Bohdan Zelinka
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NEIGHBOURHOOD DIGRAPHS

BOHDAN ZELINKA

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Abstract. The symbol $N_G(v)$ denotes the subgraph of a directed graph G induced by the set of all terminal vertices of edges outgoing from the vertex v. The paper studies graphs H for which there exists G such that $N_G(v) \cong H$ for each vertex v of G.

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At the Symposium on Graph Theory in Smolenice in 1963 A. A. Zykov [1] has proposed a problem concerning neighbourhood graphs in undirected graphs. Here we shall consider an analogous problem for directed graphs.

Let G be a digraph, let v be its vertex. By the symbol $N_G(v)$ we denote the subgraph of G induced by the set of all terminal vertices of edges outgoing from v in G; this graph will be called the neighbourhood of v in G. The problem analogous to that of Zykov is to characterize the digraph H with the property that there exists a digraph G such that $N_G(v) \cong H$ for each vertex v of G. We shall not solve this problem in general, but we shall show some classes of graphs which have the mentioned property.

Theorem 1. *Let H be a digraph whose vertex set $V(H)$ is the union of two disjoint sets A, B and whose edge set is the set of all edges with initial vertices in A and terminal vertices in B. Then there exists a digraph G with the property that $N_G(v) \cong H$ for each vertex v of G.*

Proof. Denote $a = |A|$, $b = |B|$. Let k be an integer, $k > b/a$. Consider six pairwise disjoint sets $A_1, A'_1, A_2, A'_2, A_3, A'_3$ of the cardinality ak. In the set A_1 choose a subset B_3, in the set A_2 choose a subset B_1, in the set A_3 choose a subset B_2 in such a way that $|B_1| = |B_2| = |B_3| = b$. The set $V(G) = A_1 \cup A'_1 \cup \cup A_2 \cup A'_2 \cup A_3 \cup A'_3$ will be the vertex set of the graph G. In G all vertices of A_i with all vertices of A'_i will be joined by pairs of oppositely directed edges for each $i \in \{1, 2, 3\}$. Further G will contain all edges going from a vertex of $A_i \cup A'_i$ into a vertex of B_i for each $i \in \{1, 2, 3\}$. No other edges than those described ones will be in G. The digraph G thus obtained has the required property.

In the case when $a = b$ the required graph G can be moie simple. Take four pairwise distinct sets A_1, A_2, A_3, A_4 of the cardinality a. All vertices of A_1 with all vertices of A_4 will be joined by pairs of oppositely directed edges. Further G will contain all edges going from a vertex of A_i into each vertex of A_{i+1} for each $i \in \{1, 2, 3, 4\}$, the subscript $i + 1$ being taken modulo 4.

Theorem 2. *Let H_0 be a digraph with the property that there exists a digraph G_0 such that $N_{G_0}(v) \cong H_0$ for each vertex v of G_0. Let H be the digraph obtained from H_0 by adding a new vertex w and all edges going from vertices of H_0 into w. Then there exists a digraph G such that $N_G(v) \cong H$ for each vertex v of G.*

Proof. Let G_1, G_2, G_3 be three pairwise disjoint digraphs which are all isomorphic to G_0. In G_1 choose a vertex w_3, in G_2 choose a vertex w_1, in G_3 choose a vertex w_2. The vertex set $V(G)$ of G is the union of the vertex sets of G_1, G_2, G_3; its edge set consists of all edges of these graphs and more over of all edges going from a vertex of G_i into w_i for each $i \in \{1, 2, 3\}$. The graph G has evidently the required property.

Theorem 3. *Let H_0 be a digraph with the property that there exists a digraph G_0 such that $N_{G_0}(v) \cong H_0$ for each vertex v of G. Let H be the digraph obtained from H_0 by adding a new vertex w and all edges going from w into vertices of H_0. Then there exists a digraph G such that $N_G(v) \cong H$ for each vertex v of G.*

Proof. Consider the vertex set $V = V(G_0)$ and a set V' such that $|V'| = |V|$, $V' \cap V = \emptyset$. Let φ be a bijection of V onto V'. The vertex set of G will be $V(G) = V \cup V'$. If u, v are two vertices of G_0 such that there exists the edge from u into v in G_0, then G will contain the edge from u into v and the edge from $\varphi(u)$ into v. Further for each $v \in V$ the vertices v and $\varphi(v)$ will be joined in G by a pair of oppositely directed edges. No other edges than those described ones will be contained in G. The graph G has evidently the required property.

REFERENCE

[1] *Theory of Graphs and Its Applications*, Proc. Symp. Smolenice 1963 (ed. M. Fiedler), Praha 1964.

B. Zelinka
Department of Metal Forming and Plastics
Institute of Mechanical and Textile Technology
Studentská 1292
461 17 Liberec 1
Czechoslovakia