a \notin G - R$. Thus, $\emptyset \in \mathfrak{H}_G^R$. As $G \in \mathfrak{H}_G^R$ in an obvious way, 2.2 (iv) holds.

3.12. Lemma. *Let G be a poset and R an irreducible set in G. Then $\mathbf{IR}_{\mathfrak{H}_G^R} = \omega_G[R] = \mathbf{P}_{\mathfrak{H}_G^R}$.*

Proof. Clearly, $\omega_G^-(a) \notin \mathfrak{H}_G^R$ for each $a \in G - R$. By this, 3.4, and 2.11, $\mathbf{IR}_{\mathfrak{H}_G^R} \subseteq \omega_G[R]$. Suppose $\omega_G^-(b) \subseteq \bar{\varepsilon}_G(a)$ for arbitrary $a \in R$, $b \in G - R$. If $b \notin \bar{\varepsilon}_G(a)$ then $b \in \varepsilon_G(a)$ and $b = a$ according to 3.2 (ii). This is a contradiction. Hence, we have $b \in \bar{\varepsilon}_G(a)$ and, therefore, $\bar{\varepsilon}_G(a) \in \mathfrak{H}_G^R$. Since $\omega_G(a) \in \mathbf{P}_{\mathfrak{H}_G^R}$ by 3.5, we obtain $\omega_G[R] \subseteq \mathbf{P}_{\mathfrak{H}_G^R}$. The statement follows by the proved inclusions and by $\mathbf{P}_{\mathfrak{H}_G^R} \subseteq \mathbf{IR}_{\mathfrak{H}_G^R}$.

3.13. Lemma. *Let G be a poset, R an irreducible set in G, $\mathfrak{G} \in \mathrm{Gs}(G)$. Then*

$$\mathbf{IR}_{\mathfrak{G}} \subseteq \omega_G[R] \text{ if and only if } \mathfrak{G} \subseteq \mathfrak{H}_G^R.$$

Proof. Suppose $\mathbf{IR}_{\mathfrak{G}} \subseteq \omega_G[R]$. Take an $A \in \mathfrak{G}$ arbitrarily. If $\omega_G^-(a) \subseteq A$, $a \notin A$ for some $a \in G - R$ then $\omega_G^-(a) = A \cap \omega_G(a) \in \mathfrak{G}$. Since $\omega_G(a) \in \mathbf{IR}_{\mathfrak{G}}$ by 3.4, we have $\omega_G(a) \in \omega_G[R]$. This result and 2.10 (i) give $a \in R$ which is a contradiction. Hence, $a \in A$, $A \in \mathfrak{H}_G^R$, and $\mathfrak{G} \subseteq \mathfrak{H}_G^R$.

Let us now assume $\mathfrak{G} \subseteq \mathfrak{H}_G^R$. If $A \in \mathbf{IR}_{\mathfrak{G}}$ then $A = \omega_G(a)$ for some $a \in G$ by 2.11. According to 3.4, $\omega_G^-(a) \in \mathfrak{G} \subseteq \mathfrak{H}_G^R$ and $A = \omega_G(a) \in \mathbf{IR}_{\mathfrak{H}_G^R} = \omega_G[R]$ by 3.4, 3.12. Thus, $\mathbf{IR}_{\mathfrak{G}} \subseteq \omega_G[R]$.

3.14. Definition. Let G be a poset and R an irreducible set in G. We put $\varphi_G^R = \xi_G \mathfrak{H}_G^R$.

3.15. Remark. From the definition of $\mathfrak{G}_G^R$ it follows that $\mathfrak{G}_G^G = \Omega_G$ for any poset G. Then, clearly, $\varphi_G^G(A) = A$ for each $A \in \Omega_G$.

The following implication is of a great importance.

3.16. Lemma. *Let G be a poset, $\mathfrak{G} \in \mathrm{Gs}(G)$, R an irreducible set in G with the property $\omega_G[R] = \mathbf{IR}_{\mathfrak{G}}$, $N \subseteq G$ a finite set such that $\omega_G[N] \subseteq \mathbf{S}_{\mathfrak{G}} \cap \mathbf{P}_{\mathfrak{G}} \cap \mathbf{A}_{\mathfrak{G}}$. Then for arbitrary $a \in G$, $A \in \mathfrak{G}$*

$$\omega_G^-(a) \subseteq A, \quad \varphi_G^R(A \cup N) = \bar{\varepsilon}_G(a) \Rightarrow \omega_G(a) \in \mathbf{P}_{\mathfrak{G}}.$$

Proof. Let us put $\varphi = \xi_G \mathfrak{G}$. Then $\varphi \in \mathrm{Op}(G)$, $\mathfrak{C}_G \varphi = \mathfrak{G}$ by 2.8. Suppose $\omega_G^-(a) \subseteq A$, $\varphi_G^R(A \cup N) = \bar{\varepsilon}_G(a)$ for $a \in G$, $A \in \mathfrak{G}$. As $a \notin \varphi_G^R(A \cup N)$, it holds $a \notin A = \varphi(A)$.

Let us denote $\{a_1, a_2, \ldots, a_n\} = N - A$. From $a_i \in \bar{\varepsilon}_G(a)$ it follows that $a \nleqq a_i$; $\omega_G^-(a) \subseteq A$ gives $a_i \notin \omega_G^-(a)$ which is equivalent to $a_i \nless a$. Thus $a \parallel a_i$ and we have $a \notin \omega_G(a_i) \cup \varepsilon_G(a_i)$ for $i = 1, 2, \ldots, n$.

Suppose $a \notin \varphi(A \cup \{a_1, a_2, \ldots, a_j\})$ for some $j \in \{0, 1, \ldots, n-1\}$. Then, according to $a \notin \omega_G(a_{j+1}) \cup \varepsilon_G(a_{j+1})$ and 3.8, we obtain $a \notin \varphi(A \cup \{a_1, a_2, \ldots, a_j\}) \vee \omega_G(a_{j+1})$. By 2.9, 2.4 (ii), it holds $\varphi(A \cup \{a_1, a_2, \ldots, a_j\}) \vee \varphi(\{a_{j+1}\}) = \varphi(\varphi(A \cup \{a_1, a_2, \ldots, a_j\}) \cup \varphi(\{a_{j+1}\})) = \varphi(A \cup \{a_1, a_2, \ldots, a_j\} \cup \varphi(\{a_{j+1}\})) = \varphi(A \cup \{a_1, a_2, \ldots, a_{j+1}\})$. Since $\varphi(\{a_{j+1}\}) = \omega_G(a_{j+1})$, we have $a \notin \varphi(A \cup \{a_1, a_2, \ldots, a_{j+1}\})$.

By induction, we obtain $a \notin \varphi(A \cup N)$ which is equivalent to $\varphi(A \cup N) \subseteq \bar{\varepsilon}_G(a)$. With respect to 3.13, it holds $\mathfrak{G} \subseteq \mathfrak{H}_G^R$ which gives $\varphi_G^R \leqq \varphi$ by 2.8. Now, $\bar{\varepsilon}_G(a) = \varphi_G^R(A \cup N) \subseteq \varphi(A \cup N) \subseteq \bar{\varepsilon}_G(a)$ and, clearly, $\varphi(A \cup N) = \bar{\varepsilon}_G(a)$. Then $\bar{\varepsilon}_G(a) \in \mathfrak{G}$ and $\omega_G(a) \in \mathbf{P}_{\mathfrak{G}}$ by 3.5.

4. SOLUTION OF THE PROBLEM

4.1. Definition. Let G be a poset. We denote by $\mathbf{M}_G$ the set of all minimal elements in G.

4.2. Theorem. *Let G be a poset and $\mathfrak{G} \in \mathrm{Gs}(G)$. Then*

$$\mathbf{A}_{\mathfrak{G}} = \omega_G[\mathbf{M}_G].$$

Proof. For $a \in \mathbf{M}_G$ we have $\omega_G(a) = \{a\}$ and, clearly, $\omega_G(a) \in \mathbf{A}_{\mathfrak{G}}$. If $a \notin \mathbf{M}_G$ then there exists $b < a$ in G. We obtain $\emptyset \subset \omega_G(b) \subset \omega_G(a)$ by 2.10(i) and $\omega_G(a) \notin \notin \mathbf{A}_{\mathfrak{G}}$. The statement now follows by 2.11.

4.3. Definition. Let G be a poset and R an irreducible set in G. We say that N is an *R-nonhomonymous set* (in G) if N is finite, $N \subseteq \mathbf{M}_G$, and if no element from $\mathbf{M}_G$ is the smallest one in $R - N$.

4.4. Definition. Let G be a poset, R an irreducible set in G, N an R-nonhomonymous set in G. Then

(i) We denote by $\mathbf{P}_G(R, N)$ the set of all $a \in R$ such that either $\varphi_G^R(\omega_G^-(a) \cup N) = \bar{\varepsilon}_G(a)$ or there exists $b \in G$ satisfying $\omega_G^-(a) \subset \omega_G(b)$, $\varphi_G^R(\omega_G(b) \cup N) = \bar{\varepsilon}_G(a)$.

(ii) We say that C is an *R, N-primitive set* (in G) if $\mathbf{P}_G(R, N) \cup N \subseteq C \subseteq R$.

4.5. Definition. Let G be a poset. We call an ordered triple (R, N, C) *suitable* (in G) if R is an irreducible, N an R-nonhomonymous, C an R, N-primitive set in G.

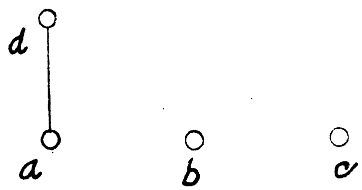

Figure 1

4.6. Example. Let G be the poset from Fig. 1. We construct a suitable triple (R, N, C) in G. As $\mathit{IR}_G = G$, there is only one irreducible set R in G, namely $R = \{a, b, c, d\}$. We can easily check that N is $\{a, b, c, d\}$-nonhomonymous in G iff $N \in \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Let us put $N = \{c\}$. According to 3.15, $\mathbf{P}_G(\{a, b, c, d\}, \{c\}) = \{x;\ x \in \{a, b, c, d\}$ and either $\omega_G^-(x) \cup \{c\} = \bar{\varepsilon}_G(x)$ or there exists $y \in G$ such that $\omega_G^-(x) \subset \omega_G(y)$ and $\omega_G(y) \cup \{c\} = \bar{\varepsilon}_G(x)\}$. We see $\emptyset = \omega_G^-(a) \subset \omega_G(b) = \{b\}$ and $\omega_G(b) \cup \{c\} = \{b, c\} = \bar{\varepsilon}_G(a)$. That is why $a \in \mathbf{P}_G(\{a, b, c, d\}, \{c\})$. Similarly, we verify $b \in \mathbf{P}_G(\{a, b, c, d\}, \{c\})$, $c \notin \mathbf{P}_G(\{a, b, c, d\}, \{c\})$, $d \notin \mathbf{P}_G(\{a, b, c, d\}, \{c\})$, so that $\mathbf{P}_G(\{a, b, c, d\}, \{c\}) = \{a, b\}$. Now, C is $\{a, b, c, d\}$, $\{c\}$-primitive iff $\{a, b, c\} \subseteq \subseteq C \subseteq \{a, b, c, d\}$. We put $C = \{a, b, c\}$; then $(R, N, C) = (\{a, b, c, d\}, \{c\}, \{a, b, c\})$ is a suitable triple in G.

4.7. Theorem. *Let G be a poset satisfying the minimal condition, $\mathfrak{G} \in \mathrm{Gs}(G)$, $\mathbf{N} \in \mathfrak{N}_{\mathfrak{G}}$. If $\omega_G[R] = \mathbf{IR}_{\mathfrak{G}}$, $\omega_G[N] = \mathbf{N}$, $\omega_G[C] = \mathbf{P}_{\mathfrak{G}}$ then (R, N, C) is a suitable triple in G.*

Proof. R is an irreducible set in G: Suppose $a \in \mathit{IR}_G$. If a is not a smallest element in G then $a \in \mathbf{IR}_G$ and a is not the l. u. bound of $\omega_G^-(a)$ in G by 3.3. Thus, there exists an upper bound b of $\omega_G^-(a)$ in G such that $a \nleqq b$. We obtain $\omega_G^-(a) \subseteq \omega_G(a) \cap \omega_G(b)$.

Since $a \notin \omega_G(b)$, it holds also the inverse inclusion and $\omega_G^-(a) = \omega_G(a) \cap \omega_G(b) \in \mathfrak{G}$. If a is the smallest element in G then $\omega_G^-(a) = \emptyset \in \mathfrak{G}$. In both cases we obtain $\omega_G(a) \in \in \mathbf{IR}_{\mathfrak{G}}$ by 3.4. As ω_G is an injection by 2.10(i), we have $a \in R$ and $IR_G \subseteq R$; this gives the statement.

N is an R-nonhomonymous set in G: N is finite by 2.10(i) and $N \subseteq \mathbf{M}_G$ by 4.2. Suppose that there exists $a \in \mathbf{M}_G$ which is the smallest element in $R - N$. Let $A \in \in \mathfrak{G} - \omega_{\mathfrak{G}}(\mathsf{V}_{\mathfrak{G}}\mathsf{N})$ be arbitrary. Obviously, $\omega_G[G]$ satisfies the minimal condition. It is a σ-dense subset in $\mathfrak{G}$ by 2.10(i). It follows by 1.5 that $\mathbf{IR}_{\mathfrak{G}}$ is a σ-dense subset in $\mathfrak{G}$. Thus, there exists $\mathfrak{A} \subseteq \mathbf{IR}_{\mathfrak{G}} = \omega_G[R]$ such that $A = \mathsf{V}_{\mathfrak{G}}\mathfrak{A}$. Since $A \notin \omega_{\mathfrak{G}}(\mathsf{V}_{\mathfrak{G}}\mathsf{N})$, it holds $\mathfrak{A} \nsubseteq \mathsf{N}$. As $\omega_G[R - N] \supseteq \mathfrak{A} - \mathsf{N} \neq \emptyset$, we can find $b \in R - N$ with the property $\omega_G(b) \in \mathfrak{A}$. But then $a \leqq b$ and $\omega_G(a) \subseteq \omega_G(b) \subseteq A$. We have proved that $\omega_G(a)$ is the smallest element in $\mathfrak{G} - \omega_{\mathfrak{G}}(\mathsf{V}_{\mathfrak{G}}\mathsf{N})$. As, at the same time, $\omega_G(a) \in \mathbf{A}_{\mathfrak{G}}$ by 4.2, we have a contradiction with $\mathsf{N} \in \mathfrak{N}_{\mathfrak{G}}$.

C is an R, N-primitive set in G: Suppose $a \in \mathbf{P}_G(R, N)$. Then $a \in R$, $\omega_G(a) \in \mathbf{IR}_{\mathfrak{G}}$ and, by 3.4, $\omega_G^-(a) \in \mathfrak{G}$. Further, $\varphi_G^R(A \cup N) = \varepsilon_G(a)$ for some $A \in \omega_G[G] \cup \{\omega_G^-(a)\}$ such that $\omega_G^-(a) \subseteq A$. As $\omega_G[G] \cup \{\omega_G^-(a)\} \subseteq \mathfrak{G}$, it holds $A \in \mathfrak{G}$. By 3.16, we obtain $\omega_G(a) \in \mathbf{P}_{\mathfrak{G}} = \omega_G[C]$. Then $a \in C$ according to 2.10(i) and we have proved $\mathbf{P}_G(R, N) \subseteq \subseteq C$. The remaining inclusions $N \subseteq C$, $C \subseteq R$ hold trivially.

In the following, we find a $\mathbf{O}$-generating system $\mathfrak{G}$ on G satisfying $\omega_G[R] = \mathbf{IR}_{\mathfrak{G}}$, $\omega_G[N] \in \mathfrak{N}_{\mathfrak{G}}$, $\omega_G[C] = \mathbf{P}_{\mathfrak{G}}$ for a given suitable triple (R, N, C) in a given poset G. According to 3.13, $\mathfrak{G} \subseteq \mathfrak{H}_G^R$. By 3.16, each $A \in \mathfrak{H}_G^R$, such that there exists $a \in R - C$ with the properties $\omega_G^-(a) \subseteq A$, $\varphi_G^R(A \cup N) = \varepsilon_G(a)$, is necessarily in $\mathfrak{H}_G^R - \mathfrak{G}$. This leads to the

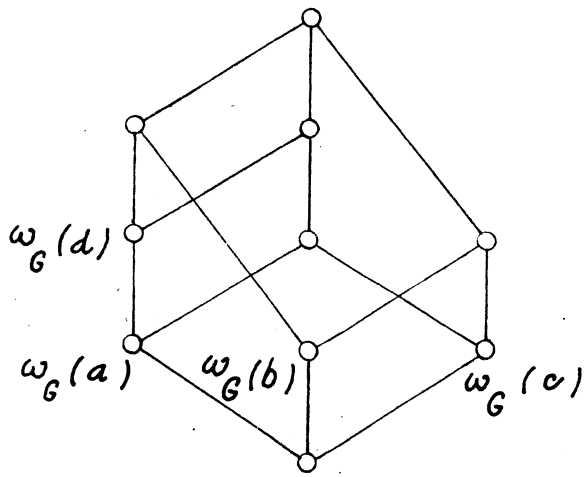

Figure 2

4.8. Definition. Let G be a poset and (R, N, C) a suitable triple in G. We put

$$\mathfrak{D}_G(R, N, C) = \{A; A \in \mathfrak{H}_G^R \text{ and there exists } a_A \in R - C \text{ such that}$$
$$\omega_G^-(a_A) \subseteq A, \varphi_G^R(A \cup N) = \bar{\varepsilon}_G(a_A)\} \text{ and}$$
$$\mathfrak{J}_G(R, N, C) = \mathfrak{H}_G^R - \mathfrak{D}_G(R, N, C).$$

We shall often write $\mathfrak{D}$, $\mathfrak{J}$ instead of $\mathfrak{D}_G(R, N, C)$, $\mathfrak{J}_G(R, N, C)$, respectively.

4.9. Example. Let us take the suitable triple $(R, N, C) = (\{a, b, c, d\}, \{c\}, \{a, b, c\})$ in the poset G from 4.6. Then $\mathfrak{D}_G(\{a, b, c, d\}, \{c\}, \{a, b, c\}) = \{A; A \in \mathfrak{H}_G^G$ and $\omega_G^-(d) \subseteq A$, $\varphi_G^G(A \cup \{c\}) = \bar{\varepsilon}_G(d)\} = \{A; A \in \Omega_G$ and $\{a\} \subseteq A$, $A \cup \{c\} = \{a, b, c\}\}$ according to 3.15. It is clear that $\mathfrak{D}_G(\{a, b, c, d\}, \{c\}, \{a, b, c\}) = \{\{a, b\}, \{a, b, c\}\}$ and $\mathfrak{J} = \mathfrak{J}_G(\{a, b, c, d\}, \{c\}, \{a, b, c\}) = \Omega_G - \mathfrak{D}_G(\{a, b, c, d\}, \{c\}, \{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$. We can easily verify $\mathfrak{J} \in \mathrm{Gs}(G)$. In Fig. 2 we can see that $\omega_G[\{a, b, c, d\}] = \mathbf{IR}_{\mathfrak{J}}$, $\omega_G[\{c\}] \in \mathfrak{N}_{\mathfrak{J}}$, $\omega_G[\{a, b, c\}] = \mathbf{P}_{\mathfrak{J}}$.

We prove that the conclusions of 4.9, namely $\mathfrak{J} = \mathfrak{J}_G(R, N, C) \in \mathrm{Gs}(G)$, $\omega_G[R] = \mathbf{IR}_{\mathfrak{J}}$, $\omega_G[N] \in \mathfrak{N}_{\mathfrak{J}}$, $\omega_G[C] = \mathbf{P}_{\mathfrak{J}}$, are true for any suitable triple (R, N, C) in any poset G.

4.10. Lemma. *Let G be a poset and (R, N, C) a suitable triple in G. Then* $\mathfrak{J}_G(R, N, C) \in \mathrm{Gs}(G)$.

Proof. We verify the validity of 2.3(i), (ii) for $\mathfrak{D}$.

If there exists $A = \omega_G(a)$ in $\mathfrak{D}$ then $\varphi_G^R(\omega_G(a) \cup N) = \bar{\varepsilon}_G(a_A)$ and $\omega_G^-(a_A) \subseteq \omega_G(a)$. Each of the cases $\omega_G^-(a_A) = \omega_G(a)$, $\omega_G^-(a_A) \subset \omega_G(a)$ implies $a_A \in \mathbf{P}_G(R, N) \subseteq C$ which contradicts $a_A \in R - C$. Thus, $\omega_G[G] \cap \mathfrak{D} = \emptyset$. If $\emptyset \in \mathfrak{D}$ then $\omega_G^-(a_\emptyset) = \emptyset$, $\varphi_G^R(N) = \bar{\varepsilon}_G(a_\emptyset)$. It is clear that $a_\emptyset \in \mathbf{M}_G$. For an arbitrary $a \in R - N$, $\omega_G(a) \in \mathbf{P}_{\mathfrak{H}_G^R}$ by 3.12. This, the obvious fact $N \in \Omega_G$, and 3.5 give $a \notin \varphi_G^R(N)$. Since $\varphi_G^R(N) = \bar{\varepsilon}_G(a_\emptyset)$, we have $a \notin \bar{\varepsilon}_G(a_\emptyset)$. Then $a \in \varepsilon_G(a_\emptyset)$ and, consequently, $R - N \subseteq \varepsilon_G(a_\emptyset)$. By this and by $a_\emptyset \in R - N$, it follows that a is the smallest element in $R - N$. But then N is not an R-nonhomonymous set which is a contradiction. We have proved $\emptyset \notin \mathfrak{D}$. Since $G \notin \mathfrak{D}$ in a trivial way, 2.3(i) is satisfied.

Clearly, $a_A \notin A$ for each $A \in \mathfrak{D}$. Let us assume $B \in \mathfrak{H}_G^R$, $A \subseteq B$, $a_A \notin B$. Then $\omega_G^-(a_A) \subseteq B$ and, as $B \subseteq \bar{\varepsilon}_G(a_A)$ by 3.2(iii), it follows $A \cup N \subseteq B \cup N \subseteq \bar{\varepsilon}_G(a_A) = \varphi_G^R(A \cup N)$. This and 2.4(i) imply $\varphi_G^R(B \cup N) = \bar{\varepsilon}_G(a_A)$. Thus, $B \in \mathfrak{D}$ and we have proved 2.3(ii).

The statement follows by 3.11 and 2.3.

The following lemma formulates an interesting property of the operator φ_G^R.

4.11. Lemma. *Let G be a poset, R an irreducible set in G, $A \in \mathfrak{H}_G^R$, $a \in G$. If $\omega_G^-(a) \subseteq A$ and $B \subseteq A \cup \varepsilon_G(a)$ for some $B \in \Omega_G$ then $\varphi_G^R(B) \subseteq A \cup \varepsilon_G(a)$.*

Proof. Let us admit that there exists $b_0 \in \varphi_G^R(B) - [A \cup \varepsilon_G(a)]$. Since $b_0 \in \varphi_G^R(B) - B$, we obtain $\omega_G(b_0) \notin \mathbf{P}_{\mathfrak{H}_G^R}$ by 3.5 and $b_0 \in G - R$ by 3.12.

Let there exist an integer $i \geqq 0$ and elements $b_0, b_1, \ldots, b_i$ such that $b_0 > b_1 > \ldots > b_i$ and $b_j \in \varphi_G^R(B) - [A \cup \varepsilon_G(a)]$, $b_j \in G - R$ for $j = 0, 1, \ldots, i$. Then, clearly, $\omega_G^-(b_i) \cap \varepsilon_G(a) = \emptyset$ and we also have $\omega_G^-(b_i) \nsubseteq A$. Indeed, $\omega_G^-(b_i) \subseteq A$, $b_i \in G - R$, $A \in \mathfrak{H}_G^R$ give $b_i \in A$ which is not true. Thus, there exists $b_{i+1} < b_i$ such that $b_{i+1} \notin A \cup \varepsilon_G(a)$. Since $b_{i+1} \in \varphi_G^R(B) - B$, we obtain $b_{i+1} \in G - R$ by 3.5, 3.12.

By induction, we construct an infinite descending chain $b_0 > b_1 > \ldots$ which is a subset of $\varphi_G^R(B) - [A \cup \varepsilon_G(a)]$. Let us put $C = \varphi_G^R(B) - \bigcup_{i \geqq 0} \varepsilon_G(b_i)$. It is clear that $C \in \Omega_G$. If $\omega_G^-(b) \subseteq C$ for $b \in G - R$ then $\omega_G^-(b) \subseteq \varphi_G^R(B)$ and $b \in \varphi_G^R(B)$. If $b \in \bigcup_{i \geqq 0} \varepsilon_G(b_i)$ then there exists $j \in \{0, 1, \ldots\}$ such that $b \in \varepsilon_G(b_j)$. It follows that $b_{j+1} \in \omega_G^-(b) \subseteq C$ and we have a contradiction. Thus, $b \in C$ and $C \in \mathfrak{H}_G^R$. Since $B \subseteq C$, we have $\varphi_G^R(B) \subseteq \varphi_G^R(C) = C$; this contradicts $C \subseteq \varphi_G^R(B)$.

4.12. Lemma. *Let G be a poset, (R, N, C) a suitable triple in G. If $a \in N$ then $\mathscr{V}(\mathfrak{J}_G(R, N, C), a)$.*

Proof. Let us take an $A \in \mathfrak{J}$ arbitrarily. We prove $A \vee \omega_G(a) \subseteq A \cup \varepsilon_G(a)$ by transfinite induction.

(1) We put $B^0 = \varphi_G^R(A \cup \{a\})$. Since $A \in \mathfrak{H}_G^R$, $\omega_G^-(a) = \emptyset$, it holds $A \cup \{a\} \in \Omega_G$ and $B^0 \subseteq A \cup \varepsilon_G(a)$ by 4.11. In case $B^0 \in \mathfrak{D}$ we have $\varphi_G^R(B^0 \cup N) = \bar{\varepsilon}_G(a_{B^0})$ and $\omega_G^-(a_{B^0}) \subseteq B^0$. If $\omega_G^-(a_{B^0}) \subseteq A$ then $A \in \mathfrak{D}$. Indeed, $\bar{\varepsilon}_G(a_{B^0}) = \varphi_G^R(B^0 \cup N) = \varphi_G^R(\varphi_G^R(A \cup \{a\}) \cup N) = \varphi_G^R(A \cup N)$ by 2.4(ii) because $a \in N$. It is a contradiction. Thus, there exists $b \in \omega_G^-(a_{B^0}) - A$. As $b \in B^0 - A$ and $B^0 \subseteq A \cup \varepsilon_G(a)$, we have $b \in \varepsilon_G(a)$ and $a_{B^0} \in \varepsilon_G(a)$, too.

(2) Let $\lambda \neq 0$ be an ordinal number. Suppose $B^\mu \in \mathfrak{D}$, $B^\mu \subseteq A \cup \varepsilon_G(a)$, $a_{B^\mu} \in \varepsilon_G(a)$ for each $\mu < \lambda$ and $B^\mu \subset B^\nu$ for all $\mu < \nu < \lambda$.

(a) If λ is a successor ordinal then we put $B^\lambda = \varphi_G^R(B^{\lambda-1} \cup \{a_{B^{\lambda-1}}\})$. Since $a_{B^{\lambda-1}} \in B^\lambda - B^{\lambda-1}$, we have $B^{\lambda-1} \subset B^\lambda$ and $B^\mu \subset B^\nu$ for all $\mu < \nu < \lambda + 1$. Clearly, $B^{\lambda-1} \cup \{a_{B^{\lambda-1}}\} \subseteq A \cup \varepsilon_G(a)$ and it holds $B^{\lambda-1} \cup \{a_{B^{\lambda-1}}\} \in \Omega_G$ by 3.2(iv) because $B^{\lambda-1} \in \Omega_G$ and $\omega_G^-(a_{B^{\lambda-1}}) \subseteq B^{\lambda-1}$.

(b) If λ is a limit ordinal then we put $B^\lambda = \varphi_G^R(\bigcup_{\mu<\lambda} B^\mu)$. For each $\mu < \lambda$ there exists ν such that $\mu < \nu < \lambda$ and we have $B^\mu \subset B^\nu \subseteq B^\lambda$. It follows that $B^\mu \subset B^\nu$ for all $\mu < \nu < \lambda + 1$. Simultaneously, $\bigcup_{\mu<\lambda} B^\mu \in \Omega_G$ and $\bigcup_{\mu<\lambda} B^\mu \subseteq A \cup \varepsilon_G(a)$.

(c) Both in (a) and in (b) we obtain $B^\lambda \subseteq A \cup \varepsilon_G(a)$ by 4.11. If $B^\lambda \in \mathfrak{D}$ then $B^0 \subset B^\lambda$ gives $\bar{\varepsilon}_G(a_{B^0}) = \varphi_G^R(B^0 \cup N) \subseteq \varphi_G^R(B^\lambda \cup N) = \bar{\varepsilon}_G(a_{B^\lambda})$. Thus, $a_{B^0} \leqq a_{B^\lambda}$ according to 3.2(i); by this and by $a_{B^0} \in \varepsilon_G(a)$, it follows that $a_{B^\lambda} \in \varepsilon_G(a)$.

(3) If B^λ is defined then $B^0 \subset B^1 \subset \ldots \subset B^\lambda$, $B^\mu \subseteq G$ for each $\mu \leqq \lambda$, and $B^\mu \in \mathfrak{D}$ for each $\mu < \lambda$. This and the connections between cardinals and ordinals (see [2]) give the existence of an ordinal μ such that B^μ is not defined. Then, necessarily, there

exists an ordinal $\nu < \mu$ satisfying $B^{\nu} \in \mathfrak{J}$. As $A \subseteq B^0$, $\omega_G(a) \subseteq B^0$, we have $A \subseteq B^{\nu}$, $\omega_G(a) \subseteq B^{\nu}$ and $A \vee \omega_G(a) \subseteq B^{\nu}$. On the other hand, by (2)(c), $B^{\nu} \subseteq A \cup \varepsilon_G(a)$ and we have $A \vee \omega_G(a) \subseteq A \cup \varepsilon_G(a)$.

4.13. Theorem. *Let G be a poset and (R, N, C) a suitable triple in G. Then* $\omega_G[R] = \mathbf{IR}_{\mathfrak{J}_G(R,N,C)}$, $\omega_G[N] \in \mathfrak{N}_{\mathfrak{J}_G(R,N,C)}$, $\omega_G[C] = \mathbf{P}_{\mathfrak{J}_G(R,N,C)}$.

Proof. (1) $\omega_G[R] = \mathbf{IR}_{\mathfrak{J}}$: As $\mathfrak{J} \subseteq \mathfrak{H}_G^R$, we obtain $\mathbf{IR}_{\mathfrak{J}} \subseteq \omega_G[R]$ by 3.13. By 4.10, $\emptyset \in \mathfrak{J} = \mathfrak{H}_G^R - \mathfrak{D}$. By this and by $N \subseteq \mathbf{M}_G$, it follows that $\omega_G^-(a) = \emptyset \notin \mathfrak{D}$ for each $a \in N$. Let $a \in R - N$ be arbitrary. According to 3.12, $\omega_G(a) \in \mathbf{P}_{\mathfrak{H}_G^R}$. This, $a \notin \omega_G^-(a) \cup N$, 3.5, give $a \notin \varphi_G^R(\omega_G^-(a) \cup N)$. If $\omega_G^-(a) \in \mathfrak{D}$ then $\varphi_G^R(\omega_G^-(a) \cup N) = \varepsilon_G(b)$ for $b = a_{\omega_G^-(a)}$. That means $\omega_G^-(a) \subseteq \varepsilon_G(b)$, $a \in \varepsilon_G(b)$; by 3.2(ii) we obtain $b = a$. But then $b \in \mathbf{P}_G(R, N) \subseteq C$ and we have a contradiction with $b \in R - C$. We have proved $\omega_G^-(a) \notin \mathfrak{D}$ for each $a \in R$. Since $\mathfrak{J} = \mathfrak{H}_G^R - \mathfrak{D}$ and $\omega_G^-(a) \in \mathfrak{H}_G^R$ by 3.12, 3.4, we obtain $\omega_G^-(a) \in \mathfrak{J}$ for each $a \in R$. Then $\omega_G[R] \subseteq \mathbf{IR}_{\mathfrak{J}}$ by 3.4.

(2) $\omega_G[C] = \mathbf{P}_{\mathfrak{J}}$: It follows from (1) that $\mathbf{P}_{\mathfrak{J}} \subseteq \omega_G[R]$. Let us take $a \in R - C$. It holds $\varepsilon_G(a) \in \mathfrak{H}_G^R$ by 3.12, 3.5. As $a \notin C$, $N \subseteq C$, we have $a \in N$. If $b \notin \varepsilon_G(a)$ for some $b \in N$ then $a < b$ and $b \notin \mathbf{M}_G$ which is not true. For this reason $\varepsilon_G(a) \cup N = \varepsilon_G(a)$ and $\varphi_G^R(\varepsilon_G(a) \cup N) = \varepsilon_G(a)$; this and $\omega_G^-(a) \subseteq \varepsilon_G(a)$ give $\varepsilon_G(a) \in \mathfrak{D}$. Then, clearly, $\varepsilon_G(a) \notin \mathfrak{J}$ and $\omega_G(a) \notin \mathbf{P}_{\mathfrak{J}}$ according to 3.5. We conclude $\mathbf{P}_{\mathfrak{J}} \subseteq \omega_G[C]$.

Let us take an $a \in C$ arbitrarily. Then $\varepsilon_G(a) \in \mathfrak{H}_G^R$ by $C \subseteq R$, 3.12, 3.5. If $\omega_G^-(b) \subseteq \varepsilon_G(a)$ for $b \in R - C$ then $b \in \varepsilon_G(a)$. Indeed, by $b \in \varepsilon_G(a)$ and 3.2(ii), it follows that $a = b$ which is a contradiction. Now, $\varepsilon_G(a) \notin \mathfrak{D}$ in an obvious way and $\varepsilon_G(a) \in \mathfrak{J} = \mathfrak{H}_G^R - \mathfrak{D}$. Then $\omega_G(a) \in \mathbf{P}_{\mathfrak{J}}$ according to 3.5. We have proved $\omega_G[C] \subseteq \mathbf{P}_{\mathfrak{J}}$.

(3) $\omega_G[N] \in \mathfrak{N}_{\mathfrak{J}}$: By (2) and by 4.2 we obtain $\omega_G[N] \subseteq \mathbf{P}_{\mathfrak{J}} \cap \mathbf{A}_{\mathfrak{J}}$. This inclusion, 4.12, and 3.8 give $\omega_G[N] \subseteq \mathbf{S}_{\mathfrak{J}}$. Thus $\omega_G[N] \subseteq \mathbf{S}_{\mathfrak{J}} \cap \mathbf{P}_{\mathfrak{J}} \cap \mathbf{A}_{\mathfrak{J}}$ and, clearly, $\omega_G[N]$ is a finite set. Let us assume that there exists $A \in \mathbf{A}_{\mathfrak{J}}$ which is the smallest element in $\mathfrak{J} - \omega_{\mathfrak{J}}(\bigvee_{\mathfrak{J}}\omega_G[N])$. Regarding 1.10 we have $\mathbf{IR}_{\mathfrak{J}} \cap \omega_{\mathfrak{J}}(\bigvee_{\mathfrak{J}}\omega_G[N]) = \omega_G[N]$. Then $(\mathbf{IR}_{\mathfrak{J}} - \omega_G[N]) \cap \omega_{\mathfrak{J}}(\bigvee_{\mathfrak{J}}\omega_G[N]) = \emptyset$ and, consequently, $\mathbf{IR}_{\mathfrak{J}} - \omega_G[N] \subseteq \mathfrak{J} - \omega_{\mathfrak{J}}(\bigvee_{\mathfrak{J}}\omega_G[N])$. By this and by the properties of A we obtain that A is the smallest element in $\mathbf{IR}_{\mathfrak{J}} - \omega_G[N]$. As $A \in \mathbf{A}_{\mathfrak{J}}$, there exists $a \in \mathbf{M}_G$ such that $A = \omega_G(a)$ according to 4.2. By these results and by (1) it follows that a is the smallest element in $R - N$. We have a contradiction with the fact that N is an R-nonhomonymous set in G.

4.14. Corollary. *Let G be a poset satisfying the minimal condition and (I, R, N, C) an ordered fourtuple of subsets of G. Then there exists a σ_0-dense embedding e of G into a complete lattice S such that $e[I] = \mathbf{A}_S$, $e[R] = \mathbf{IR}_S$, $e[N] \in \mathfrak{N}_S$, $e[C] = \mathbf{P}_S$ if and only if $I = \mathbf{M}_G$ and (R, N, C) is a suitable triple in G.*

Proof. Let there exist a σ_0-dense embedding e of G into a complete lattice S such that $e[I] = \mathbf{A}_S$, $e[R] = \mathbf{IR}_S$, $e[N] \in \mathfrak{N}_S$, $e[C] = \mathbf{P}_S$. By 2.10(ii), there exist $\mathfrak{G} \in \mathrm{Gs}(G)$ and an isomorphism $\iota: S \to \mathfrak{G}$ such that $\iota e = \omega_G$. Then $\omega_G[I] = \iota e[I] =$

$= \iota[\mathbf{A}_S] = \mathbf{A}_{\mathfrak{G}}$ and, similarly, $\omega_G[R] = \mathbf{IR}_{\mathfrak{G}}$, $\omega_G[N] \in \mathfrak{N}_{\mathfrak{G}}$, $\omega_G[C] = \mathbf{P}_{\mathfrak{G}}$. By this and by 4.2, 4.7, $I = \mathbf{M}_G$ and (R, N, C) is a suitable triple in G.

Suppose that $I = \mathbf{M}_G$ and (R, N, C) is a suitable triple in G. If we put $S = \mathfrak{J}_G(R, N, C)$ and $e = \omega_G \colon G \to S$ then $e[I] = \mathbf{A}_S$ by 4.2 and $e[R] = \mathbf{IR}_S$, $e[N] \in \mathfrak{N}_S$, $e[C] = \mathbf{P}_S$ by 4.13.

5. MAIN THEOREM

Let (V, L) be a language. We put

$$\Sigma\mathfrak{A}(V, L) = \{\bigcup \sigma_L[A]; A \subseteq V\}.$$

For each language (V, L), $\Sigma\mathfrak{A}(V, L)$ is a finite lattice with $\emptyset$ as the smallest element and with union as the operation of join. If a language (V, L) contains no parasitary elements then the identical map from $\mathfrak{A}(V, L)$ into $\Sigma\mathfrak{A}(V, L)$ is a σ_0-dense embedding.

Let S be a lattice. We call an ordered pair $(r, (V, L))$ an *1-representation* of S if (V, L) is a language and $r \colon S \to \Sigma\mathfrak{A}(V, L)$ an isomorphism.

Using the statements [4] II, 3.1 and [4] II, 3.3, we can easily prove

5.1. Theorem. *Let S be a nonempty finite lattice and (H, R, P, I, N, F, C) an ordered seventuple of subsets of S. Then there exists an 1-representation $(r, (V, L))$ of S such that (V, L) contains no parasitary elements and $r[M] = \mathbf{M}(V, L)$ for $M = H, R, P, I, N, F, C$ if and only if $H \subseteq S - IR_S$, $R = \mathbf{IR}_S$, $P = \mathbf{IR}_S - \mathbf{A}_S$, $I = \mathbf{A}_S$, $N \in \mathfrak{N}_S$, $F = \mathbf{A}_S - N$, $C = \mathbf{P}_S$.*

5.2. Main theorem. *Let G be a finite poset and (H, R, P, I, N, F, C) an ordered seventuple of subsets of G. Then there exists a p-representation $(r, (V, L))$ of G such that (V, L) contains no parasitary elements and $r[M] = \mathbf{M}(V, L)$ for $M = H, R, P, I, N, F, C$ if and only if $I = \mathbf{M}_G$, (R, N, C) is a suitable triple in G, $H = G - R$, $P = R - \mathbf{M}_G$, $F = \mathbf{M}_G - N$.*

Proof. Let there exist a p-representation $(r, (V, L))$ of G such that (V, L) contains no parasitary elements and $r[M] = \mathbf{M}(V, L)$ for $M = H, R, P, I, N, F, C$. The ordered pair $(1_{\Sigma\mathfrak{A}(V,L)}, (V, L))$ is an 1-representation of $\Sigma\mathfrak{A}(V, L)$ and $\mathbf{H}(V, L) \subseteq \Sigma\mathfrak{A}(V, L) - IR_{\Sigma\mathfrak{A}(V,L)}$, $\mathbf{R}(V, L) = \mathbf{IR}_{\Sigma\mathfrak{A}(V,L)}$, $\mathbf{P}(V, L) = \mathbf{IR}_{\Sigma\mathfrak{A}(V,L)} - \mathbf{A}_{\Sigma\mathfrak{A}(V,L)}$, $\mathbf{I}(V, L) = \mathbf{A}_{\Sigma\mathfrak{A}(V,L)}$, $\mathbf{N}(V, L) \in \mathfrak{N}_{\Sigma\mathfrak{A}(V,L)}$, $\mathbf{F}(V, L) = \mathbf{A}_{\Sigma\mathfrak{A}(V,L)} - \mathbf{N}(V, L)$, $\mathbf{C}(V, L) = \mathbf{P}_{\Sigma\mathfrak{A}(V,L)}$ according to 5.1. By these results, by the fact that $r \colon G \to \Sigma\mathfrak{A}(V, L)$ is a σ_0-dense embedding, by 4.14, it follows that $I = \mathbf{M}_G$ and (R, N, C) is a suitable triple in G. By the definition of a pure homonym we have $H = G - R$. The assertions $P = R - \mathbf{M}_G$, $F = \mathbf{M}_G - N$ hold trivially.

Let now $I = \mathbf{M}_G$, (R, N, C) be a suitable triple in G, $H = G - R$, $P = R - \mathbf{M}_G$, $F = \mathbf{M}_G - N$. By 4.14, there exists a σ_0-dense embedding e of G into a complete lattice S such that $e[I] = \mathbf{A}_S$, $e[R] = \mathbf{IR}_S$, $e[N] \in \mathfrak{N}_S$, $e[C] = \mathbf{P}_S$. Then, clearly,

$e[H] \subseteq S - \boldsymbol{IR}_S$, $e[P] = \mathbf{IR}_S - \mathbf{A}_S$, $e[F] = \mathbf{A}_S - e[N]$, and S is a nonempty finite lattice by 1.7(ii). According to 5.1, there exists an 1-representation $(r', (V, L))$ of S such that (V, L) contains no parasitary elements and $r'[e[M]] = \mathbf{M}(V, L)$ for $M = H, R, P, I, N, F, C$. If we put $r = r'e$ then the ordered pair $(r, (V, L))$ is a p-representation of G and $r[M] = \mathbf{M}(V, L)$ for $M = H, R, P, I, N, F, C$.

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