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NATURAL TRANSFORMATIONS OF THE SECOND TANGENT FUNCTOR INTO ITSELF

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It is well known that there exists a canonical involutive automorphism i_M: $TTM \to TTM$ on the second tangent bundle $TTM = T(TM)$ of any smooth manifold M. From the categorical point of view, i is a natural transformation of the second tangent functor into itself, [3], [5]. In the present paper, we determine all smooth natural transformations of the second tangent functor into itself. We remark that our approach is of general character, so that it can be applied to other problems of similar types.

1. Smooth natural transformations of r-th order prolongation functors. Let $\mathcal{M}$ be the category of smooth manifolds and maps and $\mathcal{FM}$ the category of smooth fibred manifolds. A functor F: $\mathcal{M} \to \mathcal{FM}$ transforming any manifold M into a fibred manifold FM over M and any smooth map f: $M \to N$ into a smooth fibred manifold morphism Ff: $FM \to FN$ over f is said to be a prolongation functor, if it satisfies two simple additional conditions of locality and regularity, [2]. Given another prolongation functor G: $\mathcal{M} \to \mathcal{FM}$, a natural transformation i: $F \to G$ is called smooth, if i_M: $FM \to GM$ is a smooth morphism over id_M for any manifold M. A prolongation functor F is said to be of the order r, if the restrictions of Ff and Fg to the fibre of FM over x coincide for any two smooth maps f, g: $M \to N$ having r-th order contact at $x \in M$, i.e. $j^r_x f = j^r_x g$ implies $Ff \mid F_x M = Fg \mid F_x M$ In this case, any r-jet $A \in J^r_x(M, N)_y$ determines a map FA: $F_x M \to F_y N$.

We define a category L^r over $\mathbf{N}$ by $L^r(m, n) = J^r_0(\mathbf{R}^m, \mathbf{R}^n)_0$ and the composition in L^r is the composition of r-jets. By a smooth action φ of L^r on a sequence (S_k), $k \in \mathbf{N}$ of smooth manifolds we mean a double sequence of smooth maps $\varphi_{m,n}$: $L^r(m, n) \times S_m \to S_n$ satisfying $\varphi_{m,p}(B \circ A, x) = \varphi_{n,p}(B, \varphi_{m,n}(A, x))$ and $\varphi_{m,m}(j^r_0 id, x) = x$ for all $x \in S_m$, $A \in L^r(m, n)$, $B \in L^r(n, p)$, provided $B \circ A$ denotes the composition of jets. Given an r-th order prolongation functor F, we set $S_k(F) = F_0\mathbf{R}^k$ and the rule $(A, x) \mapsto (FA)(x)$, $x \in S_m(F)$, $A \in L^r(m, n)$ determines a smooth action $\varphi(F)$ of L^r on $(S_k(F))$. Conversely, given a smooth action φ of L^r on a sequence (S_k) of manifolds, there exists exactly one prolongation functor $F(\varphi)$: $\mathcal{M} \to \mathcal{F}\mathcal{M}$ such that $S_k = S_k(F(\varphi))$ and $\varphi = \varphi(F(\varphi))$, see [1].

Given another smooth action ψ of L^r on another sequence (W_k) of manifolds, we define an equivariant map λ: $\varphi \to \psi$ as a sequence λ_k: $S_k \to W_k$ of smooth maps satisfying $\lambda_n(\varphi_{m,n}(A, x)) = \psi_{m,n}(A, \lambda_m(x))$ for all $x \in S_m$ and $A \in L^r(m, n)$. If we have a smooth natural transformation i: $F \to G$ of two r-th order prolongation functors, then $\lambda_k(i) := i_{\mathbf{R}^k} \mid F_0\mathbf{R}^k$, $k \in \mathbf{N}$ determine an equivariant map $\lambda(i)$: $\varphi(F) \to \varphi(G)$. Conversely, Janyška has proved, [1], that for any equivariant map λ: $\varphi \to \psi$ there exists exactly one smooth natural transformation i: $F(\varphi) \to F(\psi)$ such that $\lambda_k = \lambda_k(i)$. (We remark that analogous results for a lifting functor, i.e. for a prolongation functor defined on the category $\mathcal{M}_n$ of n-dimensional manifolds and their embeddings, are deduced in [4].)

2. Differential equations for the natural transformations of TT into itself. The second tangent functor TT is a second order prolongation functor. Let a^p_i, a^p_{ij} or b^i, c^i, d^i be the canonical coordinates on $L^2(m, n)$ or $TT_0\mathbf{R}^m = \mathbf{R}^{3m}$, respectively, $i, j, \ldots = 1, \ldots, m$; $p = 1, \ldots, n$. Then the action $\varphi(TT)$ of L^2 on $(S_k(TT))$ is given by

$$\bar{b}^p = a^p_i b^i, \; \bar{c}^p = a^p_i c^i, \; \bar{d}^p = a^p_{ij} b^i c^j + a^p_i d^i. \tag{1}$$

Assume that the coordinate expression of the map λ_m: $\mathbf{R}^{3m} \to \overline{\mathbf{R}}^{3m}$ induced by a smooth natural transformation of TT into itself is

$$\bar{b}^i = f^i(b, c, d), \; \bar{c}^i = g^i(b, c, d), \; \bar{d}^i = h^i(b, c, d). \tag{2}$$

Consider the group $L^2_m \subset L^2(m, m)$ of all invertible 2-jets. Then (2) must be an L^2_m-equivariant map. Thus, the fundamental vector fields induced on $\mathbf{R}^{3m}$ and $\overline{\mathbf{R}}^{3m}$ by any vector of the Lie algebra of L^2_m must be λ_m-related. This is equivalent to the following system of partial differential equations

$$\left\{ \begin{aligned} &\frac{\partial f^i}{\partial b^k} A^k_j b^j + \frac{\partial f^i}{\partial c^k} A^k_j c^j + \frac{\partial f^i}{\partial d^k} A^k_j d^j = A^i_j f^j, \\ &\frac{\partial g^i}{\partial b^k} A^k_j b^j + \frac{\partial g^i}{\partial c^k} A^k_j c^j + \frac{\partial g^i}{\partial d^k} A^k_j d^j = A^i_j g^j, \\ &\frac{\partial h^i}{\partial b^k} A^k_j b^j + \frac{\partial h^i}{\partial c^k} A^k_j c^j + \frac{\partial h^i}{\partial d^k} A^k_j d^j = A^i_j h^j, \end{aligned} \right. \tag{3}$$

$$0 = \frac{\partial f^i}{\partial d^j} A^j_{kl} b^k c^l, \qquad 0 = \frac{\partial g^i}{\partial d^j} A^j_{kl} b^k c^l, \tag{4}$$

$$\frac{\partial h^i}{\partial d^j} A^j_{kl} b^k c^l = A^i_{jk} f^j g^k, \tag{5}$$

where A^i_j, A^i_{jk} are the canonical coordinates on the Lie algebra of L^2_m.

If we set $A^i_j = \delta^i_j$ in (3), we find that f^i, g^i and h^i are smooth homogeneous functions of degree one defined on the whole $\mathbf{R}^{3m}$, which implies that f^i, g^i and h^i are linear functions. If we now substitute any linear functions into (3), we see that all matrices of the coefficients by individual series of variables commute with A^i_j, so that these matrices are of the form $k\delta^i_j$, $k \in \mathbf{R}$. Hence (3) implies

$$f^i = a_1 b^i + a_2 c^i + a_3 d^i,$$
$$g^i = a_4 b^i + a_5 c^i + a_6 d^i, \tag{6}$$
$$h^i = a_7 b^i + a_8 c^i + a_9 d^i.$$

Then (4) gives $a_3 = 0 = a_6$ and (5) has the following form

$$a_9 A^i_{jk} b^j c^k = A^i_{jk}(a_1 b^j + a_2 c^j)(a_4 b^k + a_5 c^k). \tag{7}$$

This is equivalent to

$$a_1 a_4 = 0, a_2 a_5 = 0, a_9 = a_1 a_5 + a_2 a_4. \tag{8}$$

Obviously, (8) can be fulfilled in four ways: I. $a_2 = 0 = a_4$, $a_9 = a_1 a_5$; II. $a_1 = 0 = a_5$, $a_9 = a_2 a_4$; III. $a_1 = 0 = a_2$, $a_9 = 0$; IV. $a_4 = 0 = a_5$, $a_9 = 0$. Hence the system (3)–(5) has the following four families of solutions

$$\bar{b}^i = k_1 b^i, \quad \bar{c}^i = k_2 c^i, \quad \bar{d}^i = k_3 b^i + k_4 c^i + k_1 k_2 d^i, \tag{9}$$
$$\bar{b}^i = k_1 c^i, \quad \bar{c}^i = k_2 b^i, \quad \bar{d}^i = k_3 b^i + k_4 c^i + k_1 k_2 d^i, \tag{10}$$
$$\bar{b}^i = 0, \quad \bar{c}^i = k_1 b^i + k_2 c^i, \quad \bar{d}^i = k_3 b^i + k_4 c^i, \tag{11}$$
$$\bar{b}^i = k_1 b^i + k_2 c^i, \quad \bar{c}^i = 0, \quad \bar{d}^i = k_3 b^i + k_4 c^i \tag{12}$$

with arbitrary $k_1, k_2, k_3, k_4 \in \mathbf{R}$.

We now find easily by direct evaluation that any of expressions (9)–(12) determines an equivariant map of action (1) into itself. According to [1], we have deduced all smooth natural transformations of the second tangent functor into itself. (The reader can verify directly that if one defines, for any manifold M, a map i_M: $TTM \to TTM$ by any of the coordinate expressions (9)–(12) with constant k_1, k_2, k_3, k_4, one obtains a natural transformation of TT into itself.)

3. An interpretation of the analytic results. Let us start from a simplier problem of finding all smooth natural transformations of the tangent functor T into itself. Using a similar analytic procedure as in § 2, we find the general solution of the corresponding system of partial differential equations in the form $\bar{b}^i = kb^i$, $k \in \mathbf{R}$. Hence all smooth natural transformations of T into itself are homoteties with the same constant coefficient on every tangent bundle TM. A geometrical interpretation of such homoteties is that we transform any vector $A = \frac{d}{dt}\Big|_0 \gamma(t)$ tangent to a curve $\gamma(t)$ into $\frac{d}{dt}\Big|_0 \gamma(kt)$, i.e. we reparametrize all curves in the same way.

Any $A \in TTM$ can be deduced from a map $\gamma: \mathbf{R}^2 \to M$ by successive differentiations $A = \frac{\partial}{\partial \tau}\Big|_0 \frac{\partial}{\partial t}\Big|_0 \gamma(t, \tau)$. Even in this case, any natural transformation (9)–(12) can be determined by a suitable change of parameters t and τ. Indeed, the case (9) corresponds to a reparametrization

$$\gamma(k_1 t + k_3 t\tau, k_2 \tau + k_4 t\tau). \tag{13}$$

For (10), we find

$$\gamma(k_2 \tau + k_3 t\tau, k_1 t + k_4 t\tau). \tag{14}$$

Setting $k_1 = 1 = k_2$, $k_3 = 0 = k_4$, we obtain the classical canonical involution. The case (11) corresponds to

$$\gamma(k_1 \tau + k_3 t\tau, k_2 \tau + k_4 t\tau) \tag{15}$$

and for (12) we have a similar reparametrization

$$\gamma(k_1 t + k_3 t\tau, k_2 t + k_4 t\tau). \tag{16}$$

We remark that natural equivalences arise only in (9) and (10) with $k_1 k_2 \neq 0$.

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