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ASSOCIATION SCHEMES II*

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3. Association schemes with three classes

3.1. Theorem. *Let* $R = \{R_0, R_1, R_2, R_3\}$ *and* (X, R) *satisfy* A1, A2, A3. *Then* (X, R) *satisfies* A4.

Proof. In the symmetric case ($R_1 = R_1^{-1}$, $R_2 = R_2^{-1}$, $R_3 = R_3^{-1}$) there holds the assertion (1.10.).

Let e.g. $R_1^{-1} = R_2$. According to 1.6. we have

$$p_{12}^0 = v_1 = v_2 = p_{21}^0, \tag{32}$$

according to 1.5.

$$p_{01}^0 = p_{10}^0 = p_{01}^2 = p_{10}^2 = p_{01}^3 = p_{10}^3 = 0, \tag{33}$$

$$p_{01}^1 = p_{10}^1 = 1, \tag{34}$$

$$p_{02}^0 = p_{20}^0 = p_{02}^1 = p_{20}^1 = p_{02}^3 = p_{20}^3 = 0, \tag{35}$$

$$p_{02}^2 = p_{20}^2 = 1, \tag{36}$$

$$p_{03}^0 = p_{30}^0 = p_{03}^1 = p_{30}^1 = p_{03}^2 = p_{30}^2 = 0, \tag{37}$$

$$p_{03}^3 = p_{30}^3 = 1, \tag{38}$$

according to 1.16.

$$v_1 p_{12}^1 = v_1 p_{11}^1 = v_1 p_{21}^1, \tag{39}$$

$$v_3 p_{13}^3 = v_1 p_{33}^1 = v_3 p_{31}^3, \tag{40}$$

$$v_2 p_{13}^2 = v_1 p_{23}^1 = v_3 p_{11}^3, \tag{41}$$

$$v_3 p_{12}^3 = v_1 p_{31}^1, \tag{42}$$

$$v_3 p_{21}^3 = v_1 p_{13}^1, \tag{43}$$

$$v_2 p_{12}^2 = v_1 p_{21}^1, \tag{44}$$

* For Chapter 1, 2 see Association schemes I. Arch. Math. 4, 1981, 173—183

according to 1.13. ($i = 1, j = 2$)

(45) $$v_1 + v_1 p_{12}^1 + v_2 p_{12}^2 + v_3 p_{12}^3 = v_1 v_2,$$

and according to 1.15. ($i = 1$, $k = 1$)

(46) $$p_{10}^1 + p_{11}^1 + p_{12}^1 + p_{13}^1 = v_1.$$

Using relations (42), (44) we get from (45)

$$v_1 + v_1 p_{12}^1 + v_1 p_{21}^1 + v_1 p_{31}^1 = v_1,$$

hence using (39) and dividing above expression by v_1, we get

(47) $$1 + 2p_{12}^1 + p_{31}^1 = v_1.$$

Using (34), (39), we get from (46)

(48) $$1 + 2p_{12}^1 + p_{13}^1 = v_1.$$

Comparing (47) and (48), we obtain

(49) $$p_{13}^1 = p_{31}^1.$$

From the relations (42), (43), (49) it follows

(50) $$p_{12}^3 = p_{21}^3.$$

Transforming relations (39), (40), we obtain

(51) $$p_{12}^1 = p_{21}^1,$$

(52) $$p_{13}^3 = p_{31}^3.$$

By 1.9. we have

(53) $$p_{12}^1 = p_{12}^2,$$

(54) $$p_{13}^1 = p_{32}^2,$$

(55) $$p_{32}^1 = p_{13}^2,$$

(56) $$p_{21}^1 = p_{21}^2,$$

(57) $$p_{31}^1 = p_{23}^2,$$

(58) $$p_{23}^1 = p_{31}^2,$$

(59) $$p_{23}^3 = p_{31}^3,$$

(60) $$p_{32}^3 = p_{13}^3.$$

From the relations (51), (53), (56) it follows

(61) $$p_{12}^2 = p_{21}^2,$$

from (32), (41), (55)

(62) $$p_{23}^1 = p_{32}^1,$$

from (55), (58), (62)

(63) $$p_{13}^2 = p_{31}^2,$$

from (49), (54), (57)

$$p_{23}^2 = p_{32}^2, \tag{64}$$

from (52), (59), (60)

$$p_{23}^3 = p_{32}^3. \tag{65}$$

The relations (32), (33), (34), (35), (36), (37), (38), (49), (50), (51), (52), (61), (62), (63), (64), (65) give the assertion.

3.2. Theorem. *Let* $\boldsymbol{R} = \{R_0, R_1, R_2, R_3\}$, $R_1^{-1} = R_2$, *and* $(X, \boldsymbol{R})$ *be an association scheme. Then*

$$p_{11}^1 = p_{12}^1 = p_{22}^2 = p_{12}^2, \tag{66}$$

$$p_{13}^1 = p_{23}^2 = v_1 - 1 - 2p_{11}^1, \tag{67}$$

$$p_{22}^1 = p_{11}^2, \tag{68}$$

$$p_{23}^1 = p_{13}^2 = v_1 - p_{11}^1 - p_{22}^1, \tag{69}$$

$$p_{33}^1 = 1 + 3p_{11}^1 + p_{22}^1 - 2v_1 + v_3, \tag{70}$$

$$p_{11}^3 = p_{22}^3 = \frac{v_1^2}{v_3} - \frac{v_1}{v_3} p_{11}^1 - \frac{v_1}{v_3} p_{22}^1, \tag{71}$$

$$p_{12}^3 = \frac{v_1^2}{v_3} - \frac{v_1}{v_3} - 2\frac{v_1}{v_3} p_{11}^1, \tag{72}$$

$$p_{13}^3 = p_{23}^3 = \frac{v_1}{v_3} + 3\frac{v_1}{v_3} p_{11}^1 + \frac{v_1}{v_3} p_{22}^1 - 2\frac{v_1^2}{v_3} + v_1, \tag{73}$$

$$p_{33}^3 = v_3 - 1 - 2\frac{v_1}{v_3} - 6\frac{v_1}{v_3} p_{11}^1 - 2\frac{v_1}{v_3} p_{22}^1 + 4\frac{v_1^2}{v_3} - 2v_1. \tag{74}$$

Proof. Theorem 1.9. implies

$$p_{11}^1 = p_{22}^2, \tag{75}$$

$$p_{33}^1 = p_{33}^2, \tag{76}$$

$$p_{11}^3 = p_{22}^3, \tag{77}$$

$$p_{22}^1 = p_{11}^2, \qquad \text{i.e. (68).}$$

From (39), (53), (75) there follows (66).

According to 1.13. ($i = 3, j = 3$)

$$v_3 + v_1 p_{33}^1 + v_2 p_{33}^2 + v_3 p_{33}^3 = v_3^2, \tag{78}$$

and by 1.15. for $i = 2$, $k = 1$

$$p_{21}^1 + p_{22}^1 + p_{23}^1 = v_2 \tag{79}$$

and for $i = 3$, $k = 1$

$$p_{31}^1 + p_{32}^1 + p_{33}^1 = v_3. \tag{80}$$

From (32), (39), (79) it follows

(81) $$p_{11}^1 + p_{22}^1 + p_{23}^1 = v_1,$$

from (39), (48)

(82) $$1 + 2p_{11}^1 + p_{13}^1 = v_1,$$

from (80) and A4

(83) $$p_{13}^1 + p_{23}^1 + p_{33}^1 = v_3,$$

from (32), (76), (78)

(84) $$v_3 + 2v_1p_{33}^1 + v_3p_{33}^3 = v_3^2,$$

from (82)

(85) $$p_{13}^1 = v_1 - 1 - 2p_{11}^1,$$

from (81)

(86) $$p_{23}^1 = v_1 - p_{11}^1 - p_{22}^1.$$

(54), (85) and A4 imply (67),
(58), (86) and A4 imply (69).

Substituting p_{13}^1 and p_{23}^1 from (67), (69) to (83) we get (70) and substituting p_{33}^1 from (70) to (84) we get (74).

By 1.16. we have

(87) $$v_3p_{11}^3 = v_1p_{23}^1.$$

The relations (86) and (87) imply

(88) $$p_{11}^3 = \frac{v_1^2}{v_3} - \frac{v_1}{v_3}p_{11}^1 - \frac{v_1}{v_3}p_{22}^1.$$

(77) and (88) imply (71),
(42), (85) and A4 imply (72).

From (40), (70) we get

(89) $$p_{13}^3 = \frac{v_1}{v_3} + 3\frac{v_1}{v_3}p_{11}^1 + \frac{v_1}{v_3}p_{22}^1 - 2\frac{v_1}{v_3} + v_1.$$

(60), (89) and A4 imply (73)

Remark. If for an association scheme with 3 classes, where $R^{-1} = R_2$ the numbers $v_1, v_3, p_{11}^1, p_{12}^1$ are known, then the others p_{ij}^k are explicitly determined by 3.2. (The numbers p_{ij}^k which do not occur in 3.2. are determined by the axiom A4 and the theorems 1.5., 1.7.).

4. General case

4.1. Theorem. *Let* $|X| \leqq 5$, *and* (X, R) *satisfy* A1, A2, A3. *Then* (X, R) *satisfies* A4.

Proof. According to 1.11. we have $\sum_{i=0}^{n} v_i = |X|$. Since valencies are natural numbers, it holds $n < |X|$.

For $n = 1$ we have $R_1^{-1} = R_1$ and according to 1.10. the assertion is valid.

For $n = 2$ the assertion is implied by 2.1., for $n = 3$ by 3.1.

For $n = 4$ it must be $|X| \geqq 5$, for $|X| = 5$ we have $v_0 = v_1 = v_2 = v_3 = v_4 = 1$ (1.11.). According to 1.18. R contains no symmetric relation (excepting R_0). Without loss of generality we can put

$$(+) \qquad R_1^{-1} = R_2, \qquad R_3^{-1} = R_4.$$

We shall now construct a scheme with above mentioned properties satisfying A1, A2, A3 by successively completing the matrix

0	1	2	3	4
2	0			
1		0		
4			0	
3				0

In the second row the number one can be placed either in the 3rd, 4th or 5th column but the last two cases are equivalent. By this there is explicitly determined completing the second row and the second column (the regularity of the relations R_1, R_2, R_3, R_4 and equalities (+) must be observed). Thus we have two possibilities:

0	1	2	3	4
2	0	1	4	3
1	2	0		
4	3		0	
3	4			0

0	1	2	3	4
2	0	4	1	3
1	3	0		
4	2		0	
3	4			0

The first matrix cannot be completed without loss of regularity of some relation (in the third row of 4th column there cannot be placed any number without loss of regularity of the corresponding relation). In the second matrix, the number 4 can be placed only by one way (in the 4th column), and thus the whole matrix is determined:

0	1	2	3	4
2	0	4	1	3
1	3	0	4	2
4	2	3	0	1
3	4	1	2	0

Calculating all p_{ij}^k (see chapter 5) we get

$$\begin{array}{llll}
p_{12}^1 = p_{21}^1 = 0 & p_{23}^1 = p_{32}^1 = 1 & p_{11}^1 = 0 \\
p_{12}^2 = p_{21}^2 = 0 & p_{23}^2 = p_{32}^2 = 0 & p_{11}^2 = 0 \\
p_{12}^3 = p_{21}^3 = 0 & p_{23}^3 = p_{32}^3 = 0 & p_{11}^3 = 1 \\
p_{12}^4 = p_{21}^4 = 0 & p_{23}^4 = p_{32}^4 = 0 & p_{11}^4 = 0 \\
p_{13}^1 = p_{31}^1 = 0 & p_{24}^1 = p_{42}^1 = 0 & p_{22}^1 = 0 \\
p_{13}^2 = p_{31}^2 = 0 & p_{24}^2 = p_{42}^2 = 0 & p_{22}^2 = 0 \\
p_{13}^3 = p_{31}^3 = 0 & p_{24}^3 = p_{42}^3 = 1 & p_{22}^3 = 0 \\
p_{13}^4 = p_{31}^4 = 1 & p_{24}^4 = p_{42}^4 = 0 & p_{22}^4 = 1 \\
p_{14}^1 = p_{41}^1 = 0 & p_{34}^1 = p_{43}^1 = 0 & p_{33}^1 = 0 \\
p_{14}^2 = p_{41}^2 = 1 & p_{34}^2 = p_{43}^2 = 0 & p_{33}^2 = 1 \\
p_{14}^3 = p_{41}^3 = 0 & p_{34}^3 = p_{43}^3 = 0 & p_{33}^3 = 0 \\
p_{14}^4 = p_{41}^4 = 0 & p_{34}^4 = p_{43}^4 = 0 & p_{33}^4 = 0 \\
p_{44}^1 = 1 \quad p_{44}^2 = 0 & p_{44}^3 = 0 & p_{44}^4 = 0
\end{array}$$

The other necessary equalities are given by 1.5. and 1.7. Consequently A4 is satisfied. The foregoing construction implies that other schemes satisfying A1, A2, A3 for $|X| = 5$, $n = 4$ are different only in numbering relations.

4.2. Theorem. *Let* (X, R) *satisfy* A1, A2, A3, $R_i, R_j, R_k \in R$, $R_i \neq R_i^{-1}$, $R_j = R_j^{-1}$, $R_k = R_k^{-1}$, $v_i = v_j = v_k = 1$, $p_{ij}^k = 1$. *Then* $p_{ji}^k = 0$ *(i.e.* (X, R) *does not satisfy* A4).

Proof. Let $(x, y) \in R_k$ hold. Then there exists one and only one $z \in X$ such that $(x, z) \in R_i$, $(z, y) \in R_j$. Since $R_j = R_j^{-1}$ and $R_k = R_k^{-1}$, it holds $(y, x) \in R_k$, $(y, z) \in R_j$ and at the same time for $z' \neq z$ it holds $(y, z') \notin R_j$, because $v_j = 1$. Further $(z, x) \notin R_i$ is valid, because $R_i^{-1} \neq R_i$ and $(z, x) \in R_i^{-1}$. Thus $p_{ji}(y, x) = 0$ and hence according to A3 we have $p_{ji}^k = 0 \neq p_{ij}^k$.

Remark. The situation described in Theorem 4.2. is expressed by the graph on Fig. 2. Completing this graph so that it may satisfy A1, A2, A3 we get the graph on Fig. 3. Thus for $|X| = 6$ it is now possible to construct such a system of relations which satisfies A1, A2, A3 and does not satisfy A4. The matrix record is

0	1	2	3	4	5
1	0	4	5	2	3
3	4	0	2	5	1
2	5	3	0	1	4
4	3	5	1	0	2
5	2	1	4	3	0

For the values of i, j, k stated in Tab. 1 it holds, $1 = p_{ij}^k \neq p_{ji}^k = 0$.

i	1	2	3	1	1	4	2	4	4	3	5	1	5	2	3	5	4	5
j	2	1	1	3	4	1	4	2	3	4	1	5	2	5	5	3	5	4
k	4	5	4	5	2	3	1	5	1	5	2	3	1	4	1	4	2	3

Tab. 1

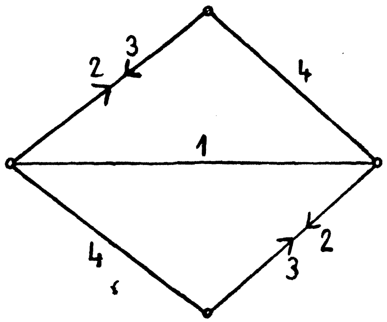

Fig. 2

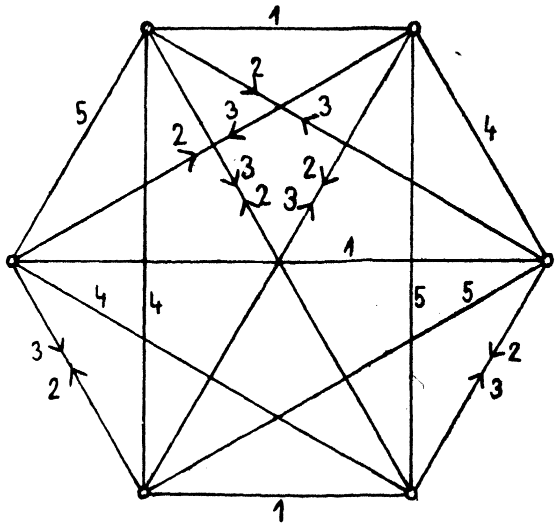

Fig. 3

Remark. As far as $(X, \boldsymbol{R})$ satisfies A1, A2 and all the relations from $\boldsymbol{R}$ are regular, it may occur even the situation when there is not satisfied the axiom A3. In this situation there may occur the case where for some i, j, k do not exist simultaneously the numbers p_{ij}^k, p_{ji}^k (example in Remark to 1.4.) but also the case where one of the numbers p_{ij}^k, p_{ji}^k exists and the other does not. For example the system described by the matrix

0	1	2	3	4	5
1	0	3	5	2	4
3	2	0	4	5	1
2	5	4	0	1	3
4	3	5	1	0	2
5	4	1	2	3	0

satisfies A1, A2, all the relations are regular, but e.g. the number p_{12}^4 does not exist and simultaneously $p_{21}^4 = 0$.

5. Association schemes for $|X| \leqq 6$

In the present chapter, there is given a survey of all association schemes for $|X| \leqq 6$ excepting those arised from aftermentioned schemes, by renumbering relations or renaming elements of carrier.

The used construction of association schemes is described in the proof of Theorem 4.1. For computing the values p_{ij}^k (proof of their existence and verifying the axiom A4) there was used the computer EC 1033. The program is based on the relation (2.5.) from [4].

The results obtained by means of computer show that the example quoted in Remark to 4.2. is for $|X| = 6$ the only case when $(X, \boldsymbol{R})$ satisfies A1, A2, A3 and does not satisfy A4. For schemes with four classes it is valid for $|X| \leqq 6$:

If $(X, \boldsymbol{R})$ satisfies A1, A2, A3, then it satisfies A4, too.

A survey of association schemes

This survey does not contain association schemes with one class (for every $|X| \geqq 2$ there exists an association scheme with one class). With each association scheme there are quoted nonzero numbers p_{ij}^k [excepting the numbers of the type p_{i0}^i, p_{0i}^i (see 1.5.)].

$|X| = 3$

0	1	2
2	0	1
1	2	0

(nonsym.)

$v_1 = v_2 = 1$

$p_{22}^1 = p_{11}^2 = 1$

$|X| = 4$

0	1	2	1
1	0	1	2
2	1	0	1
1	2	1	0

(sym.)

$v_1 = 2 \qquad v_2 = 1$

$p_{12}^1 = p_{21}^1 = 1 \qquad p_{11}^2 = 2$

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

(sym.)

$v_1 = v_2 = v_3 = 1$

$p_{23}^1 = p_{32}^1 = p_{13}^2 = p_{31}^2 = p_{12}^3 = p_{21}^3 = 1$

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

(nonsym.)

$v_1 = v_2 = v_3 = 1$

$p_{23}^1 = p_{32}^1 = p_{11}^2 = p_{33}^2 = p_{12}^3 = p_{21}^3 = 1$

$|X| = 5$

0	1	1	2	2
1	0	2	1	2
1	2	0	2	1
2	1	2	0	1
2	2	1	1	0

(sym.)

$v_1 = v_2 = 2$

$p_{12}^1 = p_{21}^1 = p_{22}^1 = p_{11}^2 = p_{12}^2 = p_{21}^2 = 1$

0	1	2	3	4
2	0	4	1	3
1	3	0	4	2
4	2	3	0	1
3	4	1	2	0

(nonsym.)

$v_1 = v_2 = v_3 = v_4 = 1$

$p_{23}^1 = p_{32}^1 = p_{44}^1 = p_{14}^2 = p_{41}^2 = p_{33}^2 =$
$= p_{11}^3 = p_{24}^3 = p_{42}^3 = p_{13}^4 = p_{31}^4 = p_{22}^4 = 1$

$|X| = 6$

0	1	1	1	1	2
1	0	2	1	1	1
1	2	0	1	1	1
1	1	1	0	2	1
1	1	1	2	0	1
2	1	1	1	1	0

(sym.)

$v_1 = 4 \qquad v_2 = 1$

$p_{12}^1 = p_{21}^1 = 1$

$p_{11}^1 = 2 \qquad p_{11}^2 = 4$

0	1	1	1	2	2
1	0	2	2	1	1
1	2	0	2	1	1
1	2	2	0	1	1
2	1	1	1	0	2
2	1	1	1	2	0

(sym.)

$v_1 = 3 \qquad v_2 = 2$

$p_{22}^2 = 1 \qquad p_{12}^1 = p_{21}^1 = 2$

$p_{11}^2 = 3$

0	1	1	1	2	3
1	0	2	3	1	1
1	3	0	2	1	1
1	2	3	0	1	1
3	1	1	1	0	2
2	1	1	1	3	0

(nonsym.)

$v_1 = 3 \qquad v_2 = v_3 = 1$

$p_{12}^1 = p_{21}^1 = p_{13}^1 = p_{31}^1 = p_{33}^2 = p_{22}^3 = 1$

$p_{11}^2 = p_{11}^3 = 3$

0	1	1	2	2	3
1	0	1	3	2	2
1	1	0	2	3	2
2	3	2	0	1	1
2	2	3	1	0	1
3	2	2	1	1	0

(sym.)

$v_1 = v_2 = 2 \qquad v_3 = 1$

$p^1_{11} = p^1_{22} = p^1_{23} = p^1_{32} = p^2_{12} = p^2_{21} = p^2_{13} = p^2_{31} = 1$

$p^3_{12} = p^3_{21} = 2$

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