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Archivum Mathematicum, Vol. 16 (1980), No. 1, 15--37

Persistent URL: `http://dml.cz/dmlcz/107052`

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVI: 15—38, 1980

ON A CERTAIN NONLINEAR PROBLEM FOR TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS

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(Received March 13, 1979)

The paper deals with the question on existence and uniqueness of a solution of the differential system

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2) \qquad (i = 1, 2) \tag{0.1}$$

which is defined in $[0, +\infty[$ and satisfies the conditions

$$\varphi(x_1(0), x_2(0)) = 0, x_1(t) \geqq 0, x_2(t) \geqq 0 \qquad \text{for } t \geqq 0. \tag{0.2}$$

The important special case of this problem

$$u'' = f(t, u, u'), \tag{0.3}$$

$$\varphi(u(0), | u'(0) |) = 0, u(t) \geqq 0, u'(t) \leqq 0 \qquad \text{for } t \geqq 0 \tag{0.4}$$

is studied separately.

Concerning the history of the question it is necessary to refer to the classical paper by A. Kneser [1] who was the first to establish the existence and the uniqueness of the solution of the equation

$$u'' = f(t, u)$$

under the conditions

$$u(0) = c_0, u(t) \geqq 0, u'(t) \leqq 0 \qquad \text{for } t \geqq 0. \tag{0.5}$$

Later on it has been found that this problem has applications in the study of the distribution of electrons in the heavy atom [2, 3]. Sufficiently general conditions of the solvability and unique solvability of the problem (0.3), (0.5) are given in [4, 5, 6, 10]. From papers, devoted to the analogous problems for differential systems we refer to [7, 8, 11].[1])

[1]) See also [9], pp. 591—596.

In this paper the new sufficient conditions of existence and uniqueness of the solution of the problems (0.1), (0.2) and (0.3), (0.4) are established and the behaviour of the solution is studied when $t \to +\infty$.[2])

1. STATEMENTS OF EXISTENCE THEOREMS

We shall use the following notations:

$$R =]-\infty, +\infty[; \; R_+ = [0, +\infty[; \; R^2 = R \times R; \; R_+^2 = R_+ \times R_+;$$

$L(I)$ is the set of real functions which are summable according to Lebesgue on I.

$L_{loc}(I)$ is the set of real functions which are summable according to Lebesgue on each compact interval contained in I.

In what follows it is assumed that $\varphi : R_+^2 \to R$ is a continuous function and $f_i : R_+ \times R_+^2 \to R \; (i = 1, 2)$ satisfy the local Carathéodory conditions, i.e. $f_i(., x_1, x_2) : R_+ \to R \; (i = 1, 2)$ are measurable for every $(x_1, x_2) \in R_+^2$; $f_i(t, ., .) : R_+^2 \to R$ $(i = 1, 2)$ are continuous for almost every $t \in R_+$ and

$$\sup \{ |f_i(., x_1, x_2)| : 0 \leqq x_1 \leqq \varrho; 0 \leqq x_2 \leqq \varrho \} \in L_{loc}(R_+) \qquad (i = 1, 2)$$

for any $\varrho \in R_+$.

Solutions of the problem (0.1), (0.2) (of the problem (0.3), (0.4)) are sought in the class of vector functions $(x_1, x_2) : R_+ \to R^2$ (in the class of functions $u : R_+ \to R$) which are absolutely continuous (absolutely continuous with their first derivatives) on each compact interval contained in R_+.

The existence theorems proved below concern the cases when f_1 and f_2 satisfy one of the following two conditions

$$f_i(t, 0, 0) = 0 \quad (i = 1, 2), \quad f_1(t, x_1, x_2) \leqq 0, \quad f_2(t, x_1, 0) \leqq 0$$
$$\text{for } t \geqq 0, x_1 \geqq 0, x_2 \geqq 0 \tag{1.1}$$

or

$$f_i(t, 0, 0) = 0, \quad f_i(t, x_1, x_2) \leqq 0 \quad (i = 1, 2)$$
$$\text{for } t \geqq 0, x_1 \geqq 0, x_2 \geqq 0, \tag{1.2}$$

and φ satisfies one of the following three conditions

$$\varphi(0, 0) < 0, \quad \varphi(x_1, x_2) > 0 \quad \text{for } x_1 > r, x_2 \geqq 0, \tag{1.3}$$

$$\varphi(0, 0) < 0, \quad \varphi(x_1, x_2) > 0 \quad \text{for } x_1 \geqq 0, x_2 > r \tag{1.4}$$

2) In contrast to [7, 8, 11] the existence theorems for the problem (0.1), (0.2) which are proved in this paper include the case, when one of the functions f_1 or f_2 changes the sign.

or

$$(1.5) \qquad \varphi(0,0) < 0, \qquad \varphi(x_1, x_2) > 0 \qquad \text{for } x_1 + x_2 > r,\ x_1 \geqq 0, x_2 \geqq 0,$$

where $r \in R_+$.

Theorem 1.1. *Let the conditions (1.1) and (1.3) be fulfilled and let there exist reals* $a_0 \in R_+$ *and* $a > a_0$ *such that*

$$(1.6) \qquad f_1(t, x_1, x_2) \leqq -\delta(x_2) \qquad \text{for } a_0 \leqq t \leqq a, 0 \leqq x_1 \leqq r, x_2 \geqq 0,$$

$$(1.7) \qquad f_2(t, x_1, x_2) \geqq -[h(t) + |f_1(t, x_1, x_2)|]\, \omega(x_2) \quad \text{for } 0 \leqq t \leqq a, 0 \leqq x_1 \leqq r, x_2 \geqq 0$$

and

$$(1.8) \qquad f_2(t, x_1, x_2) \leqq [h(t) + |f_1(t, x_1, x_2)|]\, \omega(x_2) \quad \text{for } t \geqq a_0, 0 \leqq x_1 \leqq r, x_2 \geqq 0$$

where $h \in L_{loc}(R_+)$, $\delta : R_+ \to R_+$ *and* $\omega : R_+ \to]0, +\infty[$ *are continuous and satisfy the conditions*

$$(1.9) \qquad \lim_{x \to +\infty} \delta(x) = +\infty,$$

$$(1.10) \qquad \int_0^{+\infty} \frac{dx}{\omega(x)} = +\infty.$$

Then the problem (0.1), (0.2) *is solvable.*

Corollary 1. Let the conditions (1.2) and (1.3) be fulfilled and let there exist $a > 0$ such that

$$f_1(t, x_1, x_2) \leqq -\delta(x_2), \quad f_2(t, x_1, x_2) \geqq -l[h(t) + |f_1(t, x_1, x_2)|](1 + x_2)$$
$$\text{for } 0 \leqq t \leqq a, 0 \leqq x_1 \leqq r, x_2 \geqq 0,$$

where $l \in R_+$, $h \in L([0, a])$ and $\delta : R_+ \to R_+$ is a function satisfying the condition (1.9). Then the problem (0.1), (0.2) is solvable.

Remark 1. The conditions (1.6) and (1.9) may be somewhat relaxed when replaced by the condition

$$f_1(t, x_1, x_2) \leqq -\delta(t, x_2) \qquad \text{for } a_0 \leqq t \leqq a, 0 \leqq x_1 \leqq r, x_2 \geqq 0,$$

where $\delta : [a_0, a] \times R_+ \to R_+$ is nondecreasing with respect to the second argument, $\delta(., x) \in L([a_0, a])$ for any $x \in R_+$ and

$$\lim_{x \to +\infty} \int_{a_0}^{a} \delta(t, x)\, dt > r.$$

But the above-mentioned conditions cannot be completely omitted. Indeed, the problem

$$\frac{dx_1}{dt} = 0, \qquad \frac{dx_2}{dt} = -x_1 - x_2,$$

$$x_1(0) = 1, \qquad x_1(t) \geqq 0, \qquad x_2(t) \geqq 0 \qquad \text{for } t \geqq 0$$

is not solvable, although it fulfils all the conditions of Theorem 1.1 except (1.6) and (1.9).

Remark 2. The restriction (1.10) is essential and cannot be relaxed. As an example consider the problem

$$\text{(1.11)} \qquad \frac{dx_1}{dt} = -(x_1 + x_2), \qquad \frac{dx_2}{dt} = -(x_1 + x_2)(1 + x_2)^{1+\varepsilon},$$

$$\text{(1.12)} \qquad x_1(0) = \frac{1}{\varepsilon}, \qquad x_1(t) \geqq 0, \qquad x_2(t) \geqq 0 \qquad \text{for } t \geqq 0,$$

where $\varepsilon > 0$ and assume that it has a solution (x_1, x_2). Then

$$x_i(t) \leqq x_i(0) \exp(-t) \to 0 \qquad \text{when } t \to +\infty \ (i = 1, 2),$$

and

$$\frac{d}{dt}[1 + x_2(t)]^{-\varepsilon} = -\varepsilon \frac{dx_1(t)}{dt} \qquad \text{for } t \geqq 0.$$

The integration of the last identity from 0 to $+\infty$ leads to the contradiction

$$1 - [1 + x_2(0)]^{-\varepsilon} = 1.$$

So for any $\varepsilon > 0$ the problem (1.11), (1.12) is unsolvable in spite of the fact that it fulfils all the conditions of Theorem 1.1 except (1.10) instead of which we have

$$\int_0^{+\infty} \frac{x^\varepsilon}{\omega(x)} dx = \int_0^{+\infty} \frac{x^\varepsilon}{(1 + x)^{1+\varepsilon}} dx = +\infty.$$

If we put

$$x_1(t) = u(t), \qquad x_2(t) = -u'(t),$$

then the problem (0.3), (0.4) turns into (0.1), (0.2), where

$$f_1(t, x_1, x_2) = -x_2, \qquad f_2(t, x_1, x_2) = -f(t, x_1, -x_2).$$

Therefore Theorem 1.1 implies the following

Corollary 2. Let the conditions (1.3) and

$$\text{(1.13)} \qquad f(t, x_1, 0) \geqq 0 \qquad \text{for } t \geqq 0, x_1 \geqq 0$$

be fulfilled and let there exist the reals $a_0 \in R_+$ and $a > a_0$ such that

$$f(t, x_1, x_2) \leqq [h(t) + |x_2|]\, \omega(|x_2|) \qquad \text{for } 0 \leqq t \leqq a, 0 \leqq x_1 \leqq r, x_2 \leqq 0$$

and

$$f(t, x_1, x_2) \geqq -[h(t) + |x_2|]\, \omega(|x_2|) \qquad \text{for } t \geqq a_0, 0 \leqq x_1 \leqq r, x_2 \leqq 0,$$

where $h \in L_{\text{loc}}(R_+)$; $\omega : R_+ \to]0, +\infty[$ is a continuous function satisfying the condition (1.10). Then the problem (0.3), (0.4) is solvable.

This statement generalizes the P. Hartman and A. Wintner theorem [6] on the solvability of the problem (0.3), (0.5).

Theorem 1.2. *Let the conditions* (1.1) *and* (1.3) *be fulfilled and let there exist* $a > 0$ *such that*

$$f_1(t, x_1, x_2) \leqq -\delta x_2^{\mu}, \qquad f_2(t, x_1, x_2) \geqq -h_0(t)(1 + x_2)^{\lambda} \qquad \text{for } 0 \leqq t \leqq a, 0 \leqq x_1 \leqq r, x_2 \geqq 0 \tag{1.14}$$

and

$$f_2(t, x_1, x_2) \leqq [h(t) + |f_1(t, x_1, x_2)|]\, \omega(x_2) \qquad \text{for } t > 0, 0 \leqq x_1 \leqq r, x_2 \geqq 0 \tag{1.15}$$

where $\delta > 0$, $\mu > 0$, $\lambda > 1$, $h_0 \in L([0, a])$, $h_0(t) > 0$ *for* $0 < t < a$,

$$\int_0^a \Big[\int_0^t h_0(\tau)\, d\tau\Big]^{\frac{\mu}{1-\lambda}} dt = +\infty, \tag{1.16}$$

$h \in L_{\text{loc}}(]0, +\infty[)$ *and* $\omega : R_+ \to]0, +\infty[$ *is a continuous function satisfying* (1.10). *Then the problem* (0.1), (0.2) *is solvable.*

Corollary 1. Let the conditions (1.2) and (1.3) be fulfilled and let there exist reals $a \in]0, 1[$, $\delta > 0$, $\mu > 0$, $l > 0$ and $\lambda > 1$ such that

$$f_1(t, x_1, x_2) \leqq -\delta x_2^{\mu}, \qquad f_2(t, x_1, x_2) \geqq -l t^{\frac{\lambda-1}{\mu}-1} |\ln t|^{\frac{\lambda-1}{\mu}} (1 + x_2)^{\lambda} \qquad \text{for } 0 \leqq t \leqq a, 0 \leqq x_1 \leqq r, x_2 \geqq 0.$$

Then the problem (0.1), (0.2) is solvable.

Corollary 2. Let the conditions (1.3) and (1.13) be fulfilled and let there exist $a > 0$ such that

$$f(t, x_1, x_2) \leqq h_0(t)(1 + |x_2|)^{\lambda} \qquad \text{for } 0 \leqq t \leqq a, 0 \leqq x_1 \leqq r, x_2 \leqq 0$$

and

$$f(t, x_1, x_2) \geqq -[h(t) + |x_2|]\, \omega(|x_2|) \qquad \text{for } t > 0, 0 \leqq x_1 \leqq r, x_2 \leqq 0,$$

where $\lambda > 1$, $h_0 \in L([0, a])$, $h_0(t) > 0$ for $0 < t < a$,

$$\int_0^a \Big[\int_0^t h_0(\tau)\,d\tau\Big]^{\frac{1}{1-\lambda}} dt = +\infty,$$

$h \in L_{loc}(]0, +\infty[)$ and $\omega : R_+ \to]0, +\infty[$ is a continuous function satisfying (1.10). Then the problem (0.3), (0.4) is solvable.

Theorem 1.3. *Let the conditions* (1.1) *and* (1.4) *be fulfilled and for a certain* $a > 0$

$$f_1(t, x_1, x_2) \geqq -[h(t) + |f_2(t, x_1, x_2)|]\,\omega(x_1), \qquad f_2(t, x_1, x_2) \leqq -\delta(x_1)$$

(1.17) $$\text{for } 0 \leqq t \leqq a,\ x_1 \geqq 0,\ 0 \leqq x_2 \leqq r,$$

where $h \in L([0, a])$, $\delta : R_+ \to R_+$ *and* $\omega : R_+ \to]0, +\infty[$ *are continuous functions satisfying* (1.9) *and* (1.10). *Suppose that for each* $\varrho > 0$ *there exist* $h_\varrho \in L_{loc}([a, +\infty[)$ *and a continuous function* $\omega_\varrho : R_+ \to]0, +\infty[$ *such that*

(1.18) $$\int_0^{+\infty} \frac{dx}{\omega_\varrho(x)} = +\infty$$

and

(1.19) $$f_2(t, x_1, x_2) \leqq [h_\varrho(t) + |f_1(t, x_1, x_2)|]\,\omega_\varrho(x_2)$$
$$\text{for } t \geqq a,\ 0 \leqq x_1 \leqq \varrho,\ x_2 \geqq 0.$$

Then the problem (*0.1*), (*0.2*) *is solvable.*

Corollary. Let the conditions (1.4) and (1.13) be fulfilled and for a certain $a > 0$

$$f(t, x_1, x_2) \geqq \delta(x_1) \qquad \text{for } 0 \leqq t \leqq a,\ x_1 \geqq 0,\ -r \leqq x_2 \leqq 0,$$

where $\delta : R_+ \to R_+$ is a function satisfying (1.9). Suppose that for each $\varrho > 0$ there exist $h_\varrho \in L_{loc}([a, +\infty[)$ and a continuous function $\omega_\varrho : R_+ \to]0, +\infty[$ satisfying (1.18) such that

$$f(t, x_1, x_2) \geqq -[h_\varrho(t) + |x_2|]\,\omega_\varrho(|x_2|) \qquad \text{for } t \geqq a,\ 0 \leqq x_1 \leqq \varrho,\ x_2 \leqq 0.$$

Then the problem (0.3), (0.4) is solvable.

Theorem 1.4. *Let the conditions* (1.1), (1.5) *and*

$$f_2(t, x_1, x_2) \leqq [h(t) + |f_1(t, x_1, x_2)|]\,\omega(x_2)$$
$$\text{for } t \geqq 0,\ 0 \leqq x_1 \leqq r,\ x_2 \geqq 0$$

be fulfilled, where $h \in L_{loc}(]0, +\infty[)$ *and* $\omega : R_+ \to]0, +\infty[$ *is a continuous function satisfying* (1.10). *Then the problem* (0.1), (0.2) *is solvable.*

Corollary 1. If the conditions (1.2) and (1.5) are valid, then the problem (0.1), (0.2) is solvable.

Corollary 2. Let the conditions (1.5), (1.13) and

$$f(t, x_1, x_2) \geqq -[h(t) + |x_2|]\,\omega(|x_2|) \qquad \text{for } t \geqq 0, 0 \leqq x_1 \leqq r, x_2 \leqq 0$$

be fulfilled, where $h \in L_{\text{loc}}(]0, +\infty[)$ and $\omega : R_+ \to]0, +\infty[$ is a continuous function satisfying (1.10). Then the problem (0.3), (0.4) is solvable.

2. THE LEMMA ON A PRIORI ESTIMATES

Lemma 2.1. *Let* $0 \leqq a_0 < a < +\infty$, $r > 0$, $h \in L_{\text{loc}}(R_+)$, $h(t) \geqq 0$ *be valid for* $t \geqq 0$. *Suppose that* $\omega : R_+ \to]0, +\infty[$ *and* $\delta_0 : R_+ \to R_+$ *are continuous functions satisfying the conditions* (1.10) *and*

$$\lim_{x \to +\infty} \delta_0(x) > \frac{r}{a - a_0},$$

and $\Omega : R_+ \to]0, +\infty[$ is a function defined by means of the equality

$$\Omega(x) = \int_0^x \frac{ds}{\omega(s)}. \tag{2.1}$$

Then there exists $r^* > r$ such that for any $b > a$ and for any absolutely continuous functions $x_i : [0, b] \to R$ $(i = 1, 2)$ the inequalities

$$x_1(0) \leqq r, x_1(t) \geqq 0, x_1'(t) \leqq 0, x_2(t) \geqq 0 \qquad \text{for } 0 \leqq t \leqq b, \tag{2.2}$$

$$x_1'(t) \leqq -\delta_0(x_2(t)) \qquad \text{for } a_0 \leqq t \leqq a, \tag{2.3}$$

$$x_2'(t) \geqq -[h(t) - x_1'(t)]\,\omega(x_2(t)) \qquad \text{for } 0 \leqq t \leqq a \tag{2.4}$$

and

$$x_2'(t) \leqq [h(t) - x_1'(t)]\,\omega(x_2(t)) \qquad \text{for } a_0 \leqq t \leqq b \tag{2.5}$$

imply the following estimate

$$x_2(t) \leqq \Omega^{-1}\Big(r^* + \int_0^t h(\tau)\,d\tau\Big) \qquad \text{for } 0 \leqq t \leqq b. \tag{2.6}$$

Proof. Choose $r_0 > 0$ such that

$$\delta_0(x) > \frac{r}{a - a_0} \qquad \text{for } x > r_0. \tag{2.7}$$

According to (2.2) and (2.3) we get

$$r \geqq x_1(a_0) - x_1(a) \geqq \int_{a_0}^{a} \delta_0(x_2(t))\,dt.$$

Hence (2.7) implies the existence of $t_0 \in [a_0, a]$ such that

$$x_2(t_0) \leqq r_0. \tag{2.8}$$

Using (2.1), (2.4) and (2.5) we have

$$\frac{d}{dt}\Omega(x_2(t)) \geqq -h(t) + x_1'(t) \qquad \text{for } 0 \leqq t \leqq t_0$$

and

$$\frac{d}{dt}\Omega(x_2(t)) \leqq \quad h(t) - x_1'(t) \qquad \text{for } t_0 \leqq t \leqq b.$$

From this inequalities by (2.2) and (2.8) it follows that

$$\Omega(x_2(t)) \leqq \Omega(x_2(t_0)) + \int_t^{t_0} h(\tau)\,d\tau + x_1(t) \leqq$$

$$\leqq \Omega(r_0) + r + \int_0^a h(\tau)\,d\tau \qquad \text{for } 0 \leqq t \leqq t_0$$

and

$$\Omega(x_2(t)) \leqq \Omega(x_2(t_0)) + \int_{t_0}^t h(\tau)\,d\tau + x_1(t_0) \leqq$$

$$\leqq \Omega(r_0) + r + \int_0^t h(\tau)\,d\tau \qquad \text{for } t_0 \leqq t \leqq b.$$

Thus

$$\Omega(x_2(t)) \leqq r^* + \int_0^t h(\tau)\,d\tau \qquad \text{for } 0 \leqq t \leqq b,$$

where $r^* = \Omega(r_0) + r + \int_0^a h(\tau)\,d\tau$ does not depend on b, x_1 and x_2. Hence according to (1.10) the estimate (2.6) is valid. This completes the proof.

3. THE LEMMA ON THE SOLVABILITY OF A CERTAIN AUXILIARY BOUNDARY VALUE PROBLEM

Consider the auxiliary two point boundary value problem

$$\varphi(x_1(0), x_2(0)) = 0, \qquad x_k(b) = 0, \tag{3.1}$$

where $k \in \{1, 2\}$, $b \in]0, +\infty[$ for the system (0.1).

Lemma 3.1. *Let there hold*

$$\varphi(0, 0) < 0, \ \varphi(x_1, x_2) > 0 \qquad \textit{for } x_k \geqq 0, x_{3-k} > r, \tag{3.2}$$

(3.3) $$f_i(t,0,0) = 0\ (i = 1,2),\ f_1(t,0,x) \leqq 0,\ f_2(t,x,0) \leqq 0 \quad \text{for } 0 \leqq t \leqq b,\ x \geqq 0$$

and

(3.4) $$\sum_{i=1}^{2} | f_i(t, x_1, x_2) | \leqq f^*(t) \qquad \text{for } 0 < t < b,\ x_1 \geqq 0,\ x_2 \geqq 0,$$

where $r \in]0, +\infty[$ *and* $f^* \in L([0, b])$. *Then the problem* (0.1), (3.1) *has at least one solution* (x_1, x_2) *such that*

(3.5) $$x_1(t) \geqq 0,\ x_2(t) \geqq 0 \qquad \text{for } 0 \leqq t \leqq b.$$

Proof. At first let us prove Lemma under the additional assumption that the right-hand sides of the system (0.1) satisfy the local Lipschitz condition, i.e. for each $\varrho > 0$

(3.6) $$\sum_{i=1}^{2} | f_i(t, x_1, x_2) - f_i(t, y_1, y_2) | \leqq l_\varrho(t)(| x_1 - y_1 | + | x_2 - y_2 |) \quad \text{for } 0 \leqq t \leqq b,\ 0 \leqq x_j, y_j \leqq \varrho\ (j = 1, 2),$$

where $l_\varrho \in L([0, b])$.

Let us put

$$\sigma(s) = \begin{cases} 0 & \text{for } s < 0, \\ s & \text{for } s \geqq 0, \end{cases}$$

(3.7) $$\tilde{f}_i(t, x_1, x_2) = f_i(t, \sigma(x_1), \sigma(x_2)) \qquad (i = 1, 2)$$

and consider the system

(3.8) $$\frac{dx_i}{dt} = \tilde{f}_i(t, x_1, x_2) \qquad (i = 1, 2)$$

under initial conditions

(3.9) $$x_k(b) = 0,\ x_{3-k}(b) = \alpha.$$

According to (3.4) and (3.6) for any $\alpha \in R$ the problem (3.8), (3.9) has the unique solution $(x_1(.;\alpha), x_2(.;\alpha))$ which is defined on the whole segment $[0, b]$.

Put

$$l_1(t;\alpha) = \begin{cases} \dfrac{\tilde{f}_1(t, 0, x_2(t;\alpha)) - \tilde{f}_1(t, x_1(t;\alpha), x_2(t;\alpha))}{x_1(t;\alpha)} & \text{for } x_1(t;\alpha) \neq 0, \\ 0 & \text{for } x_1(t;\alpha) = 0, \end{cases}$$

$$l_2(t;\alpha) = \begin{cases} \dfrac{\tilde{f}_2(t, x_1(t;\alpha), 0) - \tilde{f}_2(t, x_1(t;\alpha), x_2(t;\alpha))}{x_2(t;\alpha)} & \text{for } x_2(t;\alpha) \neq 0, \\ 0 & \text{for } x_2(t;\alpha) = 0. \end{cases}$$

From (3.3) and (3.7) there follows

$$\frac{dx_1(t;\alpha)}{dt} = \tilde{f}_1(t, 0, x_2(t;\alpha)) - l_1(t;\alpha)\,x_1(t;\alpha) \leqq$$
$$\leqq -l_1(t;\alpha)\,x_1(t;\alpha) \qquad \text{for } 0 \leqq t \leqq b$$

and

$$\frac{dx_2(t;\alpha)}{dt} = \tilde{f}_2(t, x_1(t;\alpha), 0) - l_2(t;\alpha)\,x_2(t;\alpha) \leqq$$
$$\leqq -l_2(t;\alpha)\,x_2(t;\alpha) \qquad \text{for } 0 \leqq t \leqq b.$$

Thus

$$x_k(t;\alpha) \geqq 0,\ x_{3-k}(t;\alpha) \geqq \alpha \exp\Big[\int_t^b l_{3-k}(\tau;\alpha)\,d\tau\Big] \geqq 0 \qquad \text{for } 0 \leqq t \leqq b,\ \alpha \geqq 0.$$

Therefore $(x_1(.;\alpha), x_2(.;\alpha))$ is a solution of the system (0.1) for any $\alpha \in R_+$.

Let us put

$$\tilde{\varphi}(\alpha) = \varphi(x_1(0;\alpha), x_2(0;\alpha))$$

and

$$\alpha^* = r + \int_0^b f^*(t)\,dt.$$

By (3.2) and (3.4)

$$x_{3-k}(0;\alpha^*) = \alpha^* - \int_0^b f_{3-k}(t, x_1(t;\alpha^*), x_2(t;\alpha^*))\,dt \geqq r$$

and

$$\tilde{\varphi}(\alpha^*) \geqq 0.$$

On the other hand, $\tilde{\varphi}$ is continuous on $[0, \alpha^*]$ and

$$\tilde{\varphi}(0) = \varphi(x_1(0;0), x_2(0;0)) = \varphi(0,0) < 0.$$

So there exists $\alpha_0 \in]0, \alpha^*]$ such that

$$\tilde{\varphi}(\alpha_0) = 0.$$

Obviously, $(x_1(.;\alpha_0), x_2(.;\alpha_0))$ is a solution of the problem (0.1), (3.1).

To complete the proof of Lemma it suffices to get rid of the additional assumption (3.6).

Let $\tilde{f}_1$ and $\tilde{f}_2$ be the functions given by the equalities (3.7) and let $\omega_m : R \to R_+$ $(m = 1, 2, \ldots)$ be a sequence of continuously differentiable functions such that

$$\omega_m(x) = 0 \qquad \text{for } |x| \geqq \frac{1}{m}, \quad \int_{-\infty}^{+\infty} \omega_m(x)\,dx = 1 \qquad (m = 1, 2, \ldots).$$

Put

$$g_{im}(t, x_1, x_2) = \int_{-\infty}^{\infty} \omega_m(y_1 - x_1)\,dy_1 \int_{-\infty}^{\infty} \omega_m(y_2 - x_1)\,\tilde{f}_i(t, y_1, y_2)\,dy_2,$$

$$h_{1m}(t, x) = \int_{-\infty}^{\infty} \omega_m(y - x) \tilde{f}_1(t, 0, y) \, dy,$$

$$h_{2m}(t, x) = \int_{-\infty}^{\infty} \omega_m(y - x) \tilde{f}_2(t, y, 0) \, dy,$$

$$f_{1m}(t, x_1, x_2) = g_{1m}(t, x_1, x_2) - g_{1m}(t, 0, x_2) - | h_{1m}(t, x_2) - h_{1m}(t, 0) |$$

and

$$f_{2m}(t, x_1, x_2) = g_{2m}(t, x_1, x_2) - g_{2m}(t, x_1, 0) - | h_{2m}(t, x_1) - h_{2m}(t, 0) |.$$

From (3.3) and (3.4) it follows for each natural m

$$f_{im}(t, 0, 0) = 0 \quad (i = 1, 2), \quad f_{1m}(t, 0, x) \leqq 0, \quad f_{2m}(t, x, 0) \leqq 0$$
$$\text{for } 0 \leqq t \leqq b, \quad x \in R$$

and

$$\sum_{i=1}^{2} | f_{im}(t, x_1, x_2) | \leqq 4f^*(t) \qquad \text{for } 0 \leqq t \leqq b, (x_1, x_2) \in R^2. \tag{3.10}$$

Besides, for any $t \in [0, +\infty[$

$$\lim_{m \to \infty} f_{im}(t, x_1, x_2) = \tilde{f}_i(t, x_1, x_2) \qquad (i = 1, 2) \tag{3.11}$$

uniformly on each bounded set of the space R^2.

From the structure of f_{1m} and f_{2m} it is obvious that these functions satisfy the local Lipschitz condition with respect to their two last arguments. Thus according to the already proved for each natural m, the system

$$\frac{dx_i}{dt} = f_{im}(t, x_1, x_2) \qquad (i = 1, 2)$$

has a solution (x_{1m}, x_{2m}) satisfying (3.1) and (3.5).

Using (3.2), (3.10) and (3.11) it is easy to prove that the sequence of the vector functions $\{(x_{1m}, x_{2m})\}_{m=1}^{\infty}$ contains a uniformly converging on $[0, b]$ subsequence such that its limit is a solution of the problem (0.1), (3.1). This completes the proof.

4. THE PROOFS OF THE EXISTENCE THEOREMS

Proof of Theorem 1.1. Without loss of generality assume that $h(t) \geqq 0$ for $t \geqq 0$.

Choose a number $r_0 \in]r, +\infty[$ and a nondecreasing continuous function $\delta_0 : R_+ \to R_+$ such that

$$\delta(x) \geqq \delta_0(x) \qquad \text{for } x \geqq 0$$

and

$$\delta_0(x) = \delta_0(r_0) > \frac{r}{a - a_0} \qquad \text{for } x \geqq r_0 .$$

Suppose that Ω is the function which is defined by means of the equality (2.1), Ω^{-1} is its inverse one and r^* is the positive constant appearing in Lemma 2.1. Let us put

$$\varrho_0(t) = \Omega^{-1}(r^* + \int_0^t h(\tau)\,d\tau), \qquad \varrho(t) = \varrho_0(t) + r_0,$$

$$\sigma_0(t, s) = \begin{cases} 1 & \text{for } 0 \leqq s \leqq \varrho_0(t), \\ 1 - \dfrac{s - \varrho_0(t)}{r_0} & \text{for } \varrho_0(t) < s \leqq \varrho(t), \\ 0 & \text{for } s > \varrho(t), \end{cases}$$

$$\sigma_1(s) = \begin{cases} s & \text{for } 0 \leqq s \leqq r, \\ r & \text{for } s > r, \end{cases} \qquad \sigma_2(t, s) = \begin{cases} s & \text{for } 0 \leqq s \leqq \varrho(t), \\ \varrho(t) & \text{for } s > \varrho(t), \end{cases}$$

$$\tilde{f}_1(t, x_1, x_2) = f_1(t, \sigma_1(x_1), \sigma_2(t, x_2)),$$

$$\tilde{f}_2(t, x_1, x_2) = \sigma_0(t, x_2) f_2(t, \sigma_1(x_1), x_2)$$

and consider the differential system

$$\text{(4.1)} \qquad \frac{dx_i}{dt} = \tilde{f}_i(t, x_1, x_2) \qquad (i = 1, 2).$$

From the definition of $\tilde{f}_1$ and $\tilde{f}_2$ and from the conditions (1.1) and (1.3) it follows

$$\text{(4.2)} \qquad \tilde{f}_i(t, x_1, x_2) = f_i(t, x_1, x_2) \qquad \text{for } t \geqq 0,$$
$$0 \leqq x_1 \leqq r, \qquad 0 \leqq x_2 \leqq \varrho_0(t) \qquad (i = 1, 2),$$

$$\text{(4.3)} \qquad \tilde{f}_1(t, x_1, x_2) \leqq -\delta_0(x_2) \leqq 0 \qquad \text{for } a_0 \leqq t \leqq a,\ x_1 \geqq 0,\ x_2 \geqq 0,$$

$$\text{(4.4)} \qquad \tilde{f}_2(t, x_1, x_2) \geqq -[h(t) + |\tilde{f}_1(t, x_1, x_2)|]\,\omega(x_2)$$
$$\text{for } 0 \leqq t \leqq a,\ x_1 \geqq 0,\ x_2 \geqq 0,$$

$$\text{(4.5)} \qquad \tilde{f}_2(t, x_1, x_2) \leqq [h(t) + |\tilde{f}_1(t, x_1, x_2)|]\,\omega(x_2)$$
$$\text{for } t \geqq a_0,\ x_1 \geqq 0,\ x_2 \geqq 0$$

and

$$\sum_{i=1}^{2} |\tilde{f}_i(t, x_1, x_2)| \leqq f^*(t) \qquad \text{for } t \geqq 0, x_1 \geqq 0, x_2 \geqq 0,$$

where

$$f^*(t) = \max \{ \sum_{i=1}^{2} | f_i(t, x_1, x_2) | : 0 \leqq x_1 \leqq r, 0 \leqq x_2 \leqq \varrho(t)\}$$

and

$$f^* \in L_{loc}([0, +\infty[).$$

According to Lemma 3.1 for each natural p the system (4.1) has a solution (x_{1p}, x_{2p}) which is defined on the segment $[0, a + p]$ and satisfies the conditions

$$\text{(4.6)} \quad \varphi(x_{1p}(0), x_{2p}(0)) = 0, \qquad x_{1p}(t) \geqq 0, \qquad x_{2p}(t) \geqq 0 \qquad \text{for } 0 \leqq t \leqq a + p.$$

By (1.1) and (1.3)

$$\text{(4.7)} \qquad x'_{1p}(t) \leqq 0, \qquad x_{1p}(t) \leqq x_{1p}(0) \leqq r \qquad \text{for } 0 \leqq t \leqq a + p.$$

On the other hand, since the inequalities (4.3)–(4.5) hold we have

$$x'_{1p}(t) \leqq -\delta_0(x_{2p}(t)) \leqq 0 \qquad \text{for } a_0 \leqq t \leqq a,$$

$$\text{(4.8)} \qquad x'_{2p}(t) \geqq -[h(t) - x'_{1p}(t)]\, \omega(x_{2p}(t)) \qquad \text{for } 0 \leqq t \leqq a,$$

$$\text{(4.9)} \qquad x'_{2p}(t) \leqq [h(t) - x'_{1p}(t)]\, \omega(x_{2p}(t)) \qquad \text{for } a_0 \leqq t \leqq a + p.$$

According to Lemma 2.1 from (4.6)–(4.9) there follows the estimate

$$\text{(4.10)} \qquad x_{2p}(t) \leqq \varrho_0(t) \qquad \text{for } 0 \leqq t \leqq a + p.$$

The conditions (4.2), (4.7) and (4.10) imply that (x_{1p}, x_{2p}) is a solution of the system (0.1) on $[0, a + p]$.

Using (4.7) and (4.10) it is easy to prove that from the sequence of vector functions $\{(x_{1p}, x_{2p})\}_{p=1}^{\infty}$ we can choose a subsequence $\{(x_{1p_m}, x_{2p_m})\}_{m=1}^{\infty}$ such that this subsequence uniformly converges on each segment from $[0, +\infty[$, and

$$(x_1, x_2) = \lim_{m \to +\infty} (x_{1p_m}, x_{2p_m})$$

is a solution of the system (0.1) on $[0, +\infty[$. On the other hand, from (4.6) it is obvious that (x_1, x_2) satisfies the conditions (0.2). This completes the proof.

Proof of Theorem 1.2. Without any loss of generality a may be chosen so small that

$$(\lambda - 1) \int_0^a h_0(\tau)\, d\tau < 1.$$

By (1.16) there exist numbers $a_0 \in \,]0, a[$ and $\varepsilon \in \,]0, 1[$ such that

$$\text{(4.11)} \qquad \varrho(t) = [\varepsilon + (\lambda - 1) \int_0^t h_0(\tau)\, d\tau]^{\frac{1}{1-\lambda}} - 1 > 0 \qquad \text{for } 0 \leqq t \leqq a$$

and

$$\text{(4.12)} \qquad \delta \int_{a_0}^{a} [\varrho(t)]^{\mu}\, dt > r.$$

Let us put

$$\sigma(s) = \begin{cases} 1 & \text{for } 0 \leqq s \leqq \varrho(0), \\ 2 - \dfrac{s}{\varrho(0)} & \text{for } \varrho(0) < s \leqq 2\varrho(0), \\ 0 & \text{for } s > 2\varrho(0) \end{cases}$$

and

$$\tilde{f}_2(t, x_1, x_2) = \begin{cases} \sigma(x_2) f_2(t, x_1, x_2) & \text{for } t \leqq a_0, \\ f_2(t, x_1, x_2) & \text{for } t > a_0. \end{cases}$$

Then since (1.15) holds, we have

$$\tilde{f}_2(t, x_1, x_2) \geqq -[\tilde{h}(t) + |f_1(t, x_1, x_2)|]\, \omega(x_2)$$
$$\text{for } 0 \leqq t \leqq a_0,\ 0 \leqq x_1 \leqq r,\ x_2 \geqq 0$$

and

$$\tilde{f}_2(t, x_1, x_2) \leqq [\tilde{h}(t) + |f_1(t, x_1, x_2)|]\, \omega(x_2)$$
$$\text{for } t > 0,\ 0 \leqq x_1 \leqq r,\ x_2 \geqq 0$$

where

$$\tilde{h}(t) = \max\left\{\frac{1}{\omega(x_2)} |f_2(t, x_1, x_2)| : 0 \leqq x_1 \leqq r,\ 0 \leqq x_2 \leqq 2\varrho(0)\right\}$$
$$\text{for } 0 \leqq t \leqq a_0,$$

$$\tilde{h}(t) = h(t) \qquad \text{for } t > a_0.$$

Thus Theorem 1.1 implies the existence of the solution (x_1, x_2) of the differential system

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2), \qquad \frac{dx_2}{dt} = \tilde{f}_2(t, x_1, x_2)$$

under the conditions (0.2).

By (1.3) and (1.14)

$$r \geqq x_1(0) \geqq x_1(a_0) - x_1(a) \geqq \delta \int_{a_0}^{a} [x_2(t)]^{\mu}\, dt \tag{4.13}$$

and

$$x_2'(t) \geqq -h_0(t)\,[1 + x_2(t)]^{\lambda} \qquad \text{for } 0 \leqq t \leqq a. \tag{4.14}$$

According to (4.11)–(4.13) we get

$$\varrho'(t) = -h_0(t)\,[1 + \varrho(t)]^{\lambda} \qquad \text{for } 0 \leqq t \leqq a \tag{4.15}$$

and there exists $t_0 \in [a_0, a]$ such that

$$x_2(t_0) < \varrho(t_0). \tag{4.16}$$

But from (4.14)–(4.16) it follows that

$$x_2(t) < \varrho(t) \qquad \text{for } 0 \leqq t \leqq t_0.$$

Therefore

$$x_2(t) < \varrho(0) \qquad \text{for } 0 \leqq t \leqq a_0.$$

From the last inequality and from the definition of $\tilde{f}_2$ it is obvious that (x_1, x_2) is a solution of the system (0.1). This completes the proof.

In order to prove Corollary 1 of Theorem 1.2 it is sufficient to verify that the function

$$h_0(t) = lt^{\frac{\lambda-1}{\mu}-1} |\ln t|^{\frac{\lambda-1}{\mu}}$$

satisfies the condition (1.16). But this becomes obvious if we take into consideration that

$$\lim_{t\to 0+} \frac{\int_0^t h_0(\tau)\,d\tau}{(t\,|\ln t|)^{\frac{\lambda-1}{\mu}}} = \frac{l\mu}{\lambda-1}.$$

Proof of Theorem 1.3. Choose the reals $r_0 > 0$ and $\varrho_0 > r_0$ such that

$$\delta(x) > \frac{r}{a} \quad \text{for } x > r_0 \tag{4.17}$$

and

$$\Omega(\varrho_0) = \Omega(r_0) + \int_0^a h(t)\,dt + r, \tag{4.18}$$

where Ω is the function defined by the equality (2.1).

Let us put

$$\delta_0(s) = \begin{cases} 0 & \text{for } s \leqq \varrho_0, \\ s-\varrho_0 & \text{for } s > \varrho_0, \end{cases} \qquad \tilde{\varphi}(x_1, x_2) = \varphi(x_1, x_2 + \delta_0(x_1)),$$

$$\sigma(s) = \begin{cases} 1 & \text{for } 0 \leqq s \leqq r, \\ 2 - \frac{s}{r} & \text{for } r < s < 2r, \\ 0 & \text{for } s \geqq 2r, \end{cases} \qquad \delta_1(s) = \begin{cases} 0 & \text{for } s \leqq r, \\ s - r & \text{for } s > r, \end{cases}$$

$$\tilde{f}_1(t, x_1, x_2) = \begin{cases} f_1(t, x_1, x_2) - \delta_1(x_2) & \text{for } t \leqq a, \\ f_1(t, x_1, x_2) & \text{for } t > a, \end{cases}$$

$$\tilde{f}_2(t, x_1, x_2) = \begin{cases} \sigma(x_2)\, f_2(t, x_1, x_2 - \delta_1(x_2)) & \text{for } t \leqq a, \\ f_2(t, x_1, x_2) & \text{for } t > a, \end{cases} \tag{4.19}$$

$$\varrho = \varrho_0 + r, \tilde{h}(t) = \max\left\{\frac{1}{\omega_\varrho(x_2)}\,|f_2(t, x_1, x_2)| : 0 \leqq x_1 \leqq \varrho, 0 \leqq x_2 \leqq 2r\right\}$$

$$\text{for } 0 \leqq t \leqq a$$

and

$$\tilde{h}(t) = h_\varrho(t) \quad \text{for } t > a.$$

By (1.1), (1.4), (1.17) and (1.19) we have

$$\tilde{\varphi}(x_1, x_2) > 0 \quad \text{for } x_1 > \varrho,\ x_2 \geqq 0, \tag{4.20}$$

$$\tilde{f}_i(t, 0, 0) = 0 \quad (i = 1, 2), \quad \tilde{f}_1(t, x_1, x_2) \leqq 0, \quad \tilde{f}_2(t, x_1, 0) \leqq 0$$

$$\text{for } t \geqq 0, x_1 \geqq 0, x_2 \geqq 0, \tag{4.21}$$

$$\tilde{f}_1(t, x_1, x_2) \leqq -\delta_1(x_2), \quad \tilde{f}_2(t, x_1, x_2) \geqq -[\tilde{h}(t) + |\tilde{f}_1(t, x_1, x_2)|]\, \omega_\varrho(x_2)$$

$$\text{for } 0 \leqq t \leqq a, \ 0 \leqq x_1 \leqq \varrho, x_2 \geqq 0, \tag{4.22}$$

$$\tilde{f}_2(t, x_1, x_2) \leqq [\tilde{h}(t) + |\tilde{f}_1(t, x_1, x_2)|]\, \omega_\varrho(x_2) \quad \text{for } t \geqq 0, 0 \leqq x_1 \leqq \varrho, x_2 \geqq 0 \tag{4.23}$$

and, on the other hand,

$$\tilde{\varphi}(x_1, x_2) > 0 \qquad \text{for } x_1 \geqq 0, x_2 > r, \tag{4.24}$$

$$\tilde{f}_1(t, x_1, x_2) \geqq -[h(t) + |\tilde{f}_2(t, x_1, x_2)|]\, \omega(x_1)$$

$$\text{for } 0 \leqq t \leqq a, x_1 \geqq 0, 0 \leqq x_2 \leqq r, \tag{4.25}$$

$$\tilde{f}_2(t, x_1, x_2) \leqq -\delta(x_1)\, \sigma(x_2) \qquad \text{for } 0 \leqq t \leqq a, x_1 \geqq 0, x_2 \geqq 0. \tag{4.26}$$

According to Theorem 1.1 the conditions (4.20)–(4.23) imply the existence of the solution (x_1, x_2) of the system (4.1) which satisfies the conditions

$$\tilde{\varphi}(x_1(0), x_2(0)) = 0, \qquad x_1(t) \geqq 0, \qquad x_2(t) \geqq 0 \qquad \text{for } t \geqq 0$$

and also

$$x_1'(t) \leqq 0 \qquad \text{for } t \geqq 0.$$

(4.24) and (4.26) give

$$x_2'(t) \leqq 0, \qquad x_2(t) \leqq x_2(0) \leqq r \qquad \text{for } 0 \leqq t \leqq a \tag{4.27}$$

and

$$\int_0^a \delta(x_1(t))\, dt \leqq r.$$

From the latter inequality by (4.17) it follows that

$$x_1(a) \leqq r_0. \tag{4.28}$$

(4.25), (4.27) and (4.28) yield

$$\Omega(x_1(t)) \leqq \Omega(r_0) + \int_0^a h(t)\, dt + \int_0^a |x_2'(t)|\, dt \leqq$$

$$\leqq \Omega(r_0) + \int_0^a h(t)\, dt + r \qquad \text{for } 0 \leqq t \leqq a.$$

Hence using (4.18) we have

$$x_1(t) \leqq \varrho_0 \qquad \text{for } 0 \leqq t \leqq a. \tag{4.29}$$

Now taking into consideration the definition of the functions $\tilde{\varphi}, \tilde{f}_1, \tilde{f}_2$ and the estimates (4.27) and (4.29) it becomes obvious that (x_1, x_2) is a solution of the problem (0.1), (0.2). This completes the proof.

Proof of Theorem 1.4. Let us denote

$$f_2^*(t) = \max \{|f_2(t, x_1, x_2)| : 0 \leqq x_1 \leqq r, 0 \leqq x_2 \leqq 4r\},$$

$$\sigma(s) = \begin{cases} 1 & \text{for } 0 \leqq s \leqq 2r, \\ 2 - \dfrac{s}{2r} & \text{for } 2r < s < 4r, \\ 0 & \text{for } s \geqq 4r, \end{cases} \qquad \delta_1(s) = \begin{cases} 0 & \text{for } s \leqq 2r \\ s - 2r & \text{for } s > 2r \end{cases}$$

and let $a > 0$ be so small that

$$\int_0^a f_2^*(t)\,\mathrm{d}t < r.$$

Define the functions $\tilde{f}_1$ and $\tilde{f}_2$ by means of the equalities (4.19) and consider the differential system (4.1).

According to Theorem 1.1 the problem (4.1), (4.2) has a solution (x_1, x_2).

By (1.1) and (1.5)

(4.30) $$x_1(t) \leqq x_1(0) \leqq r \qquad \text{for } t \geqq 0;\ x_2(0) < r.$$

Therefore

(4.31) $$|x_2(t)| \leqq |x_2(0)| + \int_0^a f_2^*(t)\,\mathrm{d}t < 2r \qquad \text{for } 0 \leqq t \leqq a.$$

Since (4.19), (4.30) and (4.31) hold, it is obvious that (x_1, x_2) is a solution of the system (0.1). This completes the proof.

5. THE UNIQUENESS THEOREM

The uniqueness theorem for the problem (0.1), (0.2) which is proved below considers the case, when the function f_1 satisfies the local Lipschitz condition with respect to the second argument and the function f_2 with respect to the third one, i.e. for any $\varrho \in R_+$ there exists $l(., \varrho) \in L_{loc}(R_+)$ such that

$$|f_1(t, x_1, x_2) - f_1(t, y_1, x_2)| \leqq l(t, \varrho)\,|x_1 - y_1|,$$

(5.1) $$|f_2(t, x_1, x_2) - f_2(t, x_1, y_2)| \leqq l(t, \varrho)\,|x_2 - y_2|$$

$$\text{for } t \geqq 0,\ 0 \leqq x_i \leqq \varrho,\ 0 \leqq y_i \leqq \varrho\ (i = 1, 2).$$

Theorem 5.1. *Let f_1 satisfy the local Lipschitz condition with respect to the second argument and f_2 — with respect to the third one. Suppose that*

(5.2) $$\varphi(y_1, y_2) > \varphi(x_1, x_2) \qquad \textit{for } y_1 > x_1 \geqq 0,\ y_2 \geqq x_2 \geqq 0,$$

(5.3) $$f_1(t, x_1, x_2) \leqq 0 \qquad \textit{for } t \geqq 0,\ x_1 \geqq 0,\ x_2 \geqq 0,$$

$$f_1(t, y_1, y_2) - f_1(t, x_1, x_2) \geqq -l_{11}(t)(y_1 - x_1) + l_{12}(t, x_2 - y_2)$$

(5.4) $$\text{for } t \geqq 0, 0 \leqq x_1 \leqq y_1, 0 \leqq y_2 \leqq x_2$$

and

$$f_2(t, y_1, y_2) - f_2(t, x_1, x_2) \leqq -l_{21}(t, y_1 - x_1) + l_{22}(t)(x_2 - y_2)$$

(5.5) $$\text{for } t \geqq 0, 0 \leqq x_1 \leqq y_1, 0 \leqq y_2 \leqq x_2$$

where $l_{ii} \in L(R_+)$ $(i = 1, 2)$, *the functions* $l_{12} : R_+^2 \to R_+$ *and* $l_{21} : R_+ \times [0, r] \to R_+$ *satisfy the local Carathéodory conditions, and are nondecreasing with respect to the second argument; for any* $c > 0$ *it holds*

(5.6) $$\text{mes}\,\{t \in R_+ : l_{12}(t, c) > 0\} > 0, \int_0^{+\infty} l_{12}(t, c[1 + \int_0^t l_{21}(\tau, c)\,d\tau])\,dt = +\infty.$$

Then the problem (0.1), (0.2) *has at most one solution.*

Proof. Let (x_1, x_2) and (y_1, y_2) be arbitrary solutions of the problem (0.1), (0.2). According to (5.1) and (5.3) there exist $r \in]0, +\infty[$ and $l_0 \in L_{loc}(R_+)$ such that

(5.7) $$0 \leqq x_1(t) \leqq r, \quad 0 \leqq y_1(t) \leqq r \quad \text{for } t \geqq 0$$

and

$$|f_1(t, y_1(t), y_2(t)) - f_1(t, x_1(t), y_2(t))| \leqq l_0(t)\,|y_1(t) - x_1(t)|,$$

(5.8) $$|f_2(t, y_1(t), y_2(t)) - f_2(t, y_1(t), x_2(t))| \leqq l_0(t)\,|y_2(t) - x_2(t)|$$

$$\text{for } t \geqq 0.$$

Put

$$u_1(t) = y_1(t) - x_1(t), \quad u_2(t) = y_2(t) - x_2(t).$$

Then either

(5.9) $$u_1(t_0) \neq 0$$

for a certain $t_0 \in R_+$ or

(5.10) $$u_1(t) = 0 \quad \text{for } t \geqq 0.$$

First suppose that (5.9) is fulfilled. To fix an idea we shall assume that $u_1(t_0) > 0$. Let us denote by $]t_*, t^*[$ the maximal interval containing t_0 in which

(5.11) $$u_1(t) > 0.$$

By (5.1) there occurs one of the following two cases

(5.12) $$u_1(t_*) = 0$$

or

(5.13) $$t_* = 0, \quad u_1(0) > 0, \quad u_2(0) < 0.$$

Let (5.12) hold. If we assume that

$$u_2(t) \geqq 0 \quad \text{for } t \geqq t_*$$

then (5.4) and (5.8) imply

$$u_1'(t) = [f_1(t, y_1(t), y_2(t)) - f_1(t, x_1(t), y_2(t))] +$$
$$+ [f_1(t, x_1(t), y_2(t)) - f_1(t, x_1(t), x_2(t))] \leqq l_0(t)\, u_1(t) \qquad \text{for } t_* < t < t^*.$$

Hence from (5.12) we have

$$u_1(t) \leqq 0 \qquad \text{for } t_* \leqq t < t^*$$

and this contradicts the condition (5.11).

Therefore in the both cases (5.12) and (5.13) there exists $t_1 \in]t_*, t^*[$ such that

(5.14) $$u_2(t_1) < 0, \qquad u_1(t) > 0 \qquad \text{for } t_1 \leqq t < t^*.$$

Considering (5.4), (5.5) and (5.14) it is easy to conclude that

$$u_1(t) \geqq c_0 + c_0 \int_{t_1}^{t} l_{12}(\tau, |u_2(\tau)|)\, d\tau, \; u_2(t) \leqq -c_0 - c_0 \int_{t_1}^{t} l_{21}(\tau, u_1(\tau))\, d\tau$$

(5.15) $$\text{for } t_1 \leqq t < t^*,$$

where c_0 is the minimum of the numbers

$$|u_i(t_1)| \exp[-\int_0^{+\infty} l_{ii}(\tau)\, d\tau] \qquad \text{and} \qquad \exp[-\int_0^{+\infty} l_{ii}(\tau)\, d\tau] \qquad (i = 1, 2).$$

By the definition of t^* it is clear that either

$$t^* < +\infty \qquad \text{and} \qquad u_1(t^*) = 0$$

or $t^* = +\infty$. According to (5.15) the first possibility may be eliminated. Thus $t^* = +\infty$.

Let $c \in]0, c_0[$ be so small that

$$c_0 + c_0 \int_{t_1}^{r} l_{21}(\tau, c_0)\, d\tau \geqq c + c \int_0^{t} l_{21}(\tau, c)\, d\tau \qquad \text{for } t \geqq t_1.$$

Then since (5.7) and (5.15) hold we obtain

$$\int_{t_1}^{t} l_{12}(s, c[1 + \int_0^{s} l_{21}(\tau, c)\, d\tau])\, ds \leqq \frac{r}{c_0} \qquad \text{for } t \geqq t_1.$$

But this contradicts the second of the conditions (5.6) and therefore (5.9) cannot be valid. Thus the condition (5.10) is fulfilled.

By (5.8) and (5.10) we have

$$|u_2'(t)| \leqq l_0(t)\, |u_2(t)| \qquad \text{for } t \geqq 0.$$

Hence either

(5.16) $$|u_2(t)| \geqq |u_2(0)| \exp[-\int_0^{t} l_0(\tau)\, d\tau] > 0 \qquad \text{for } t \geqq 0$$

or

$$u_2(t) = 0 \qquad \text{for } t \geqq 0. \tag{5.17}$$

Suppose that (5.16) is observed. Then (5.4), (5.5) and (5.10) imply

$$|u_2(t)| \geqq |u_2(0)| \exp\left[-\int_0^{+\infty} l_{22}(\tau)\, d\tau\right] = c > 0 \qquad \text{for } t \geqq 0$$

and

$$l_{12}(t, c) = 0 \qquad \text{for } t \geqq 0.$$

This contradicts the first of the conditions (5.6) and therefore (5.17) holds.

Thus the problem (0.1), (0.2) cannot have two distinct solutions. This completes the proof.

Remark. The conditions (5.6) are essential and cannot be omitted. For example consider the systems

$$\frac{dx_1}{dt} = -\exp(-t)\, x_2, \qquad \frac{dx_2}{dt} = 0 \tag{5.18}$$

and

$$\frac{dx_1}{dt} = \sigma(x_2) - x_1, \qquad \frac{dx_2}{dt} = -x_1, \tag{5.19}$$

where

$$\sigma(s) = \begin{cases} 0 & \text{for } 0 \leqq s \leqq 2, \\ 2 - s & \text{for } s > 2. \end{cases}$$

For any $c \in [0, 1]$ the vector function

$$x_1(t) = 1 - c + c\exp(-t), \qquad x_2(t) = c$$

is a solution of the system (5.18) and the vector function

$$x_1(t) = \exp(-t), \qquad x_2(t) = c + \exp(-t)$$

is a solution of the system (5.19) which satisfies the conditions

$$x_1(0) = 1, \qquad x_i(t) \geqq 0 \qquad \text{for } t \geqq 0 \ (i = 1, 2). \tag{5.20}$$

Hence the problem (5.18), (5.20) (the problem (5.19), (5.20)) has an infinite set of solutions, although all conditions of Theorem 5.1 are fulfilled except the second (first) of conditions (5.6).

Corollary. Let the condition (5.2) be fulfilled and let the function f satisfy the local Lipschitz condition with respect to the third argument. Suppose that there exists

$l \in L(R_+)$ such that

$$f(t, y_1, y_2) - f(t, x_1, x_2) \geqq -l(t)(y_2 - x_2) \quad \text{for } t \geqq 0,\ 0 \leqq x_1 \leqq y_1,\ x_2 \leqq y_2 \leqq 0.$$

Then the problem (0.3), (0.4) has at most one solution.

6. ON BEHAVIOUR OF SOLUTIONS OF THE PROBLEM (0.1), (0.2) WHEN $t \to +\infty$

Theorem 6.1. *Let there hold*

$$f_1(t, x_1, x_2) \leqq -g_1(t, x_2), \qquad f_2(t, x_1, x_2) \leqq -g_2(t, x_1)$$
$$\text{for } t \geqq 0,\ x_1 \geqq 0,\ x_2 \geqq 0, \tag{6.1}$$

where the functions $g_i : R_+^2 \to R_+$ $(i = 1, 2)$ satisfy the local Carathéodory conditions and are nondecreasing with respect to the second argument. Suppose that either

$$\int_0^{+\infty} g_i(t, c)\, dt = +\infty \qquad \text{for } c > 0 \qquad (i = 1, 2), \tag{6.2}$$

or there exists $k \in \{1, 2\}$ such that

$$\int_0^{+\infty} g_k(t, c)\, dt < +\infty,\ \int_0^{+\infty} g_{3-k}\Big(t, \int_t^{+\infty} g_k(\tau, c)\, d\tau\Big)\, dt = +\infty \qquad \text{for } c > 0. \tag{6.3}$$

Then any solution (x_1, x_2) of the problem (0.1), (0.2) *satisfies the condition*

$$\lim_{t \to +\infty} x_i(t) = 0 \qquad (i = 1, 2).$$

Proof. Let (x_1, x_2) be an arbitrary solution of the problem (0.1), (0.2). By (6.1) the functions x_1 and x_2 are decreasing and

$$\int_0^t g_i(\tau, x_{3-i}(\tau))\, d\tau \leqq x_i(0) \qquad \text{for } t \geqq 0 \qquad (i = 1, 2).$$

Hence it is obvious that if (6.2) is valid then (6.4) holds.

Now assume that (6.3) is fulfilled. Then it is clear that

$$\int_0^{+\infty} g_{3-k}(t, c)\, dt = +\infty \qquad \text{for } c > 0.$$

Therefore

$$\lim_{t \to +\infty} x_k(t) = 0.$$

Hence it remains to show that

$$\lim_{t \to +\infty} x_{3-k}(t) = 0.$$

Admit the contrary, i.e. that

$$x_{3-k}(t) \geqq c \qquad \text{for } t \geqq 0,$$

where c is a certain positive constant. Then by (6.1)

$$x_k(t) \geqq \int_t^{+\infty} g_k(\tau, c)\,\mathrm{d}\tau \qquad \text{for } t \geqq 0$$

and

$$\int_0^t g_{3-k}\Big(t, \int_t^{+\infty} g_k(\tau, c)\,\mathrm{d}\tau\Big)\,\mathrm{d}t \leqq x_{3-k}(0) \qquad \text{for } t \geqq 0.$$

But the last inequality contradicts the condition (6.3). This proves the theorem.

Corollary. Let

$$f(t, x_1, x_2) \geqq g(t, x_1) \qquad \text{for } t \geqq 0, x_1 \geqq 0, x_2 \leqq 0,$$

where $g : R_+^2 \to R_+$ satisfies the local Carathéodory conditions, is nondecreasing with respect to the second argument and

$$\int_0^{+\infty} t g(t, c)\,\mathrm{d}t = +\infty \qquad \text{for } c > 0.$$

Then each solution u of the problem (0.3), (0.4) satisfies the condition

$$\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty} u'(t) = 0.$$

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