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DECAYING TRAJECTORIES IN SUBLINEAR RETARDED EQUATIONS OF ARBITRARY ORDER

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Abstract. Sufficient conditions have been found to ensure that all solutions of the retarded sublinear equation

$$p_n(t)(p_{n-1}(t)(\dots p_1(t)(p_0y(t))')'\dots)')' + a(t)h(y(g(t))) = f(t),$$

where $h(t)$ satisfies sublinearity requirement approach limits as $t \to \infty$.

This phenomenon is later linked to complete nonoscillation.

Key words. Ordinary differential equation, retarded argument, oscillatory criteria.
AMS MOS Subject Clasification. Primary 34 K 15, 34 C 10.

1. Introduction

Recently Singh [7] obtained sufficient conditions which ensure that *all* solutions of the equation

$$(1)\qquad (r(t)y'(t))' + a(t)y^{\gamma}(g(t)) = f(t), \qquad 0 < \gamma \leqq 1,$$

(where γ is the ratio of odd integers) tend to finite or infinite limits as $t \to \infty$. Our main purpose in this work is to extend this study to a more general setting represented by the equation

$$(2)\qquad L_ny(t) + a(t)h(y(g(t))) = f(t),$$

where $n \geqq 2$ and L_n is a disconjugate differential operator defined by

$$(3)\qquad L_ny(t) = p_n(t)(p_{n-1}(t)(\dots(p_1(t)(p_0(t)y(t))')'\dots)')'.$$

Following our assumptions and notations in [9], we assume

(i) $p_i \in C([\alpha, \infty), (0, \infty))$, $0 \leqq i \leqq n$,

$$(4)\qquad \int_a^{\infty} p_i^{-1}(t)\,dt = \infty, \qquad 1 \leqq i \leqq n-1;$$

(ii) $a, f, g \in C([\alpha, \infty), R)$, there exists a $t_0 > \alpha$ such that

$$0 < g(t) \leqq t \quad \text{for } t \geqq t_0, \quad \text{and} \quad g(t) \to \infty \text{ as } t \to \infty;$$

(iii) $h \in C(R, R)$, h is nondecreasing, and sign $h(y) =$ sign y. There exists $\gamma \in (0,1]$ which is ratio of odd integers, and a number $Q > 0$ such that

(5) $$0 < h(t)/t^\gamma \leqq Q.$$

Let

(6) $$L_0 y(t) = p_0(t)\, y(t), L_i y(t) = p_i(t)\,(L_{i-1} y(t))', \qquad 1 \leqq i \leqq n.$$

Th domain of L_n is definded to be the set of all functions $y:[T_y, \infty) \to R$ such that $L_i y(t)$, $0 \leqq i \leqq n$, exist and are continuous on $[T_y, \infty)$. In what follows by a „solution" of equation (2) (or any other related equation) we mean a function $y(t)$ which is nontrivial in any neighborhood of infinity and satisfies equation (2) for all sufficiently large t. A solution of (2) is called oscillatory if it has arbitrarily large zeros; otherwise the solution is called nonoscillatory.

The main thrust of contemporary work in oscillation theory associated with functional equations has been to find oscillatory or nonoscillatory criteria. For this we refer the reader to Onose [3], Singh [8], Philos [4] and Staikos and Philos [10]. Lately, however, the asymptotic nature of oscillatory and nonoscillatory solutions is being studied quite extensively due to its growing importance in the stability of time dependent physical systems. A survey of some of these applications is found in a recently published Russian text by Shevelov [11].

Our work here is related to that of Kusano and Onose [2], Staikos and Philos [10], Singh and Kusano [9] and Singh [5, 6, 7], but our results here are different and more complete.

We say that a solution of equation (1) is completely nonoscillatory if $L_i y(t)$ is nonoscillatory for $i = 1, 2, \ldots, n-1$.

In section (3) of this work we find a condition for complete nonoscillation of equation (2), which is then linked to this equation having all solutions approaching limits as $t \to \infty$.

2. Main results

Let $i_k \in \{1, 2, \ldots, n-1\}$, $1 \leqq k \leqq n-1$, and $t, s \in [\alpha, \infty)$. We define $I_0 \equiv 1$

(7) $$I_k(t, s; p_{i_k}, \ldots, p_{i_1}) = \int_s^t p_{i_k}^{-1}(r)\, I_{k-1}(r, s; p_{i_k}, \ldots, p_{i_2})\, dr.$$

It is easily verified that for $1 \leqq k \leqq n-1$

(8) $$I_k(t, s; p_{i_k}, \ldots, p_{i_1}) = \int_s^t p_{i_1}^{-1}(r)\, I_{k-1}(t, r; p_{i_k}, \ldots, p_{i_2})\, dr.$$

Define

(9) $$J_i(t, s) = p_0^{-1} I_i(t, s; p_1, \ldots, p_i), \qquad J_i(t) = J_i(t, \alpha)$$

for $0 \leqq i \leqq n-1$. The following Lemma is an adaptation of our Lemma 1 in [9].

Lemma 1. *In addition to* (i)–(iii) *suppose*

$$\int^{\infty} p_n^{-1}(t)\,|f(t)|\,dt < \infty \tag{10}$$

and

$$\int^{\infty} [J_{n-1}(g(t))]^{\gamma}\, p_n^{-1}(t)\,|a(t)|\,dt < \infty. \tag{11}$$

Then every solution $y(t)$ *of equation* (2) *satisfies*

$$y(t) = 0(J_{n-1}(t)) \quad \text{as} \quad t \to \infty. \tag{12}$$

Proof. Let $y(t)$ be a solution of (2) defined on $[t_0, \infty)$, $t_0 \geqq \alpha$. There exists $T > t_0$ such that $g(t) \geqq t_0$ for $t \geqq T$. n-times integration of (2) yields

$$p_0 y(t) = \sum_{i=0}^{n-1} C_i I_i(t, T; p_1, p_2, \ldots, p_i) + \tag{13}$$

$$+ \int_T^t I_{n-1}(t, r; p_1, \ldots, p_{n-1})\, p_n^{-1}(r)\,[f(r) - a(r)\,h(y(g(r)))]\,dr,$$

where C_i, $0 \leqq i \leqq n - 1$, are constants. Since (4) implies that

$$\lim_{t\to\infty} [I_i(t, T; p_1, \ldots, p_i)]/I_{n-1}(t, T; p_i, \ldots, p_{n-1}) = 0,$$

$0 \leqq i \leqq n - 2$; using (8) in (13) we have

$$|y(g(t))| \leqq CJ_{n-1}(t) + J_{n-1}(t) \int_T^t p_n^{-1}(r)\,[|f(r)| + |a(r)|\,Q\,|y(g(r)|^{\gamma}]\,dr$$

for some constant $C > 0$ and $t \geqq T$. Since $g(t) \leqq t$ and (10) holds, we have

$$|y(g(t))|/J_{n-1}(g(t)) \leqq K + \int_T^t [J_{n-1}(g(r))]^{\gamma}\, p_n^{-1}\,|a(r)|\,[|y(g(r))|/J_{n-1}) g(r))]^{\gamma}\,dr \tag{14}$$

for $t \geqq T'$, where $K > 0$ is a constant and $T' > T$ is chosen so large that $g(t) \geqq T$ for $t \geqq T'$. Using Bihari's lemma [1], we see that

$$|y(g(t))|/J_{n-1}(g(t)) \leqq Q_0 \tag{15}$$

for some constant $Q_0 > 0$, and the proof is complete.

Theorem 1. *In addition to* (i)–(iii) *suppose*

$$1/p_0(t) \int_t^{\infty} 1/p_1(x) \int_{x_1}^{\infty} 1/p_2(x_2) \ldots 1/p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} p_n^{-1}(x)\,|a(x)|\,[J_{n-1}(x)]^{\gamma} \times \tag{16}$$

$$\times\, dx\, dx_{n-1}\, dx_{n-2} \ldots dx_1 < \infty$$

and

$$1/p_0(t) \int_t^{\infty} 1/p_1(x_1) \int_{x_1}^{\infty} 1/p_2(x_2), \ldots, 1/p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} p_n^{-1}\,|f(x)|\,dx\,dx_{n-1} \times \tag{17}$$

$$\times \ldots, dx_1 < \infty,$$

Then all solutions of equation (2) *approach limits, finite or infinite as* $t \to \infty$.

Proof. Let $y(t)$ be a solution of equation (2). Suppose to the contrary that

(18) $$\liminf_{t\to\infty} |y(t)| < \limsup_{t\to\infty} |y(t)|.$$

Then $L_i y$, $1 \leqq i \leqq n$, is oscillatory. Let $T_0 > t_0$ be large enough so that $g(t) \geqq t$ for $t \geqq T$. Since (18) holds, there exist positive numbers λ and δ, $\delta < \lambda$, and

(19) $$\liminf_{t\to\infty} |y(t)| < \delta,$$

(20) $$\limsup_{t\to\infty} |y(t)| > \lambda.$$

From (16), (17) and (4), it follows that conditions of Lemma 1 are satisfied. There exist two numbers $T > T_0$ and $\xi > 0$ such that

(21) $$|y^\gamma(g(t))| \leqq \xi\,[J_{n-1}(g(t))]^\gamma$$

for $t \geqq T$ in accordance with Lemma 1. Choose $T_1 > T$ large enough so that

(22) $$1/p_0(t)\int_t^\infty 1/p_1(x_1)\int_{x_1}^\infty 1/p_2(x_2),\ldots,1/p_{n-1}(x_{n-1})\int_{x_{n-1}}^\infty |a(x)|\,p_n^{-1}(x)\times \times[J_{n-1}(x)]^\gamma\,dx\,dx_{n-1}\,dx_{n-2},\ldots,dx_1 < (\lambda-\delta)/4$$

and

(23) $$1/p_0(t)\int_t^\infty 1/p_1(x_1)\int_{x_1}^\infty 1/p_2(x_2),\ldots,1/p_{n-1}(x_{n-1})\int_{x_{n-1}}^\infty p_n^{-1}(x)\,|f(x)|\times \times dx, dx_{n-1}, dx_{n-2},\ldots,dx_1 < (\lambda-\delta)/4$$

for $t \geqq T_1$. Since $y'(t)$ is oscillatory, there exist arbitrarily large points A_0 andB_0 such that $|y(A_0)| < \delta$ and $|y(B_0)| > \lambda$. Let $B > T_1$ be such that $|y(B)| > \lambda$. Let $[T_2, T_4]$ be the smallest closed interval containing B such that $|y(T_2)| = |y(T_3)| = \delta$, $T_2 > T_1$, and $\mathrm{Max}\,|y(t)| > \lambda$ for $t \in [T_2, T_4]$. Let $M = \mathrm{Max}\,|y(t)| = |y(T_3)|$, $T_3 \in (T_2, T_4)$. It follows that $M > \lambda$. Let

(24) $$e_1 < e_2 < e_3 < \ldots < e_{n-2},$$

where $T_4 < e_1$, be zeros of $(p_0(t)\,y(t))$, $p_1(t)\,(p_0(t)\,y(t))',\ldots,(p_{n-2}(t),\ldots,(p_1(t)\times(p_0 y(t)')',\ldots)')'$ respectively. On repeated integration of equation (1) we have

(25) $$(p_0 y(t))'\int_t^{e_1} 1/p_1(x_1)\int_{x_1}^{e_2} 1/p_2\,x_2),\ldots,1/p_{n-1}(x_{x-2})\times \times\int_{x_{n-2}}^{e_{n-2}} p_n^{-1}(x)\,a(x)\,h(y(g(x)))\,dx\,dx_{n-2}\,dx_{n-3},\ldots,dx_1 + + \int_t^{e_1} 1/p_1(x_1)\int_{x_1}^{e_2} 1/p_2(x_2),\ldots,1/p_{n-1}(x_{n-2})\int_{x_{n-2}}^{e_{n-2}} f(x)\,p_n^{-1}(x)\,dx\,dx_{n-2}\,dx_{n-3},\ldots,dx_1.$$

Integrating (25) between $[T_2, T_3]$ we have

$$(26)\qquad \pm(M-\delta) = -1/p_0(x_0)\int_{T_2}^{T_3}\int_{x_0}^{e_1} 1/p_1(x_1)\int_{x_1}^{e_2}\dots, 1/p_{n-1}(x_{n-2})\int_{x_{n-2}}^{e_{n-2}}\times$$
$$\times p_n^{-1}(x)\, a(x)\, h(y(g(x)))\, dx\, dx_{n-2}, \dots, dx_0 +$$
$$+ 1/p_0(x_0)\int_{T_2}^{T_3}\int_{x_0}^{e_1} 1/p_1(x_1)\int_{x_1}^{e_2}\dots, 1/p_{n-1}(x_{n-2})\int_{x}^{e_{n-2}} p_n^{-1}(x) f(x)\, dx\, dx_{n-2}, \dots, dx_0;$$

since $|y(T_1)| = \delta$ and $|y(T_2)| = M$. In view of (24) (see Singh [6]) and (21); (26) yields

$$(27)\qquad \lambda-\delta \leqq Q\xi\, 1/p_0(x_0)\int_{t}^{e_{n-2}}\int_{x_0}^{e_{n-2}} 1/p_1(x_1)\int_{x_1}^{e_{n-2}}\dots, 1/p_{n-1}(x_{n-2})\times$$
$$\times\int_{x_{n-2}}^{e_{n-2}} |a(x)|\, p_n^{-1}(x)\, [J_{n-1}(g(x))]^{\gamma}\, dx\, dx_{n-2}, \dots, dx_0 +$$
$$+ 1/p_0(x_0)\int_{t}^{e_{n-2}}\int_{x_0}^{e_{n-2}} 1/p_1(x_1), \dots, 1/p_{n-1}(x_{n-1})\int_{x_{n-2}}^{e_{n-2}}\times$$
$$\times |f(x)|\, p_n^{-1}(x),\, dx\, dx_{n-2}, \dots, dx_0$$

since $h(t)/t^{\gamma} \leqq Q$ and $M > \lambda$. From (22), (23) and (27) we get

$$(28)\qquad \lambda-\delta \leqq (\lambda-\delta)/4 + (\lambda-\delta)/4,$$

a contradiction. The proof is complete.

Corollary 1. *If* $p_0 \equiv p_1 \equiv \dots \equiv p_{n-1}$ *then* (16) *and* (17) *reduce to*

$$(29)\qquad \int^{\infty} t^{n-1}(g)\, t))^{\gamma(n-1)}\, |a(x)|\, dx < \infty$$

and

$$(30)\qquad \int^{\infty} t^{n-1}\, |f(x)|\, dx < \infty$$

respectively. Hence subject to (29) *and* (30), *all solutions of the equation*

$$(31)\qquad y^{(n)}(t) + a(t)\, h(y(g(t))) = f(t)$$

approach limits, finite or infinite, as $t \to \infty$.

Corollary 2. *All oscillatory solutions of equation* (2) *tend to zero as* $t \to \infty$.

Remark 1. In regard to the bounded solutions of equation (2), Conditions (16) and (17) can be relaxed. Lemma 1 is not needed.

We have the following theorem:

Theorem 2. *Suppose* (i)–(iii) *hold. Further suppose that*

$$(32)\qquad 1/p_0(t)\int^{\infty} 1/p_1(x_1), \dots, 1/p_{n-1}(x_{n-1})\int_{x_{n-1}}^{\infty} p_n^{-1}(x)\, |a(x)|\times$$
$$\times dx\, dx_{n-1}\, dx_{n-2}, \dots, dx < \infty$$

and

$$1/p_0(t) \int_t^\infty 1/p_1(x_1), \ldots, 1/p_{n-1}(x_{n-1}) \int_{x_{n-1}}^\infty p_n^{-1}(x) \, |f(x)| \times \times \mathrm{d}x \, \mathrm{d}x_{n-1}, \ldots, \mathrm{d}x_1 < \infty. \tag{33}$$

Then all bounded solutions of equation (2) *approach finite limits as* $t \to \infty$.

Proof. Let $y(t)$ be a bounded solution of equation (2). Suppose to the contrary that $\liminf_{t\to\infty} |y(t)| < \limsup_{t\to\infty} |y(t)|$. Then $L_i y$, $1 \leqq i \leqq n$ is oscillatory. Let $T_0 > t_0$ be large enough so that $g(t) \geqq t_0$ for $t \geqq T_0$. There exist positive numbers λ and δ such that $\delta < \lambda$, and (19) and (20) of the proof of Theorem 1 hold. There exists a number $T > T_0$ such that, for $t \geqq T$, (32) and (33) imply

$$M_0^\gamma Q \, 1/p_0(t) \int_t^\infty 1/p_1(x_1), \ldots, 1/p_{n-1}(x_{n-1}) \int_{x_{n-1}}^\infty p_n^{-1}(x) \, |a(x)| \times \times \mathrm{d}x \, \mathrm{d}x_{n-1}, \ldots, \mathrm{d}x_1 < (\lambda - \delta)/4 \tag{34}$$

and

$$1/p_0(t) \int^\infty 1/p_1(x_1) \int_{x_1}^\infty 1/p_2(x_2), \ldots, 1/p_{n-1}(x_{n-1}) \int_{x_{n-1}}^\infty p_n^{-1}(x) \times \times |f(x)| \, \mathrm{d}x \, \mathrm{d}x_{n-1}, \ldots, \mathrm{d}x_1 < (\lambda - \delta)/4, \tag{35}$$

where

$$M_0 = \sup \{ |y(t)| : t \geqq T \}. \tag{36}$$

(35) and (36) replace (22) and (23) respectively in the proof of Theorem 1. From this point on, the proof is the same as that of Theorem 1 from inequality (23) onward. Very minor modifications need be made. The proof is not complete.

Example 1. Consider the equation

$$x^{(\mathrm{iv})}(t) + \frac{1}{t^6} x(e^{-\pi} t) = e^{-\pi}/t^5 \tag{37}$$

for $t \geqq 1$. Condition (29) of Corollary 1 is not satisfied. However, conditions (32) and (33) of Theorem 2 are satisfied. We notice that (37) has an unbounded solution $x(t) = t$. Thus Theorem 1 does not apply to (37); but according to Theorem 2 all bounded solutions of (37) must approach finite limits.

3. Complete nonoscillation

Theorem 3. *Suppose Conditions of Theorem 1 hold. Further suppose that* $a(t) > > 0$ *and*

$$\lim_{t\to\infty} \frac{f(t)}{a(t)} = 0. \tag{38}$$

Let $y(t)$ be a solution of equation (2). *Then either $y(t) \to 0$ as $t \to \infty$ or $y(t)$ is completely nonoscillatory and approaches $+\infty$ as $t \to \infty$.*

Proof. Suppose $\lim_{t\to\infty} |y(t)| = \beta > 0$. Without any loss of generality suppose that $y(t)$ is eventually positive. Dividing equation (2) by $a(t)$ we observe that there exists a large $T > t_0$ and a constant $\lambda > 0$ such that

$$(L_n y(t))/a(t) < -\lambda \tag{39}$$

for $t \geqq T$. Since $\int^{\infty} 1/p_i(t)\,\mathrm{d}t = \infty$ $1 \leqq i \leqq n-1$, conclusion follows. The proof is complete.

Corollary 3. *Suppose conditions of Theorem 3 hold. If $L_i y(t)$, $1 \leqq i \leqq n$, is oscillatory then $y(t) \to 0$ as $t \to \infty$, where $y(t)$ is a solution of equation* (2).

Proof. Suppose to the contrary that $y(t) \to \beta \neq 0$ as $t \to \infty$. Without any loss of generality suppose that $\beta > 0$. There exists a large T such that in a manner of last theorem we have a $\lambda > 0$ such that

$$(L_n y(t))/a(t) < -\lambda \tag{40}$$

for $t \geqq T$. Suppose $L_i y(t)$ is oscillatory for some i where $1 \leqq i \leqq n$. Then (40) is contradicted, and the proof is complete.

Example 2. Consider the equation

$$y'''(t) + e^{-t}y(t) = -8e^{-2t} + e^{-3t}. \tag{41}$$

All conditions of Theorem 3 are satisfied. This equation has $y(t) = e^{-2t}$ as a solution satisfying the conclusion of Theorem 3.

Theorem 4. *Suppose conditions of Theorem 1 hold. Further suppose that $a(t) > 0$ and*

$$\liminf_{t\to\infty} \frac{|f(t)|}{a(t)} > 0. \tag{42}$$

Then all solutions of equation (2) *are nonoscillatory.*

Proof. Suppose to the contrary that a solution $y(t)$ of equation (1) is oscillatory. By Theorem 1 $y(t) \to 0$ as $t \to \infty$. Dividing equation (1) by $y(t)$ we observe in view of (42) that $L_n y(t)/a(t)$ eventually assumes a constant sign. But then $p_0(t)\,y(t)$ i.e $y(t)$ is nonoscillatory, a contradiction. The proof is complete.

Remark 2. Our next theorem links complete nonoscillation with eventually vanishing trajectories of equation (2).

Theorem 5. *Suppose $a(t) > 0$, (i)–(iii) hold and n is odd. Further suppose that $\liminf_{t\to\infty} h(t)/t^{\gamma} > 0$, and equation* (2) *has a completely nonoscillatory bounded solution $y(t)$ satisfying* $\operatorname{sign} y(t) = \operatorname{sign} y'(t)$ *and*

(43) $$(-1)^{i-1} L_i y(t) \geqq 0, \qquad i = 1, 2, \ldots, n-1.$$

Let $f(t)/a(t)$ be bounded, $g'(t) \geqq 0$ for $t \geqq t_0$ and

(44) $$1/p_0(t) \int_t^\infty 1/p_1(x_1) \int_{x_1}^\infty \ldots, 1/p_{n-1}(x_{n-1}) \int_{x_{n-1}}^\infty (|f(x)|\, p_n^{-1}(x))/y^\gamma(g(x))) \times \times \mathrm{d}x, \ldots, \mathrm{d}x_1 < \infty.$$

Then all solutions of equation (2) *approach limits finite or infinite, as $t \to \infty$.*

Proof. Without any loss of generality, let $T > t_0$ be large enough so that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geqq T$. (4) and (43) imply that

(45) $$\lim_{t\to\infty} L_i y(t) = 0, \qquad i = 1, 2, \ldots, n-1.$$

We rewrite equation (1) as

(46) $$(L_n y(t))\, p_n^{-1}(t) + a(t)\, p_n^{-1}(t)\, h(y(g(t))) = f(t)\, p_n^{-1}(t).$$

Integrating (46) between $[t, \infty)$ where $t > T$ we have

(47) $$\frac{L_{n-1}y(t)}{y^\gamma(g(t))} + K_0 \int_t^\infty a(x)\, p_n^{-1}(x)\, \mathrm{d}x + + \int_t^\infty \frac{\gamma L_{n-1}y(x)\, y'(g(x))\, g'(x)\, \mathrm{d}x}{y^{\gamma+1}(g(x))} \leqq \int_t^\infty \frac{|f(x)|\, p_n^{-1}(x)\, \mathrm{d}x}{y^\gamma(g(x))},$$

where $\liminf_{t\to\infty} \dfrac{h(r)}{r^\gamma} > K_0 > 0$.

Since third term on the left is nonnegative we have

(48) $$-\frac{L_{n-1}y(t)}{y^\gamma(g(t))} + K_0 \int_t^\infty a(x)\, p_n^{-1}(x)\, \mathrm{d}x \leqq \int_t^\infty \frac{|f(x)|\, p_n^{-1}(x)\, \mathrm{d}x}{y^\gamma(g(x))}.$$

Dividing (48) by $p_{n-1}(t)$ and integrating again we get

$$\frac{L_{n-2}y(t)}{y^\gamma(g(t))} + K_0 \int_t^\infty 1/p_{n-1}(x_1) \int_{x_1}^\infty a(x)\, p_n^{-1}(x)\, \mathrm{d}x\, \mathrm{d}x_1 - - \int_t^\infty \frac{\gamma L_{n-2}y(x)\, y'(g(x))\, g'(x)\, \mathrm{d}x}{y^{\gamma+1}(g(x))} \leqq \int_t^\infty 1/p_{n-1}(x_1) \int_{x_1}^\infty \frac{|f(x)|\, p_n^{-1}(x)\, \mathrm{d}x\, \mathrm{d}x_1}{y^\gamma(g(x))},$$

which in view of (43) yields

(49) $$\frac{L_{n-2}y(t)}{y^\gamma(g(t))} + K_0 \int_t^\infty 1/p_{n-1}(x_1) \int_{x_1}^\infty a(x)\, p_n^{-1}(x)\, \mathrm{d}x\, \mathrm{d}x_1 \leqq \leqq \int_t^\infty 1/p_{n-1}(x_1) \int_{x_1}^\infty \frac{|f(x)|\, p_n^{-1}(x)\, \mathrm{d}x\, \mathrm{d}x_1}{y^\gamma(g(x))}$$

pursuing this course we finally get

$$(50) \qquad -\frac{y(t)}{y^{\gamma}(g(t))} + K_0\, 1/p_0(t) \int_t^{\infty} 1/p_1(x_1) \int_{x_1}^{\infty} \ldots, 1/p_{n-1}(x_{n-1}) \times$$
$$\times \int_{x_{n-1}}^{\infty} a(x)\, p_n^{-1}(x)\, dx\, dx_{n-1}, \ldots, dx_1 \leqq$$
$$\leqq 1/p_0(t) \int_t^{\infty} 1/p_1(x_1) \int_{x_1}^{\infty} \ldots, 1/p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} |f(x)|\, p_n^{-1}(x) \times$$
$$\times\, y^{-\gamma}(g(x))\, dx\, dx_{n-1}, \ldots, dx_1 .$$

Now

$$(51) \qquad \frac{y(t)}{y^{\gamma}(g(t))} \geqq \frac{y(g(t))}{y^{\gamma}(g(t))} = y^{1-\gamma}(g(t)) < \infty .$$

Since $\operatorname{sign} y(t) = \operatorname{sign} y'(t)$, $\gamma < 1$ and $y(t)$ is bounded from (50), (51) and (44) we get

$$(52) \qquad 1/p_0(t) \int_t^{\infty} 1/p_1(x_1) \int_t^{\infty} \ldots, 1/p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} a(x)\, p_n^{-1}(x)\, dx\, dx_{n-1}, \ldots, dx_1 < \infty$$

and since $f(t)/a(t)$ is bounded for $t \geqq T$, we also have

$$(53) \qquad 1/p_0(t) \int_t^{\infty} 1/p_1(x_1) \int_{x_1}^{\infty} \ldots, 1/p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} |f(x)|\, p_n^{-1}(x)\, dx\, dx_{n-1}, \ldots, dx < \infty .$$

(52) and (53) are the conditions of Theorem 1. The proof is complete.

Example 3. Consider the equation

$$(54) \qquad y^{(iv)}(t) + e^{-t-\pi} y(t-\pi) = e^{-t-\pi} - e^{-2t} - e^{-t}$$

for $t \geqq \pi$. This equation has $y(t) = 1 - e^{-t}$ as a bounded completely nonoscillatory solution satisfying all the conditions of the theorem. Hence all solutions of (54) approach limits as $t \to \infty$.

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