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ANOTHER APPROACH TO THE CLASSICAL CALCULUS OF VARIATIONS II. HAMILTONIAN THEORY

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In the preceding Part I, a modified version of the general Lagrange problem $\mathscr{LP}(P, M, \mathfrak{A}, \varphi, \mathfrak{B}, \psi)$ has been presented. This problem is concerned with critical points of a certain functional (expressible by the above mentioned *exterior forms* φ, ψ) on a certain space **P** of mappings (of the above *manifold P with boundary* into the above *manifold M*) which satisfy a general system of partial differential equations (expressible by the above mentioned $C^\infty(M)$*-modules* $\mathfrak{A}$, $\mathfrak{B}$ of exterior forms on M), see also Section 1 below. In the present part we shall leave out all investigations on the boundary of P; then the space $\mathscr{EX}(P, M, \mathfrak{A}, \mathrm{d}\varphi)$ of extremals, mappings of P into M which solve the relevant Euler—Lagrange system of partial differential equations (determined by the module $\mathfrak{A}$ and the form $\mathrm{d}\varphi$), is the main object of our investigations. The Euler-Lagrange equations are expressible in invariant terms by exterior forms, however, we do not get an exterior system of equations in the common sense, provided $\mathfrak{A} \neq \{0\}$.

The case $\mathfrak{A} = \{0\}$, the trivial module consisting only of the zero form, seems to be a very special one. (The mappings from **P** *need not* satisfy any condition outside the boundary of P, cf. Section 1 below.) In this case, the extremals solve certain exterior system expressible by the single form $\mathrm{d}\varphi$. Thus, if $\mathrm{d}\varphi$ admits certain simple (*canonical*) expression in an appropriate coordinate system, the mentioned exterior system (and also the equivalent Euler—Lagrange system) is of certain special (one may say: *Hamiltonian*) type. In this sense $\mathrm{d}\varphi$ determines the *canonical structure* on the manifold M which, together with this structure, may be called a *phase space*.

Our aim is to transfer a general Lagrange problem ($\mathfrak{A} \neq \{0\}$) into an equivalent one with $\mathfrak{A} = \{0\}$, the (*inner*) *standard* problem after the terminology in I.7. We reconcile to the fact that this task may admit several solutions, a fact which is surely confirmed by the classical examples from the geodesics field theory of the multiple integral variational problem. We do not try to develop the most general theory. On the contrary, we introduce simplifying assumptions whenever possible to reach the case closely connected with the mentioned classical results. The line of possible generalizations and various modifications will be quite evident.

1. Review of Part I. Let M be a manifold, $\mathfrak{A}(\mathfrak{B})$ be a $C^\infty(M)$-module of n-forms ($(n-1)$-forms) on M, not necessarily of all forms. Let P be an n-dimensional compact oriented manifold with boundary Q, $\partial : Q \to P$ be the inclusion of the boundary. Denote by V the space of all embeddings $p : P \to M$, let $q = p \circ \partial : Q \to M$. We introduce the mapping $G : V \to W = \Pi\mathbf{R}_{\alpha,\beta}$ ($\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$; the direct product of real axes $\mathbf{R}$ indexed by the mentioned couples (α, β)) with the components $G(p)_{\alpha,\beta} = \int_P p^*\alpha + \int_Q q^*\beta$. (Note that $\boldsymbol{P} = G^{-1}(0)$ is the set of all $p \in V$ satisfying $p^*\alpha = q^*\beta \equiv 0$; $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$.) Let $\varphi(\psi)$ be an n-form ($(n-1)$-form) on M. We introduce the functional $F(p) = \int_P p^*\varphi + \int_Q q^*\psi$.

The Lagrange problem $\mathscr{LP}(P, M, \mathfrak{A}, \varphi, \mathfrak{B}, \psi)$ (briefly denoted $\mathscr{LP}$) deals with *G-critical points* of F. The latter notion (which is a suitable substitute for the commonly used concept of a *critical point of F on the set $\boldsymbol{P}$*) may be precised as follows: $p \in V$ is called a G-critical point of F if $\mathrm{d}F_p(Z_p) = 0$ for every tangent vector Z_p of (the infinite-dimensional manifold) V satisfying $\mathrm{d}G_p(Z_p) = 0$. (See Part I for an elementary approach in which the tangent maps $\mathrm{d}F$, $\mathrm{d}G$ are not explicitly used.)

The weakest result, Theorem I.6, asserts that *$p \in V$ is a C-critical point of F if and only if for every vector field Z on M there exist forms $\bar{\alpha} \in \mathfrak{A}$, $\bar{\beta} \in \mathfrak{B}$ which satisfy*

$$\int_P p^*Z \lrcorner \mathrm{d}(\varphi - \bar{\alpha}) + \int_Q q^*Z \lrcorner (\varphi - \bar{\alpha} + \mathrm{d}(\psi - \bar{\beta})) = 0.$$

This condition implicitly involves the Euler–Lagrange system and the boundary transversality conditions, but in a very latent form.

In this connection, $p \in V$ may be called an *extremal* (to the problem $\mathscr{LP}$) if $p^*\alpha \equiv 0$ ($\alpha \in \mathfrak{A}$) and, moreover, if there exists a form $\bar{\alpha} \in \mathfrak{A}$ satisfying

(1) $$p^*Z \lrcorner \mathrm{d}(\varphi - \bar{\alpha}) \equiv 0, \qquad \text{for every vector field } Z \text{ on } M.$$

Because $p^*Z \lrcorner \zeta = 0$ for every $(n+1)$-form ζ on M and every vector field Z on M *tangent to the subset $pP \subset M$* (i.e., satisfying $Z_{p(t)} = \mathrm{d}p(Z'_t)$; $t \in P$, Z' is an appropriate vector field on P), it is not necessary to consider all vector fields Z on M in the relation (1); cf. I, 8.

2. Germs of extremals. We shall widely use some local concepts. Especially, from now on, the main object will be the *space of germs of extremals* $\mathscr{EX}(M, A, \mathrm{d}\varphi)$ (briefly denoted $\mathscr{EX}$) to the above problem $\mathscr{LP}$, defined as follows:

Let $[p]_t$ be the germ at a point $t \in P$ of an embedding (equivalently: of an immersion) $p : P \to M$ of an n-dimensional manifold P into M. Then, the mentioned space $\mathscr{EX}$ consists of all germs $[g]_t$ for which $p^*\alpha \equiv 0$ ($\alpha \in \mathfrak{A}$) and (1) holds with an appropriate form $\bar{\alpha} \in \mathfrak{A}$ (in an appropriate neighbourhood of t, may be). Especially, $U = \mathscr{EX}(M, \{0\}, 0)$ is the space of germs of all embeddings (immersions) $p : P \to M$ of an n-dimensional manifold P into M. We introduce an unusual topology into the space U: A *closed subset* C of U is a finite union of certain subsets $C_i \subset U$ $(i = 1, \ldots, k)$, $C = \cup C_i$, where every summand C_i is determined by choosing a $C^\infty(M)$-module $\mathfrak{T}_i$ of exterior forms on M and consists of all germs $[p]_t \in U$ satisfying $[p^*\tau]_t \equiv 0$ $(\tau \in \mathfrak{T}_i)$. Owing to the inclusion $\mathscr{E}\mathscr{X}(M, \mathfrak{A}, \mathrm{d}\varphi) \subset \subset U$, we obtain the *relative topology* on every space $\mathscr{E}\mathscr{X}(M, \mathfrak{A}, \mathrm{d}\varphi)$.

An immersion p: $P \to M$ will be identified with the family $[p]_t$ $(t \in P)$ of germs. Therefore, $p \in \mathscr{E}\mathscr{X}$ means that $p^*\alpha \equiv 0$ $(\alpha \in \mathfrak{A})$ and (1) is true in an appropriate neighbourhood of every point $t \in P$, for an appropriate form $\bar{\alpha} \in \mathfrak{A}$ dependent on t. Also, the above p lies in an open subset of $\mathscr{E}\mathscr{X}$, if every relevant germ $[p]_t$ does.

A germ $[p]_t$ which lies in a certain open subset $O \subset \mathscr{E}\mathscr{X}$ will be called a *generic germ*. Here, the set O will be precised in all concrete cases, and we tacitly assume O being dense in the space $\mathscr{E}\mathscr{X}$ under consideration to avoid certain trivial situations. Similarly, an immersion p is *generic*, if every related germ $[p]_t$ is, in the previous sense.

3. Resolvent sequences. A Lagrange problem $\mathscr{L}\mathscr{P}(P, M^*, \mathfrak{A}^*, \varphi^*, \ldots)$ (the dotted places do not matter and need not be specified) is called a *prolongation* (by a mapping $\pi : M^* \to M$) of the original problem $\mathscr{L}\mathscr{P}$, if the following conditions are satisfied: (*i*) If $p^* \subset \mathscr{E}\mathscr{X}^*(= \mathscr{E}\mathscr{X}(M^*, \mathfrak{A}^*, \mathrm{d}\varphi^*))$ is generic, then $p = \pi \circ p^* \in \mathscr{E}\mathscr{X}$. (*ii*) To every generic $p \in \mathscr{E}\mathscr{X}$ there exists $p^* \in \mathscr{E}\mathscr{X}^*$, $p = \pi \circ p^*$.

A Lagrange problem $\mathscr{L}\mathscr{P}(P, M^b, \mathfrak{A}^b, \varphi^b, \ldots)$ (the dotted places need not be specified) is called a *restriction* (by a mapping $i : M^b \to M$) of the original problem $\mathscr{L}\mathscr{P}$, if the following conditions are satisfied: (*i*) If $p = i \circ p^b \in \mathscr{E}\mathscr{X}$ (where $p^b : P \to M^b$) is generic, then $p^b \in \mathscr{E}\mathscr{X}^b$ (an abbreviation of $\mathscr{E}\mathscr{X}(M^b, \mathfrak{A}^b, \mathrm{d}\varphi^b)$). (*ii*) If $p^b \in \mathscr{E}\mathscr{X}^b$ is generic, then $p = i \circ p^b \in \mathscr{E}\mathscr{X}$.

Remind our aim which is to transfer a general Lagrange problem $\mathscr{L}\mathscr{P}$ into a standard one. To this end, we may use the above defined concepts. Thus, starting with the general problem $\mathscr{L}\mathscr{P}$ one may derive a lot of sequences of the type

$$\mathscr{L}\mathscr{P} = \mathscr{L}\mathscr{P}_1, \ldots, \mathscr{L}\mathscr{P}_k(= \mathscr{L}\mathscr{P}(P, M_k, \mathfrak{A}_k, \varphi_k, \ldots)), \ldots, \mathscr{L}\mathscr{P}_N,$$

where every pair of neighbouring terms $\mathscr{L}\mathscr{P}_k, \mathscr{L}\mathscr{P}_{k+1}$ is related either by a prolongation, or by a restriction. Of course, we are mainly interested in the corresponding sequences of spaces of germs of extremals

$$\mathscr{E}\mathscr{X} = \mathscr{E}\mathscr{X}_1, \ldots, \mathscr{E}\mathscr{X}_k(= \mathscr{E}\mathscr{X}(M_k, \mathfrak{A}_k, \mathrm{d}\varphi_k)), \ldots, \mathscr{E}\mathscr{X}_N.$$

We succeed, if $\mathfrak{A}_N = \{0\}$, and if every generic extremal $p \in \mathscr{E}\mathscr{X}$ corresponds to (we prefer: *exactly one*) extremal $p_N \in \mathscr{E}\mathscr{X}_N$, and also reversely. Then the above sequences are called *resolvent sequences* (for the problem $\mathscr{L}\mathscr{P}$), and we may apply the terminology mentioned at the beginning of the present part: M_N is a *phase space*, $\mathrm{d}\varphi_N$ determines the *canonical structure*, etc.

A little notice: The proposed construction is a somewhat adventurous one. All current classical variational problems are resolved with $N \leqq 3$, however, there are some indications that certain *irregular problems* demand $N = 4, 5$. We postpone this question to another place.

4. Standard prolongation. Assume that certain forms $\alpha_1, \ldots, \alpha_C \in \mathfrak{A}$ exist with the property that every $\alpha \in \mathfrak{A}$ is expressible as

$$\alpha = a_1\alpha_1 + \ldots + a_C\alpha_C(a_1, \ldots, a_C \in C^\infty(M)).$$

We introduce the trivial vector bundle $M^+ = \mathbf{R}^C \times M$ with the evident bundle projection $\pi : M^+ \to M$ and, denoting $x^+ = (A, x) \in M^+$ $(A = (A_1, \ldots, A)_C \in \mathbf{R}^C$, $x \in M)$, we define the *fundamental form* α^+ on the bundle M^+ by

$$(\alpha^+)_{(A,.)} = A_1\pi^*\alpha_1 + \ldots + A_C\pi^*\alpha_C.$$

The formula $\alpha = \sigma^* \circ \alpha^+$ yields a one-to-one correspondence between the forms $\alpha \in \mathfrak{A}$ and the cross-sections $\sigma : M \to M^+$ of the bundle M^+.

Let π^*B be the $C^\infty(M^+)$-module of all $(n - 1)$-forms on M^+ generated by the forms $\pi^*\beta$ $(\beta \in \mathfrak{B})$. The problem $\mathscr{LP}(P, M^+, \{0\}, \pi^*\varphi - \alpha^+, \pi^*\mathfrak{B}, \pi^*\psi)$ (briefly $\mathscr{LP}^+$) will be called the *standard prolongation* of $\mathscr{LP}$.

There is a close relationship between critical points of $\mathscr{LP}$ and $\mathscr{LP}^+$, however, we are interested only in the related spaces of germs of extremals $\mathscr{EX}$ and $\mathscr{EX}^+$ (abbreviation for $\mathscr{EX}(M^+, \{0\}, \mathrm{d}(\pi^*\varphi - \alpha^+))$). The following result asserts that $\mathscr{LP}^+$ is a prolongation of $\mathscr{LP}$ in the sense of Section 3.

5. Theorem. *The set O, consisting of all germs $[p^+]_t \in U^+$ (U^+ is the space of germs of all immersions $p^+ : P \to M^+$), for which $[p]_t = [\pi \circ p^+]_t \in U$ still is a germ of an immersion, is open. If $p^+ \in O \cap \mathscr{EX}^+$, then $p \in \mathscr{EX}$. Conversely, let $p \in \mathscr{EX}$ and $\bar{\alpha} = \bar{a}_1\alpha_1 + \ldots + \bar{a}_C\alpha_C$ be the related form in (1). Then $p = \pi \circ p^+$, where $p^+ \in \mathscr{EX}^+$ is determined by $p^{+*}A_1 = p^*\bar{a}_1, \ldots, p^{+*}A_C = p^*\bar{a}_C$.*

Proof: $[p^+]_t \notin O$ if and only if $(\pi \circ p^+)^*\tau \equiv 0$, for all n-forms τ on M^+, the first assertion follows.

Before continuing the proof, note that every $p^+ \in O$ may be locally represented by $p^+ = \sigma \circ p$, σ being a cross-section of M^+, $p : P \to M$ an embedding. Then, a vector field Z^+ on M^+ may be locally decomposed as $Z^+ = H + V$, where H is a *horizontal* vector field (i.e. $H_{\sigma(x)} \equiv \mathrm{d}\sigma(Z_x)$; $x \in M$, Z is a vector field on M) and $V = v_1\, \partial/\partial A_1 + \ldots + v_C\, \partial/\partial A_C$ is a *vertical* vector field. We come to the proper proof.

First, let $p^+ \in O \cap \mathscr{EX}^+$. That means, a counterpart of (1) is true:

$$0 \equiv p^{+*}Z^+ \lrcorner\ \mathrm{d}(\pi^*\varphi - \alpha^+) = (\sigma \circ p)(H + V) \lrcorner\ \mathrm{d}(\pi^*\varphi - \alpha^+).$$

Clearly, $\sigma^*H \lrcorner\ \pi^*\,\mathrm{d}\varphi = Z \lrcorner\ \mathrm{d}\varphi$, $V \lrcorner\ \pi^*\,\mathrm{d}\varphi = 0$, $\sigma^*H \lrcorner\ \mathrm{d}\alpha^+ = Z \lrcorner\ \mathrm{d}\sigma^*\alpha^+$,

$\sigma^*V \lrcorner \, d\alpha^+ = \sigma^*V \lrcorner \, \Sigma(dA_k \wedge \pi^*\alpha_k + A_k\pi^* \, d\alpha_k) = \Sigma\bar{v}_k\alpha_k \; (\bar{v}_k = v_k \circ \sigma)$. We have the identity

$$0 \equiv p^*Z \lrcorner \, d(\varphi - \sigma^*\alpha^+) + p^* \, \Sigma\bar{v}_k\alpha_k,$$

satisfied for all vector fields Z and all functions $\bar{v}_1, \ldots, \bar{v}_C$ on M. Consequently, $p^*\alpha_k \equiv 0$ (hence $p^*\alpha \equiv 0$, $\alpha \in \mathfrak{A}$), $p^*Z \lrcorner \, d(\varphi - \bar{\alpha}) \equiv 0$ ($\bar{\alpha} = \sigma^*\alpha^+$). We see, $p \in \mathscr{E}\mathscr{X}$.

Second, let $p \in \mathscr{E}\mathscr{X}$. Then (1) is true with certain form $\bar{\alpha} = \bar{a}_1\alpha_1 + \ldots + \bar{a}_C\alpha_C \in \mathfrak{A}$. Defining $p^+ : P \to M^+$ by $p^+t = ((\bar{a}_1 \circ p, \ldots, \bar{a}_C \circ p), pt)$ we have $p = \pi \circ p^+$, and reverse running of the previous part of the present proof gives $p^+ \in \mathscr{E}\mathscr{X}^+$.

6. S-restriction. Let $\mathscr{L}\mathscr{P}$ be a standard problem, namely $\mathscr{L}\mathscr{P} = \mathscr{L}\mathscr{P}(P, M, \{0\}, \ldots)$ (the dotted places need not be defined). Pet $i : M^- \to M$ be a mapping of a manifcld M^- into M, S be a set of vector fields on M (we employ only the simple case $S = \emptyset$, the empty set, in the present part). Introducing the $C^\infty(M^-)$-module $\mathfrak{A}^-$ of n-forms generated by all forms of the type $i^*S \lrcorner \, d\varphi$ ($S \in S$), the problem $\mathscr{L}\mathscr{P}(P, M^-, \mathfrak{A}^-, i^*\varphi, \ldots)$ will be called an S-restriction (by i) of the mentioned problem $\mathscr{L}\mathscr{P}$. As usual, we shall deal only with the related space of germs of extremals $\mathscr{E}\mathscr{X}(M^-, A^-, di^*\varphi)$ (briefly denoted $\mathscr{E}\mathscr{X}^-$).

Retaining the previous notation, we state conditions under which an S-restriction is a restriction in the sense of Section 3.

7. Theorem. *If $p = i \circ p^- \in \mathscr{E}\mathscr{X}$ (where $p^- : P \to M^-$), then $p^- \in \mathscr{E}\mathscr{X}^-$. Let O be an open subset of U^- (the space of germs of all immersions $p^- : P \to M^-$) and assume that for every germ $[p^-]_t \in O \cap \mathscr{E}\mathscr{X}^-$ ($t \in P$) satisfying $p^*S \lrcorner \, d\varphi \equiv 0$ ($p = i \circ p^-$, $S \in S$), every vector field Z on M admits a decomposition $Z = H + V$ with $H_{i(x)} = di(Z_x^-)$ ($x = p(t)$; Z^- is an appropriate vector field on M^-), $p^*V \lrcorner \, d\varphi = 0$. Then, $p = i \circ p^- \in \mathscr{E}\mathscr{X}$, provided $p \in U$, $p^- \in O \cap \mathscr{E}\mathscr{X}^-$.*

Proof: $p^- \in \mathscr{E}\mathscr{X}^-$ means that $p^{-*}\alpha^- \equiv 0$ ($\alpha^- \in \mathfrak{A}^-$) and $(p^{-*}Z^- \lrcorner \, di^*\varphi)_t \equiv 0$ for every t. These conditions may be rewritten as follows:

$$p^*S \lrcorner \, d\varphi \equiv 0 \qquad \text{and} \qquad (p^*Z \lrcorner \, d\varphi)_t \equiv 0$$

($Z_{i(x)} = di(Z_x^-)$, $x = p^-(t)$) and they are satisfied if $p \in \mathscr{E}\mathscr{X}$. This proves the first statement.

For the second assertion, assume $[p^-]_t \in O \cap \mathscr{E}\mathscr{X}^-$. It is sufficient to prove

$$0 \equiv (p^*Z \lrcorner \, d\varphi)_t = (p^*H \lrcorner \, d\varphi)_t + (p^*V \lrcorner \, d\varphi)_t.$$

But the first summand is equal to

$$((i \circ p^-)^* \, H \lrcorner \, d\varphi)_t = (p^{-*}Z^- \lrcorner \, di^*\varphi)_t$$

and vanishes because $[p^-]_t \in \mathscr{E}\mathscr{X}^-$; the vanishing of the second summand has been postulated.

8. Regular problems. We present a rather unusual and general definition which consists, roughly speaking, in the requirement that a resolvent sequence exists with length $N \leqq 3$. More accurately, we postulate an existence of a resolvent sequence of the type

$$\mathscr{L}\mathscr{P} = \mathscr{L}\mathscr{P}_1, \mathscr{L}\mathscr{P}^+ = \mathscr{L}\mathscr{P}_2, (\mathscr{L}\mathscr{P}^+)^- = \mathscr{L}\mathscr{P}_3$$

(the standard prolongation followed by an S-restriction, necessarily $S = \emptyset$ because $\mathscr{L}\mathscr{P}_3$ is a standard problem with $\mathfrak{A}_3 = \{0\}$) and, moreover, we require that the correspondence between generic extremals from $\mathscr{E}\mathscr{X}$ and $\mathscr{E}\mathscr{X}_3$ is one-to-one. We denote $(\mathscr{L}\mathscr{P}^+)^-$ more simply by $\mathscr{L}\mathscr{P}_0$.

As usual, we are interested only in the corresponding resolvent sequence of spaces of germs of extremals $\mathscr{E}\mathscr{X} = \mathscr{E}\mathscr{X}_1$, $\mathscr{E}\mathscr{X}^+ = \mathscr{E}\mathscr{X}_2$, $\mathscr{E}\mathscr{X}^0 = (\mathscr{E}\mathscr{X}^+)^- = \mathscr{E}\mathscr{X}_3$ explicitly written as follows:

(2) $\quad \mathscr{E}\mathscr{X}(M, A, \mathrm{d}\varphi), \mathscr{E}\mathscr{X}(M^+, \{0\}, \mathrm{d}(\pi^*\varphi - \alpha^+)), \mathscr{E}\mathscr{X}(M^0, \{0\}, \mathrm{d}i^*(\pi^*\varphi - \alpha^+))$.

This chain is completely determined by the mapping $i : M^0(= (M^+)^-) \to M^+$, called a *resolvent mapping* for the problem $\mathscr{L}\mathscr{P}$. Look at the question of what kind this mapping i should be.

First, if we come out with an extremal $p \in \mathscr{E}\mathscr{X}$, then there exist extremals $p^+ \in \in \mathscr{E}\mathscr{X}^+$ *lying over* p (i.e., satisfying $p = \pi \circ p^+$); cf. the second statement of Theorem 5. If there exists a unique extremal p^+ of the mentioned type *lying in the subset* $iM^0 \subset M^+$ (i.e., factorisable by $p^+ = i \circ p^0$, $p^0 : P \to M^0$), we finish since $p^0 \in \mathscr{E}\mathscr{X}^0$; cf. the first statement of Theorem 7.

Second, starting with a generic $p^0 \in \mathscr{E}\mathscr{X}^0$, the mapping i must be of certain special type (a *geodesic mapping,* see Section 10 below) to guarantee that $p^+ = i \circ p^0 \in \mathscr{E}\mathscr{X}^+$. If this is the case, then $p = \pi \circ p^+ \in \mathscr{E}\mathscr{X}$, for generic p^+ (the first statement of Theorem 5), and the above question is completely answered.

It seems that the second requirement on the mapping i may be weakened by supposing only $p = \pi \circ i \circ p^0 \in \mathscr{E}\mathscr{X}$, for generic $p_0 \in \mathscr{E}\mathscr{X}$ (instead of $p^+ = i \circ p^0 \in \in \mathscr{E}\mathscr{X}^+$). However, this is not true:

9. Theorem. *Let $i : M^0 - M$ be a submersion. If $p^0 \in \mathscr{E}\mathscr{X}^0$, $p = \pi \circ i \circ p^0 \in \mathscr{E}\mathscr{X}$, then $p^+ = i \circ p^0 \in \mathscr{E}\mathscr{X}^+$.*

A sketch of the proof: Locally, $p^+ = \sigma \circ p$ with an appropriate cross-section $\sigma : M \to M^+$ with values in the subset $iM^0 \subset M^+$. Then the rest of the proof is similar as for the Theorem 5.

10. A bit of metamathematics. There may exist too many phase spaces M^0 and phase mappings i for a given problem $\mathscr{L}\mathscr{P}$. However, as a rule, this number may be strongly reduced if certain additional structures (foliations, group symmetries, fixed tensors, boundary conditions, etc.) are present on M, M^+, and we ask only for such phase objects which are *intrinsically related* with them. Of course, the last phrase is somewhat ambigous and needs a more careful explanation. But in order not to fall into vague categorical generalities, we appeal to the common sense of the reader and indicate only the wholly concrete case of the group symmetry.

11. Pull-back. Let $\mathscr{LP}$ be a (general) Lagrange problem, $g : M' \to M$ be a mapping of a manifold M' into M. Denote by $g^*\mathfrak{A}$ ($g^*\mathfrak{B}$) the $C^\infty(M)$-module of all n-forms ($(n-1)$-forms) generated by the forms $g^*\alpha$, $\alpha \in \mathfrak{A}$ ($g^*\beta$, $\beta \in \mathfrak{B}$). The problem $\mathscr{LP}(P, M', g^*\mathfrak{A}, g^*\varphi, g^*\mathfrak{B}, g^*\psi)$ (briefly $g^*\mathscr{LP}$) is called the *pull-back* (by the mapping g) of the problem $\mathscr{LP}$. We have the corresponding space of germs of extremals $\mathscr{EX}(M', g^*\mathfrak{A}, dg^*\varphi)$ (briefly denoted $g^*\mathscr{EX}$).

Pull-back is not a fundamental operation. Indeed, set $i = \sigma \circ g : M' \to M^+$, where $\sigma : M \to M^+$ is the zero section of M^+, and let S be the set of all vertical vector fields on M^+. One can see that the S-restriction $(\mathscr{LP}^+)^-$ is identical with $g^*\mathscr{LP}$. Especially, for $M = M'$, $g = e$, the identity mapping, we have $(\mathscr{LP}^+)^- = e^*\mathscr{LP} = \mathscr{LP}$; S-restriction is, to a certain extent, an inverse operation to the standard prolongation.

If the problem $g^*\mathscr{LP}$ is a restriction by the map g of the problem $\mathscr{LP}$ in the sense of Section 3, g is called a *geodesic mapping*. Looking more closely at this case, one can see that the requirement (i) of the restriction is always satisfied. (If $p = g \circ p' \in \mathscr{EX}$, $p' : P \to M'$, then $p' \in g^*\mathscr{EX}$; a simple fact.) Therefore, g is a geodesic mapping if and only if $p = g \circ p' \in \mathscr{EX}$ for every generic $p' \in g^*\mathscr{EX}$. This is in agreement with the more particular case of Riemannian geometry.

If the problem $\mathscr{LP}$ under consideration is a standard one, then the notion of a pull-back exactly coincides with the $\emptyset$-restriction. Especially, this is the case of the resolvent mapping discussed in Section 8.

12. Group symmetry. Let a group G of diffeomorphisms $g : M \to M$ act on the manifold M and preserve the module $\mathfrak{A}$. That means,

$$g^*(a_1\alpha_1 + \ldots + a_C\alpha_C) = g_1(a_1, \ldots, a_C)\,\alpha_1 + \ldots + g_C(a_1, \ldots, a_C)\,\alpha_C$$

holds with certain functions $g_1, \ldots, g_C \in C^\infty(\mathbf{R}^C)$, for all functions $a_1, \ldots, a_C \in C^\infty(M)$. We have a linear representation of G in the space $\mathbf{R}^C$ acting to the right:

$$Ag = (a_1(A), \ldots, a_C(A)),\ A = (A_1, \ldots, A_C) \in \mathbf{R}^C.$$

Also, we have a group G^+ of diffeomorphisms g^+ operating on M^+ to left:

$$g^+(A, x) = (Ag^{-1}, gx), \qquad (A, x) \in M^+ = \mathbf{R}^C \times M. \tag{3}$$

The fundamental form α^+ is preserved,

$$g^{+*}\alpha^+ \equiv \alpha^+ (g \in G),$$

which is equivalent to the fact that the group symmetry preserves the prolongation procedure, $g^{+*}(\mathscr{L}\mathscr{P}^+) = (g^*\mathscr{L}\mathscr{P})^+$; written in all details,

$$\mathscr{L}\mathscr{P}(P, M^+, \{0\}, g^{+*}(\pi^*\varphi - \alpha^+), g^{+*}\mathfrak{B}^+, g^{+*}\pi^*\psi) = \\ = \mathscr{L}\mathscr{P}(P, M^+, (0), \pi^*(g^*\varphi - \alpha^+), (g^*\mathfrak{B})^+, \pi^*g^*\psi).$$

Let $i : M^0 \to M^+$ be a resolvent mapping to the problem $\mathscr{L}\mathscr{P}$. We know that the corresponding standard problem $\mathscr{L}\mathscr{P}^0 = (\mathscr{L}\mathscr{P}^+)^-$, the $\emptyset$-restriction by i of $\mathscr{L}\mathscr{P}^+$, is identical with the pull-back $i^*(\mathscr{L}\mathscr{P}^+)$, cf. the note at the end of Section 11. Assume that the group G acts on the phase space M^0 to the left. Then the mapping

$$i_g = (g^+)^{-1} \circ i \circ g : M^0 \to M^+$$

is a resolvent mapping to the problem $g^*\mathscr{L}\mathscr{P}$. The corresponding standard problem to the problem $g^*\mathscr{L}\mathscr{P}$ is

$$(i_g)^* (g^*\mathscr{L}\mathscr{P})^+ = ((g^{-1})^+ \circ i \circ g)^* (g^+)^* (\mathscr{L}\mathscr{P}^+) = g^*(i^*\mathscr{L}\mathscr{P}^+) = g^*\mathscr{L}\mathscr{P}^0;$$

it possesses the *equivariancy* property. Consequently, the canonical structure is *equivariant*, too. Because the canonical structure related with the problem $g^*\mathscr{L}\mathscr{P}$ is given by the exterior differential of the form $i_g^*(\pi^*g^*\varphi - \alpha^+)$, the last assertion is easily verifiable:

$$i_g^*(\pi^*g^*\varphi - \alpha^+) = ((g^{-1})^+ \circ i \circ g)^* (g^+)^* (\pi^*\varphi - \alpha^+) = g^*i^*(\pi^*\varphi - \alpha^+).$$

13. Continuation of Section 10. Retaining the previous notation we will assume that ((*i*):) certain structures are present on M^+ which are preserved under actions of G^+, ((*ii*):) certain additional requirements expressible only in terms of the mentioned structures are imposed on the sought resolvent mappings.

Then, it may happen that ((*iii*):) there exists a unique resolvent mapping $i(g)$ satisfying (ii) to every problem $g^*\mathscr{L}\mathscr{P}$. Denote $i = i(e)$ (e being the unit of G) the resolvent mapping for the problem $e^*\mathscr{L}\mathscr{P} = \mathscr{L}\mathscr{P}$. Owing to (i), we may expect that the mapping i_g also satisfies the requirement (ii). In this case $i(g) = i_g$ follows, and we have the following result: *The resolvent mapping $i(g)$, which may be often determined without any calculations with the groups G, G^+, is automatically equivariant.*

More generally, let ((*iv*):) we have a set $I(g)$ of all resolvent mappings satisfying (ii), to the problem $g^*\mathscr{L}\mathscr{P}$. By a little adaptation of the above case, one can see that the sets $I(g)$ are permuted by the group G in a very natural sense: $i_g \in I(g)$ for every $i \in I(e)$.

Illustrative examples

14. Setting of the problem LP. As in the Part I, we set $M = \mathbf{R}^{n+m+nm}$ (variables $x^i, y^j, y^j_i, i = 1, \ldots, n; j = 1, \ldots, m$), $\mathfrak{A}$ is the $C^\infty(M)$-module generated by the contact forms α^j $(= \mathrm{d}y^j - \Sigma y^j_i \, \mathrm{d}x^i)$, $\varphi = f\,\mathrm{d}x$ $(f \in C^\infty(M)$, $\mathrm{d}x = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n)$.

The data $P, \mathfrak{B}, \psi$ need not be specified since we shall consider only the spaces of germs of extremals.

A form $\alpha \in \mathfrak{A}$ is uniquely expressible by the sum

$$\alpha = \Sigma a^{J,L}_{I,K} \alpha^J \wedge dx^{(I)} \wedge \omega^L_K, \qquad |I| = |J| + |L|, \qquad |J| \geqq 1,$$

where we use the following multiindex notation:

$$J = (j_1, \ldots, j_r), j_1 \leqq \ldots \leqq j_r, \alpha^J = \alpha^{j_1} \wedge \ldots \wedge \alpha^{j_r}, |J| = r,$$
$$I = (i_1, \ldots, i_s), i_1 < \ldots < i_s, dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_s}, |I| = s,$$

and dx^I is the complementary product defined by the property $dx^I \wedge dx^{(I)} = dx$. (Thus, $dx^{(i)} = \partial/\partial x^{(i)} \lrcorner\, dx$, $dx^{(i,i')} = \partial/\partial x^{i'} \lrcorner\, \partial/\partial x^i \lrcorner\, dx, \ldots$) At last,

$$L = (l_1, \ldots, l_t), \qquad K = (k_1, \ldots, k_t), \qquad (l_1, k_1) \leqq \ldots \leqq (l_t, k_t)$$

(lexicographic order), and

$$\omega^L_K = \omega^{l_1}_{k_1} \wedge \ldots \wedge \omega^{lt}_{kt}, \qquad \omega^l_k = dy^l_k - \Sigma y^l_{ik} dx^i,$$

where the functions $y^l_{ik} \in C^\infty(M)$ $(l = 1, \ldots, m;\ i, k = 1, \ldots, n)$ will be defined below. We shall also denote

$$f^J = \frac{\partial^r f}{\partial y^{j_1} \ldots \partial y^{j_r}}, \qquad f^L_K = \frac{\partial^t f}{\partial y^{l_1}_{k_1} \ldots \partial y^{l_t}_{k_t}},$$
$$a^J_I = a^{J,\emptyset}_{I,\emptyset},$$

for brevity.

Dealing only with the *generic* extremals, with the form $p^* dx$ nowhere vanishing, we may assume (by a appropriate choice of the functions y^l_{ik}) that $p^*\omega^l_k \equiv 0$. Then the extremals satisfy

(4) $$p^*\alpha^j \equiv 0 \qquad (j = 1, \ldots, m),$$

and also,

(5) $$p^*(f^j dx + \sum_i d\bar{a}^j_i \wedge dx^{(i)}) \equiv 0 \qquad (j = 1, \ldots, m),$$

(6) $$p^*(f^j_i + \bar{a}^j_i) dx \equiv 0 \qquad (i = 1, \ldots, n; j = 1, \ldots, m),$$

the conditions arising from (1) by successive substitutions $Z = \partial/\partial y^j$, $Z = \partial/\partial y^j_i$ (the vectors $Z = \partial/\partial x^i$ may be omitted, cf. the end of Section 1). The system (4)–(6) is equivalent to the Euler–Lagrange system.

15. Prolonged problem. We introduce the space $M^+ = \mathbf{R}^C \times M$, C is the number of new independent variables $A^{J,L}_{I,K}$ occuring in the corresponding fundamental form

$$\alpha^+ = \Sigma A^{J,L}_{I,K} \pi^*(\alpha^J \wedge dx^{(I)} \wedge dy^L_K), \qquad |I| = |J| + |L|, \qquad |J| \geqq 1.$$

Dealing only with *generic* extremals p^+ for which the form $p^{+*}\pi^* \mathrm{d}x$ nowhere vanishes, we have the following conditions:

$$(4)^+ \qquad p^{+*}\pi^*(\alpha^J \wedge \mathrm{d}x^{(I)} \wedge \mathrm{d}y^L_K) \equiv 0,$$

$$(5)^+ \qquad p^{+*}(\pi^* f^j \,\mathrm{d}x + \sum_i \mathrm{d}A^j_i \wedge \pi^* \,\mathrm{d}x^{(i)}) \equiv 0 \qquad (A^J_I \equiv A^{J,\emptyset}_{I,\emptyset}),$$

$$(5)^+ \qquad p^{+*}(\pi^* f^j_i + A^j_i)\,\pi^* \,\mathrm{d}x \equiv 0.$$

They arise from the counterpart of (1), the relation $p^{+*}Z^+ \lrcorner \,\mathrm{d}(\pi^*\varphi - \alpha^+) \equiv 0$, by successive choosing $Z^+ = \partial/\partial A^{J,L}_{I,K}$, $Z^+ = \partial/\partial \pi^* y^j$, $Z^+ = \partial/\partial \pi^* y^j_i$ (the vectors $Z^+ = \partial/\partial \pi^* x^i$ are omitted).

16. Present structures. The space M is considered as a manifold. The form $\varphi = f\,\mathrm{d}x$ determines the foliation $x^1 = c^1, \dots, x^n = c^n$ of M; at least in the non-trivial case $f \neq 0$ on open subsets of M. We have the group G of diffeomorphisms $g : M \to M$ preserving this foliation and preserving the module $\mathfrak{A}$, too. Every transformation $g \in G$ is determined by the functions $g^* x^i \in C^\infty(\mathbf{R}^n)$ (variables in $\mathbf{R}^n$ are $x^1, \dots, x^n$), $g^* y^j \in C^\infty(\mathbf{R}^{n+m})$ (variables $x^1, \dots, x^n, y^1, \dots, y^m$), which are arbitrary to a large extent, and by the functions $g^* y^j_i \in C^\infty(M)$ calculable from the condition $g^*\alpha^j \in \mathfrak{A}$. The prolonged group G^+ is given by (3), and we do not need it explicitly. Note only that every subspace of M^+ given by the conditions

$$A^{J,L}_{I,K} \equiv 0,$$

where $|I|$, $|J|$, $|K|$, $|L|$ are fixed constants, is preserved under G^+. This easily follows from the fact that the forms $g^* \,\mathrm{d}x^i$, $g^*\alpha^j$, are linear combinations of $\mathrm{d}x^{i'}$, $\alpha^{j'}$, respectively.

17. Resolvent mapping. We restrict ourselves to regular problems, and let us look for the resolvent chain (2). Moreover, we claim the following restrictions for the sought phase space and resolvent mapping: $M^0 = M$, $i : M^0 = M \to M^+$ is a cross-section, and

$$(7) \qquad i^* A^{J,L}_{I,K} \equiv 0 \qquad \text{if} \qquad |K| \geqq 1, |L| \geqq 1.$$

The requirement (7) is of intrinsical nature, it is satisfied by $i_g = (g^+)^{-1} \circ i \circ g$, too.

The cross-section i will be determined if we know the remaining functions $i^* A^{I,\emptyset}_{J,\emptyset} = i^* A^J_I$. But $(6)^+$ enforces that

$$(8) \qquad i^* A^j_i \equiv -f^j_i,$$

and there are not any requirements arising from the equations of extremals $(4)^+$–$(6)^+$ for the functions $i^* A^J_I$ with $|J| \geqq 2$.

The conditions (8) are also equivariant for the group actions. Indeed, (8) means that

$$i^* \, \partial/\partial\pi^* y_i^j \lrcorner \, \mathrm{d}(\pi^*\varphi - \alpha^+) \equiv 0,$$

and the corresponding relation for the transformed case

$$i_g^* \partial/\partial\pi^* y_i^j \lrcorner \, \mathrm{d}(\pi^* g^* \varphi - \alpha^+) \equiv 0$$

is clearly equivalent to

$$g^* i^* \, \partial/\partial\pi^* y_i^j \lrcorner \, \mathrm{d}(\pi^*\varphi - \alpha^+) \equiv 0.$$

18. Poincaré—Cartan form. Choose

$$i^* A_I^J \equiv 0, \qquad |J| \geqq 2.$$

Now, the cross-section i is completely determined and we are going to verify whether it is a resolvent mapping, following the lines mentioned in Section 8.

The first demand (cf. Section 8) is relatively a simple one: Coming out with $p \in \mathscr{E}\mathscr{X}$, there exists a unique mapping $p^+ : P \to M^+$ satisfying $p = \pi \circ p^+$, $p^+ = i \circ p$ (i.e., p^+ lying over p and in the set $iM = iM^0$), since i is a cross-section. Then,

$$p^{+*} A_i^j = p^* i^* A_i^j = -p^* f_i^j = p^* \bar{a}_i^j$$

(remind that $\bar{\alpha} = \Sigma \bar{a}_i^j \alpha^j \wedge \mathrm{d}x^{(i)} + \ldots$ is the form occuring in (1)), and the converse statement of Theorem 5 gives $p^+ \in \mathscr{E}\mathscr{X}^+$. Consequently, $p^\circ = p : P \to M = M^0$ is the unique extremal from the space $\mathscr{E}\mathscr{X}^0$ corresponding to a generic $p \in \mathscr{E}\mathscr{X}$.

The second demand concerning the mapping i looks as follows: Starting with $p^0 \in \mathscr{E}\mathscr{X}^0$, we have to prove that $p^+ \in \mathscr{E}\mathscr{X}^+$ (then $p = \pi \circ p^+ \in \mathscr{E}\mathscr{X}$, for generic p^+, and we finished; cf. Section 8). For this, we apply the second assertion of Theorem 6 (where $\mathscr{E}\mathscr{X}, \mathscr{E}\mathscr{X}^-, S, H, V$, is replaced by $\mathscr{E}\mathscr{X}^+, \mathscr{E}\mathscr{X}^0, \emptyset$, vectors tangent to $iM,^0$ linear combinations of $\partial/\partial A_{I,K}^{J,L}$, respectively). We have to verify the requirement $p^{+*} V \lrcorner \, \mathrm{d}\varphi \equiv 0$, i.e. $(4)^+$, supposing $p^0 \in \mathscr{E}\mathscr{X}^0$. But $p^0 \in \mathscr{E}\mathscr{X}_0$ implies, among others, the relations

$$p^{0*} \, \partial/\partial y_i^j \lrcorner \, \mathrm{d}i^*(\pi^*\varphi - \alpha^+) \equiv 0,$$

which gives, after simple calculations,

$$p^{0*} \sum_{i',j'} f_{i,i'}^{j,j'} \alpha^j \wedge \mathrm{d}x^{(i)} = 0 \qquad (i = 1, \ldots, n, j = 1, \ldots, m).$$

Supposing that

$$p^{0*} \det (f_{i,i'}^{j,j'}) \neq 0,$$

the last relations are equivalent to $p^{0*}\alpha^j \wedge \mathrm{d}x^{(i)} \equiv 0$, that is, $p^{0*}\alpha^j \equiv 0$. Consequently, $(4)^+$ is true since $p^+\pi^*\alpha^j = p^*\alpha^j = p^{0*}\alpha^j \equiv 0$. The result is that $p \in \mathscr{E}\mathscr{X}$, if $p^0 (= p) \in \boldsymbol{O} \cap \mathscr{E}\mathscr{X}^0$, $\boldsymbol{O}$ being the set of generic germs defined by (9).

Although the above results are well-known, the whole procedure was presented in some details because it possesses a general character and may be word-by-word applied to other (also irregular) problems. At last, we may summarise some immediate consequences as follows: If the set $\boldsymbol{O}$ given by (9) is dense in the space $\mathscr{E}\mathscr{X}$, then $\mathscr{L}\mathscr{P}$ is a regular problem and $p \in \boldsymbol{O} \cap \mathscr{E}\mathscr{X}$ if and only if $p^0 \in \boldsymbol{O} \cap \mathscr{E}\mathscr{X}^0$. The canonical structure is determined by the exterior differentiale of the (*Poincaré – Cartan*) form

$$\varphi^0 = i^*(\pi^*\varphi - \alpha^+) = \varphi - i^*\alpha^+ = f\,\mathrm{d}x + \Sigma f_i^j \alpha^j \wedge \mathrm{d}x^{(i)},$$

which is equivariant: To the problem $g^*\mathscr{L}\mathscr{P}$ there corresponds the form $g^*\varphi^0$. The *Hamilton function* H is defined by

$$\varphi^0 = H\,\mathrm{d}x + \Sigma f_i^j\,\mathrm{d}y_i^j \wedge \mathrm{d}x^{(i)},$$

(i.e., $H = f - \Sigma f_i^j y_i^j$) and in local *Legendre coordinates* $x^i, y^j, \lambda_i^j = f_i^j$ we have

$$\varphi^0 = H\,\mathrm{d}x + \Sigma \lambda_i^j\,\mathrm{d}y^j \wedge \mathrm{d}x^{(i)}.$$

The Euler–Lagrange system arising from the condition $p^{0*}Z^0 \lrcorner\, \mathrm{d}\varphi^0 = 0$ (the counterpart of (1)) by setting $Z^0 = \partial/\partial y^j$, $Z^0 = \partial/\partial \lambda_i^j$ is of particular, *Hamiltonian type:*

$$\sum_i \frac{\partial p^{0*}\lambda_i^j}{\partial x^i} \equiv p^{0*}\frac{\partial H}{\partial y^j}, \qquad \frac{\partial p^{0*}y^j}{\partial x^i} \equiv -p^{0*}\frac{\partial H}{\partial \lambda_i^j}.$$

19. Carathéodory form. Supposing (7), (8), and assuming $f \neq 0$, there exists exactly one cross-section i for which the form $i^*(\pi^*\varphi - \alpha^+)$ is decomposable into n linear factors. It is the form

$$\varphi^0 = f\vartheta^1 \wedge \ldots \wedge \vartheta^n, \qquad \vartheta^i = \mathrm{d}x^i + \sum_j \frac{f_i^j}{f}\alpha^j. \tag{10}$$

By comparing the coefficients of various products $i^*\alpha^J \wedge \mathrm{d}x^{(I)}$ we get

$$\begin{aligned}
&i^*\pi^*f = f \text{ (triviality)}, \qquad i^*A_i^j = -f_i^j \text{ (condition (8))},\\
&i^*A_{i_1 i_2}^{j_1 j_2} = -f^{-1}(f_{i_1}^{j_1} f_{i_2}^{j_2} - f_{i_2}^{j_1} f_{i_1}^{j_2}), \ldots,\\
&i^*A_I^J = -f^{1-|I|}\Sigma f_{[i_1}^{j_1} \ldots f_{ir]}^{jr}, \ldots;
\end{aligned}$$

the squared bracket denotes alternation. The cross-section i will be a resolvent mapping, if $p \in \boldsymbol{O} \cap \mathscr{E}\mathscr{X}$ implies $p^0 \in O \cap \mathscr{E}\mathscr{X}^0$ and also reversely. Arguments similar to that of the preceding section show that this is the case if $(4)^+$ is true for generic extremals $p^0 \in \mathscr{E}\mathscr{X}^0$.

Having this in mind, assume $p^0 \in \mathscr{E}\mathscr{X}^0$. Then

$$p^{0*}Z^0 \lrcorner\, \mathrm{d}(f\vartheta^1 \wedge \ldots \wedge \vartheta^n) \equiv 0$$

is true for all vector fields Z^0 on $M^0 = M$. Using the non-holonomic coordinate frame $\vartheta^i, \alpha^j, \mathrm{d}y_i^j$ ($i = 1, \ldots, n$; $j = 1, \ldots, m$), denoting $\vartheta^{(i)} = -(-1)^i\vartheta^1 \wedge \ldots \wedge \vartheta^{i-1} \wedge \vartheta^{i+1} \wedge \ldots \wedge \vartheta^n$, and choosing $Z^0 = \partial/\partial y_i^j$, the above condition gives

$$p^{0*}\sum_{ii',j'} f^{-2}\Delta_{ii'}^{jj'}\alpha^j \wedge \vartheta^{(i)} \equiv 0, \qquad \Delta_{ii'}^{jj'} = ff_{ii'}^{jj'} - f_i^j f_{i'}^{j'} + f_{i'}^j f_i^{j'}.$$

Assuming

$$p^{0*}\det(\Delta_{ii'}^{jj'}) \neq 0,$$

we obtain $p^{0*}\alpha^j \wedge \vartheta^{(i)} \equiv 0$, and in the case $p^{0*}\vartheta^1 \wedge \ldots \wedge \vartheta^n \neq 0$ we have $p^{0*}\alpha^j \equiv 0$; we see that $(4)^+$ is true.

Summarise the results: $\mathscr{L}\mathscr{P}$ is a regular problem if the set $\boldsymbol{O}$ given by the conditions

$$p^*f \neq 0, \qquad p^*\vartheta^1 \wedge \ldots \wedge \vartheta^n \neq 0, \qquad p^*\det(\Delta_{ii'}^{jj'}) \neq 0$$

is dense in the space $\mathscr{E}\mathscr{X}$ (or, which is the same, in the space $\mathscr{E}\mathscr{X}^0$). Then, the canonical structure is determined by the exterior differential of the above form φ^0, called *Carathéodory form,* which is equivariant. We may introduce the *Legendre coordinates* and *Hamilton function* by writing

$$\begin{aligned}\varphi^0 &= f^{1-n}\bigwedge_{i=1}^{n=n}\left(\sum_{i'}\Delta^{ii'}\mathrm{d}x^{i'} + \sum_j f_i^j\,\mathrm{d}y^j\right) = \\ &= f^{1-n}\det(\Delta^{ii'})\bigwedge_{i=1}^{i=n}\left(\mathrm{d}x^i + \sum_{i',j}\nabla^{ii'}f_{i'}^j\,\mathrm{d}y^j\right) = \\ &= H\bigwedge_{i=1}^{i=n}\left(\mathrm{d}x^i + \sum_j \lambda_i^j\,\mathrm{d}y^j\right),\end{aligned}$$

where we denote $\Delta^{ii'} = (f\delta_{i'}^i - \Sigma f_i^j y_{i'}^j)$, $(\nabla^{ii'}) = (\Delta^{ii'})^{-1}$ is the inverse matrix. Thus, the Legendre coordinates are $x_i, y^j, \lambda_i^j = \sum_{i'}\nabla^{ii'}f_{i'}^j$, if the last system is locally invertible. This is equivalent to the local invertibility of the system $f_i^j = \mu_i^j$, i.e. to the condition (9). In these canonical coordinates, the Euler–Lagrange system possesses the Hamiltonian form

$$\sum_i \frac{\partial p^{0*}\lambda_i^j}{\partial \xi^i} = p^{0*}\frac{\partial \ln H}{\partial y^j}, \qquad \frac{\partial p^{0*}y^j}{\partial \xi^i} = p^{0*}\frac{\partial \ln H}{\partial \lambda_i^j},$$

where $H = f^{1-n}\det(\Delta^{ii'})$, $\xi^i \equiv \mathrm{d}x^i + \Sigma\lambda_i^j\,\mathrm{d}y^j$.

20. General solution. The Carathéodory form seems to be extremely important for boundary problems and geodesic field theory, and is in fact equivariant for larger group then the above group G^+. We shall demonstrate its utility in deriving some other resolvent mappings.

Decompose the Carathéodory form into the sum of terms homogeneous in α^j and $\mathrm{d}x^i$:

$$\varphi^0 = \sum_{r=0}^{r=n}\Phi^r, \qquad \Phi^r = f^{1-r}\sum f_{[i_1}^{j_1}\ldots f_{i_r]}^{j_r}\alpha^J \wedge \mathrm{d}x^{(I)}, \qquad |J| = |I| = r.$$

Clearly, $\Phi^0 = f\,\mathrm{d}x$, $\Phi^1 = \Sigma f_i^j\alpha^j \wedge \mathrm{d}x^{(i)}$, and every form Φ^r is equivariant. In this way, we get a lot of equivariant forms $\Sigma c^r\Phi^r$, with $c^1, \ldots, c^r$ arbitrary constants, however, the canonical structure may arise only in the case $c^0 = c^1 = 1$. The related resolvent mapping i may be then determined by comparing the coefficients in the equation

$$i^*(\pi^*\varphi - \alpha^+) = \Phi^0 + \Phi^1 + \sum_{r=2}^{r=n} c^r\Phi^r,$$

Especially, $c^2 = \ldots = c^n = 0$ presents the Poincaré—Cartan form, $c^2 = \ldots = c^n = 1$ gives the Carathéodory form. Certain investigations related with the form

$$\binom{n}{r}^{-1} f\,\Sigma\eta^I \wedge dx^{(I)}, \qquad \eta^i = dx^i + \binom{n}{r}\sum_j f_i^j\alpha^j/f, \qquad |I| = r$$

(the sum is taken over all combinations of order r), also appear in literature. We get the Poincaré—Cartan (Carathéodory) form in the case $r = 1$ $(r = n)$.

There are many other equivariant phase structures, and all they admit an explicit expression in terms of certain number of arbitrary functions on the space M. However, *by imposing the requirement that the functions* $i^*A_I^J$ *may be definite functions of* $x^i, y^j, y_i^j, f, f^j, f_i^j, \ldots$, *then the only possible equivariant phase structures are given by the above forms* $\Phi^0 + \Phi^1 + c^2\Phi^2 + \ldots + c^n\Phi^n$. We delay the proof to another place.

(Part III Examples will follow.)

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