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ON SOME INTEGRODIFFERENTIAL INEQUALITIES IN ONE VARIABLE DEFINED ON AN INFINITE INTERVAL*

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Abstract. This article deals with some new linear and nonlinear integrodifferential inequalities in one variable defined on an infinite interval. The upper bounds of the unknown function and its higher derivatives will be explicitly obtained.

Key words. Ordinary differential equations – Integrodifferential inequalities.

INTRODUCTION

Gronwall in [3] studied his famous integral inequality

$$x(t) \leqq h(t) + \int_{t_0}^{t} k(s)\, x(s)\, \mathrm{d}s; \qquad t_0 \leqq t < T \leqq \infty, \tag{1}$$

where $x(t)$, $h(t)$ and $k(t)$ are real-valued, non-negative and continuous functions defined on $[t_0, T)$. He obtained an explicit upper bound for $x(t)$.

Various generalizations of this inequality in the form of integrodifferential inequalities have been thoroughly investigated, see [1, 2, 4, 5, 8–10] for example.

Motivated by certain applications in the theory of integral equations, Pachpatte [6] studied the following integral inequality

$$x(t) \leqq h(t) + g(t) \int_{t}^{\infty} k(s)\, x(s)\, \mathrm{d}s, \qquad t \in I, \tag{2}$$

where $x(t)$, $h(t)$, $g(t)$ and $k(t)$ are real-valued, nonnegative and continuous functions defined on $I \equiv [0, \infty)$. He proved that

$$x(t) \leqq h(t) + g(t) \int_{t}^{\infty} h(s)\, k(s) \exp\Big(\int_{t}^{s} g(\tau)\, k(\tau)\, \mathrm{d}\tau\Big)\, \mathrm{d}s; \qquad t \in I. \tag{3}$$

In this paper, we investigate some linear and nonlinear integrodifferential

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inequalities in which the independent variable is defined on the interval $I \equiv [0, \infty)$.

Throughout this paper, we will assume that $x^{(m)}(t)$; $m > 0$, $p(t)$, $q(t)$, $g_k(t)$ $(k = 0, 1, 2, \dots, m)$ and $g_j(t)$ $(j = 1, \dots, m)$ are real-valued, non-negative, and continuous functions defined on $I \equiv [0, \infty)$.

Theorem 1. *Consider the following linear integrodifferential inequality*

$$x^{(m)}(t) \leqq p(t) + q(t) \sum_{k=0}^{m} \int_t^{\infty} g_k(s)\, x^{(k)}(s)\, ds; \qquad m > 0. \tag{4}$$

Then, $x^{(m)}(t)$ satisfies the inequality

$$x^{(m)}(t) \leqq p(t) + q(t) \int_t^{\infty} \varphi_1(s) \exp\left(\int_t^{s} \varphi_2(\tau)\, d\tau\right) ds, \tag{5}$$

where

$$\varphi_1(t) = g_m(t)\, p(t) + \sum_{k=0}^{m-1} g_k(t) \left\{ \varphi_0(t) + \frac{1}{(m-k-1)!} \int_0^t (t-s)^{m-k-1} p(s)\, ds \right\}, \tag{6}$$

$$\varphi_2(t) = g_m(t)\, q(t) + \sum_{k=0}^{m-1} g_k(t) \frac{1}{(m-k-1)!} \int_0^t (t-s)^{m-k-1} q(s)\, ds, \tag{7}$$

and

$$\varphi_0(t) = \sum_{i=k}^{m-1} \frac{x^{(i)}(0)\, t^{i-k}}{(i-k)!}. \tag{8}$$

Proof

Let

$$M_1(t) = \sum_{k=0}^{m} \int_t^{\infty} g_k(s)\, x^{(k)}(s)\, ds, \tag{9}$$

then,

$$x^{(m)}(t) \leqq p(t) + q(t)\, M_1(t). \tag{10}$$

Integrating (10), $(m-k)$ times from 0 to t, we get

$$x^{(k)}(t) \leqq \varphi_0(t) + \frac{1}{(m-k-1)!} \int_0^t (t-s)^{m-k-1} \{p(s) + q(s)\, M_1(s)\}\, ds. \tag{11}$$

Differentiating (9) and then using (10) and (11), we have

$$M_1'(t) \geqq -\varphi_1(t) - \varphi_2(t)\, M_1(t); \qquad M_1(\infty) = 0, \tag{12}$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are defined by (6) and (7) respectively. On integrating (12) from t to ∞ and substituting for $M_1(t)$ in (10), we obtain the required result.

Theorem 2. *Consider the nonlinear integrodifferential inequality*

$$x^{(m)}(t) \leqq p(t) \left[1 + \sum_{k=0}^{m} \int_t^{\infty} g_k(s)\, x^{(m)}(s)\, x^{(k)}(s)\, ds\right], \qquad m > 0. \tag{13}$$

In addition to the assumptions mentioned earlier, we will assume that $p(t)$ is a non-

decreasing function for all $t \in I$. If

$$\lambda(t) = \int_t^\infty \varphi_4(\tau) \exp\left(\int_\tau^\infty \varphi_3(s)\,ds\right) d\tau < 1, \tag{14}$$

then,

$$x^{(m)}(t) \leqq p(t) \exp\left(\int_t^\infty \varphi_3(s)\,ds\right)/[1 - \lambda(t)], \tag{15}$$

where

$$\varphi_3(t) = \sum_{k=0}^{m-1} g_k(t)\, p(t)\, \varphi_0(t), \tag{16}$$

$$\varphi_4(t) = g_m(t)\, p^2(t) + \sum_{k=0}^{m-1} g_k(t)\, p(t) \frac{1}{(m-k-1)!} \int_0^t (t-s)^{m-k-1}\, p(s)\,ds \tag{17}$$

and $\varphi_0(t)$ is defined as in (8).

Proof. Let

$$M_2(t) = 1 + \sum_{k=0}^{m} \int_t^\infty g_k(s)\, x^{(m)}(s)\, x^{(k)}(s)\,ds, \tag{18}$$

then,

$$x^{(m)}(t) \leqq p(t)\, M_2(t). \tag{19}$$

Following the same steps as in theorem (1), we get

$$M_2'(t) \geqq -\varphi_3(t)\, M_2(t) - \varphi_4(t)\, M_2^2(t); \qquad M_2(\infty) = 1. \tag{20}$$

On integrating (20) from t to ∞ and substituting the obtained value of $M_2(t)$ into (19) inequality (15) follows.

In the following theorem, we assume that W is a positive, continuous, non-decreasing and submultiplicative function such that for any constant $n > 0$

$$G_n(u) = \int_0^u \frac{ds}{W^n(s)}. \tag{21}$$

Theorem 3. *Consider the following nonlinear inequality*

$$x^{(m)}(t) \leqq p(t) + \sum_{j=0}^{m} q_j(t) \int_t^\infty g_j(s)\, W^n\left(\sum_{k=0}^{m} x^{(k)}(s)\right) ds; \qquad m > 0, \tag{22}$$

where $p(t)$ is a real valued, non-negative, continuous and nondecreasing function on I. Then,

$$x^{(m)}(t) \leqq p(t) \prod_{j=0}^{m} q_j(t)\, G_n^{-1}\left\{G_n(1) + \int_t^\infty \varphi_5(s)\,ds\right\}, \tag{23}$$

as long as

$$G_n(1) + \int_t^\infty \varphi_5(s)\,ds \in \operatorname{Dom} G_n^{-1}, \tag{24}$$

where

$$\varphi_5(t) = \sum_{j=0}^{m} \frac{g_j(t)}{p(t)} W^n[p(t)\prod_{j=0}^{m} q_j(t) + \sum_{k=0}^{m-1} (\varphi_0(t) +$$

$$+ \frac{1}{(m-k-1)!} \int_0^t (t-s)^{m-k-1} p(s) \prod_{j=0}^{m} q_j(s)\, ds], \tag{25}$$

and $\varphi_0(t)$ *is defined in* (8).

Proof. Since $p(t)$ is positive non-decreasing then (22) can be rewritten as

$$x^{(m)}(t) \leqq p(t) \prod_{j=0}^{m} q_j(t)\, M_3(t), \tag{26}$$

where

$$M_3(t) = 1 + \sum_{j=0}^{m} \int_t^{\infty} \frac{q_j(s)}{p(s)} W^n(\sum_{k=0}^{m} x^{(k)}(s))\, ds. \tag{27}$$

Integrating (26) $(m-k)$ times from 0 to t, we have

$$x^{(k)}(t) \leqq \varphi_0(t) + \frac{1}{(m-k-1)!} \int_0^t (t-s)^{m-k-1} \{p(s) \prod_{j=0}^{m} q_j(s)\, M_3(s)\}\, ds. \tag{28}$$

Differentiating (27) then on using (26) and (27) and according to the non-decreasing and submultiplicative nature of W, we get

$$M_3'(t) \geqq -\varphi_5(t)\, W^n(M_3(t)). \tag{29}$$

Integrating the above inequality from t to ∞ and substituting the value of $M_3(t)$ in (26) we easily get inequality (23).

Assuming the same conditions of theorem (3) one can easily prove the following.

Theorem 4. *Consider the nonlinear integrodifferential inequality*

$$x^{(m)}(t) \leqq p(t) + \sum_{j=0}^{m} q_j(t) \int_t^{\infty} g_j(s)\, W^{n_1}(x^{(m)}(s))\, W^{n_2}(\sum_{k=0}^{m} x^{(k)}(s))\, ds,$$

$$n_1, n_2 > 0. \tag{30}$$

Then,

$$x^{(m)}(t) \leqq p(t) \prod_{j=0}^{m} q_j(t)\, G^{-1}_{n_1+n_2}\{G_{n_1+n_2}(1) + \int_t^{\infty} \varphi_6(s)\, ds\}, \tag{31}$$

as long as

$$G_{n_1+n_2}(1) + \int_t^{\infty} \varphi_6(s)\, ds \in \operatorname{Dom} G^{-1}_{n_1+n_2},$$

where

$$\varphi_6(t) = \sum_{j=0}^{m} \frac{g_j(t)}{p(t)} W^{n_1}(p(t) \prod_{j=0}^{m} q_j(t))\, W^{n_2}\Big(p(t) \prod_{j=0}^{m} q_j(t) +$$

$$+\sum_{k=0}^{m-1}\left(\varphi_0(t)+\frac{1}{(m-k-1)!}\int_0^t (t-s)^{m-k-1}\,p(s)\prod_{j=0}^{m} q_j(s)\,\mathrm{d}s\right)\Bigg),$$

and $\varphi_0(t)$ *is defined by* (8).

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