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A NOTE ON SYSTEMS OF IDEALS

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1. Introduction

In [2] the concept of an ideal operation was defined and in the same paper it was shown that the commutator of a group G can serve as an ideal operation for the system of all normal subgroups (SNS) in G. However, this construction with the commutator is not possible for all groups (the restriction is given in theorem 2.7 of [2]). In the present note we show that the groups allowing the construction mentioned above must be nilpotent. Furthermore, we give an example of an ideal operation which fits for the SNS in an arbitrary group.

We now summarize our notation. Here G denotes a multiplicative group, $C_G(H)$ is the centralizer of H in G and if H has only one element x, then we write $C_G(x)$. If A is a subset of G, then $S(A)$ is the subgroup generated by A and $N(A)$ is the normal subgroup generated by A. If A has only one element x, then we write $S(x)$ and $N(x)$.

2. Results

We first formulate some definitions and results of [2] in our notation.

Let G be a group. If * is a binary operation in G such that

$$G * N(A) \subseteq N(A)$$

and

$$N(A) * B \subseteq N(A * B) \qquad \text{for all } A, B \subseteq G,$$

then we say that * is an ideal operation for the SNS in G.

For our purposes it is enough to establish the corollary of theorem 2.7 in [2]. We state it as

Theorem 1 ([2], p. 242). Let G be a group. If the operation commutator is an ideal operation for the SNS in G, then $C_G(y)$ is normal in G for all $y \in G$.

If a group G allows the operation commutator to be an ideal operation for the SNS in G, then we say that G is a K-group. Clearly, all abelian groups are K-groups. We give more examples of K-groups.

Example 1. Let G be a finite group with only normal subgroups. Then G is a K-group.

Example 2. Let G be a p-group with only abelian subgroups. Then G is a K-group.

The groups mentioned in our examples have one common feature: they are all nilpotent. Now we show that, in fact, all K-groups are nilpotent.

However, we first introduce the Fitting subgroup $F(G)$ of G which is the product of all nilpotent normal subgroups of G. Now $F(G)$ is itself a nilpotent normal subgroup of G ([1], pp. 276–277).

Theorem 2. Let G be a K-group. Then

(i) $N(x)$ is abelian for all $x \in G$,

(ii) G is nilpotent.

Proof. Let x be an arbitrary element of G. Now

$$N(x) = S(g^{-1}hg \mid g \in G, h \in S(x)).$$

Since $C_G(x) \geqq S(x)$, we have $C_G(x) \geqq N(x)$, for $C_G(x)$ is normal in G, by theorem 1. Thus $C_G(N(x)) \geqq S(x)$. Furthermore, $C_G(N(x)) \geqq g^{-1}S(x)\,g$ for all $g \in G$. Thus $C_G(N(x)) \geqq N(x)$, so $N(x)$ is abelian. Thus (i) is true.

Now $N(x)$ is abelian, hence nilpotent. We conclude that $F(G) \geqq N(x)$ for all $x \in G$, and so $F(G) = G$. It follows that G is nilpotent. The proof is complete.

Now it is natural to ask whether there exist ideal operations for the SNS in an arbitrary group G.

If a binary operation $*$ in G is an ideal operation for the SNS in G, then

$$1 * g = g * 1 = 1 \qquad \text{for all } g \in G.$$

It is easy to see that a binary operation $*$ satisfying $a * b = 1$ for all $a, b \in G$ is an ideal operation for the SNS in G. We say that it is a trivial ideal operation.

Finally, we give an example of a nontrivial ideal operation. We define the binary operation $*$ in G as follows:

$$1 * g = g * 1 = 1 \qquad \text{for all } g \in G$$

and

$$a * b = b \qquad \text{for all } a, b \in G;\ a \neq 1,\ b \neq 1.$$

REFERENCES

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