Ján Futák

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ON BOUNDEDNESS AND STABILITY OF SOLUTIONS OF NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS IN HILBERT SPACES

JÁN FUTÁK

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Abstract. In this paper there are investigated an existence problem of bounded solutions and a stability of a trivial solution of non-linear differential equations of the 2nd order in Hilbert spaces.

Key words: differential equations in Hilbert spaces, existence and boundedness of solutions, stability, differential equations with delay.

MS Classification. 34 G 20

Let H be a Hilbert separable space with a norm $\|\,.\,\| = (.\,;.)^{1/2}$ where $(.\,;.)$ is inner product. Let $[H]$ denote the Banach algebra of bounded linear operators from H to H with the operator norm $|\,.\,|$, which we refer to as the uniform operator topology.

Let $L_{\text{loc}}(R_+, H)$ denote the space of locally Lebesgue integrable functions u: $R_+ \to H$ with topology of convergence in the mean on every compact subinterval of R_+ and let $C'_{\text{loc}}(R_+, H)$ mean the space of continuously differentiable functions u: $R_+ \to H$ with the locally uniform convergence. Let T: $C'_{\text{loc}}(R_+, H) \to L_{\text{loc}}(R_+, H)$ be a continuous operator of volterra type.

We shall consider a differential equation

$$x''(t) + A(t)\,x'(t) + B(t)f(t, x(t)) + T(x)(t) = 0, \tag{1}$$

where A, B: $R_+ \to [H]$ are locally absolutely continuous (in the uniform operator topology) and symetrical operators, f: $R_+ x H \to H$ and $\frac{\partial f}{\partial t}$ are continuous functions.

We assume further that for each $t \in R_+$ that $B(t)$ is a uniformly positive operator (it guarantees the existence of an inverse operator $B^{-1}(t)$ and also that the least eigenvalue of $B(t)$ is positive – see [1] p. 50).

Let $\mu(K)$ denote the eigenvalue of the operator K. Put $\lambda(t) = \max \mu(B(t))$, $\omega(t) = \max \mu(B^{-1}(t))$ and $\nu(t) = \max \mu[(B^{-1}(t))' - 2B^{-1}(t)\,A(t)]$. Throughout the paper we assume that $\lambda(t)$ is a nondecreasing function on R_+.

We define

$$F_i(t, x_1, x_2, \dots, x_n, \dots) = \int_0^{x_i} f_i(t, x_1, x_2, \dots, x_{i-1}, s, x_{i+1}, \dots)\, \mathrm{d}s,$$

where f_i, $i = 1, 2, \dots$ are components of $f(t, x)$ and $x = (x_1, x_2, \dots, x_n, \dots) \in H$.

Assume that the series $\sum_{i=1}^{\infty} F_i(t, x_1, x_2, \dots, x_n, \dots)$ is convergent and let

$$F(t, x) = \sum_{i=1}^{\infty} F_i(t, x_1, x_2, \dots, x_n, \dots).$$

Next suppose everywhere that

$$\sum_{i=1}^{\infty} \frac{\partial F_i}{\partial x_k} = 0 \qquad \text{for } k \neq i,\ k = 1, 2, \dots$$

Let $[0, t^*)$ be the interval of existence of a solution of (1). By solution of (1) we understand any function x: $[0, t^*) \to H$ which is locally absolutely continuous on $[0, t^*)$ together with its first derivative, it satisfies (1) everywhere on $[0, t^*)$ and is maximally extended to the right.

Definition. (see [2] and [4]). *Let $f(t, 0) \equiv 0$, $T(0)(t) \equiv 0$ and b: $R_+ \to (0, \infty)$ be some continuous function. The solution $x = 0$ of (1) is said to be $b-$stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $x(t)$ of (1) for which*

$$\| x(0) \|^2 + \| x'(0) \|^2 < \delta$$

is defined on R_+ and

$$\| x(t) \|^2 + \frac{\| x'(t) \|^2}{b(t)} < \varepsilon \qquad \text{for } t \in R_+.$$

On boundedness of solutions and $b-$stability of a trivial solution of scalar second order differential equations is dealt with in [4]. Boundedness of solutions and $b-$stability of a trivial solution of nonlinear second order differential systems is investigated in [2]. In this paper the results from [2] and [4] are generalized.

Define for arbitrary function u: $R_+ \to H$ and $t \in R_+$

$$\| u \|_t = \max_{0 \leq s \leq t} \| u(s) \|.$$

Theorem 1. *Let F_0: $R_+ \to R_+$ be a continuous function and α: $R_+ \to R_+$ be a measurable function such that*

$$\lim_{t \to \infty} F_0(t) = \infty \qquad \text{and} \qquad \int_0^{\infty} \lambda(t)\, [\alpha(t)\, \omega(t) + v_+(t)]\, \mathrm{d}t < \infty, \tag{2}$$

where $v_+ = \max\{v, 0\}$.

Moreover let

(i) *$F(t, u) \geq F_0(\| u \|)$ for $t \in R_+$, $u \in H$,*

(ii) $\dfrac{\partial F(t,u)}{\partial t} \leqq \alpha(t) F_0(\| u \|)$ *for* $t \in R_+, u \in H$,

(iii) $\| Tx \|_t \leqq \alpha(t) [\sqrt{\lambda(t)} + \sqrt{\lambda(t) F_0(\| x \|_t)} + \| x' \|_t]$

for $t \in R_+$, $x \in C'_{\text{loc}}(R_+, H)$.

Then every solution of (1) exists on R_+ *and is bounded.*

Proof. Let $x(t)$ be arbitrary solution of (1) with its interval existence $[0, t^*)$. Without loss of generality suppose that the function $F_0(t)$ is nondecreasing.

If (1) is multiplied (from the left-hand side) by $B^{-1}(t)$ and the scalar function $x'(t)$ and then integrated from 0 to t, $t \in [0, t^*)$, we obtain

(3) $\int_0^t (B^{-1}x''; x')\,ds + \int_0^t (B^{-1}Ax'; x')\,ds + \int_0^t (f(s,x); x')\,ds + \int_0^t (B^{-1}Tx; x')\,ds = 0.$

As there holds

$$(B^{-1}x''; x') = \frac{1}{2}(B^{-1}x'; x')' - \frac{1}{2}([B^{-1}]'x'; x')$$

and

$$(f(t,x); x') = \frac{dF(t,x)}{dt} - \frac{\partial F(t,x)}{\partial t},$$

from (3) we have

(4) $$(B^{-1}x'; x') + 2F(t,x) = C_0 + \int_0^t (([B^{-1}]' - 2B^{-1}A)x'; x')\,ds + 2\int_0^t \frac{\partial F(s,x)}{\partial s}\,ds - 2\int_0^t (B^{-1}Tx; x')\,ds,$$

where $C_0 = (B^{-1}(0)x'(0); x'(0)) + 2F(0, x(0))$.

With regard to hypotheses (i), (ii) from (4) it follows

(5) $$(B^{-1}x'; x') + 2F_0(\| x \|) \leqq C_0 + 2\int_0^t \alpha(s) F_0(\| x \|)\,ds + \int_0^t \nu(s) \| x'(s) \|^2\,ds + 2\int_0^t (B^{-1}Tx; x')\,ds, \quad \text{for } t \in [0, t^*).$$

From (5) with regard to (iii) we obtain estimates

(6) $$\frac{1}{\lambda(t)} \| x' \|_t^2 \leqq C_0 + 2\int_0^t \alpha(s) F_0(\| x \|_s)\,ds + \int_0^t \nu_+(s) \| x' \|_s^2\,ds + 2\int_0^t \alpha(s)\lambda(s)\omega(s)\left[1 + \sqrt{F_0(\| x \|_s)} + \frac{\| x' \|_s}{\sqrt{\lambda(s)}}\right]\frac{\| x' \|_s}{\sqrt{\lambda(s)}}\,ds, \qquad t \in [0, t^*),$$

$$(7)\qquad 2F_0(\| x \|_t) \leqq C_0 + 2\int_0^t \alpha(s) F_0(\| x \|_s)\,ds + \int_0^t v_+(s) \| x' \|_s^2\,ds +$$
$$+ 2\int_0^t \alpha(s)\lambda(s)\omega(s)\left[1 + \sqrt{F_0(\| x \|_s)} + \frac{\| x' \|_s}{\sqrt{\lambda(s)}}\right]\frac{\| x' \|_s}{\sqrt{\lambda(s)}}\,ds, \qquad t \in [0, t^*).$$

After addition of (6) and (7) we get

$$(8)\quad \frac{1}{\lambda(t)} \| x' \|_t^2 + 2F_0(\| x \|_t) \leqq 2C_0 + 4\int_0^t \alpha(s) F_0(\| x \|_s)\,ds + 2\int_0^t v_+(s) \| x' \|_s^2\,ds +$$
$$+ 4\int_0^t \alpha(s)\lambda(s)\omega(s)\left[1 + \sqrt{F_0(\| x \|_s)} + \frac{\| x' \|_s}{\sqrt{\lambda(s)}}\right]\frac{\| x' \|_s}{\sqrt{\lambda(s)}}\,ds \qquad \text{for } t \in [0, t^*).$$

According to the inequality $2(1 + a + b)\,b \leqq 1 + a^2 + 4b^2$ and $\alpha(t) \leqq \alpha(t)\,\lambda(t)\,\omega(t)$ from (8) it follows

$$(9)\qquad \frac{\| x' \|_t^2}{\lambda(t)} + 2F_0(\| x \|_t) \leqq C_1 + 8\int_0^t \lambda(s)\,[\alpha(s)\,\omega(s) + v_+(s)]\,\times$$
$$\times\left[\frac{\| x' \|_s^2}{\lambda(s)} + 2F_0(\| x \|_s)\right]ds,$$

where $C_1 = 2C_0 + 2\int_0^\infty \lambda(s)\,\alpha(s)\,\omega(s)\,ds$.

By using of Gronwall inequality ([3], p. 37) we have

$$(10)\qquad \frac{\| x' \|_t^2}{\lambda(t)} + 2F_0(\| x \|_t) \leqq C_1 \exp\Big(8\int_0^\infty \lambda(t)\,[\alpha(t)\,\omega(t) + v_+(t)]\,dt\Big) \leqq C_2,$$
$$t \in [0, t^*).$$

With respect to (2) we have from (10) that $t^* = \infty$ and $\sup_{t \in R_+} \| x(t) \| < \infty$.
Thus the proof is complete.

Remark. If $v(t) \leqq 0$ for $t \in R_+$, then the assertion of Theorem 1 is true, just the integral in (2) is the form

$$\int_0^\infty \lambda(t)\,\alpha(t)\,\omega(t)\,dt < \infty.$$

Example. Consider a differential equation on $H = R = (-\infty, +\infty)$, of the form

$$(1')\qquad x''(t) + a(t)\,x'(t) + b(t) f(t, x(t)) + T(x)(t) = 0,$$

where $a, b\colon R_+ \to R$ are locally absolutely continuous functions, $b(t) > 0$ and non-decreasing on R_+, $f\colon R_+ \times R \to R$ and $\dfrac{\partial f(t, x)}{\partial t}$ are continuous functions and $T\colon C'_{\text{loc}}(R_+, R) \to L_{\text{loc}}(R_+, R)$ is a continuous operator (see [4]) of volterra type.
Then

$$F(t, x) = \int_0^x f(t, s)\,ds, \; \lambda(t) = b(t) \qquad \text{and} \qquad v(t) = \left(\frac{1}{b(t)}\right)' - 2\frac{a(t)}{b(t)}.$$

The integral in (2) is the form

$$\int_0^\infty \left[\alpha(t) + b(t)\left\{\left(\frac{1}{b(t)}\right)' - 2\frac{a(t)}{b(t)}\right\}_+\right] dt < \infty.$$

Thus all assumptions of Theorem 1 for equation (1') be satisfied.

If e.g.

$$a(t) \geqq -\frac{b'(t)}{2b(t)} \quad \text{then } v(t) \leqq 0 \quad \text{for } t \in R_+ \text{ (see Remark).}$$

Theorem 2. *Let* $F_0 : [0, r] \to R_+$, $(r > 0)$ *be a continuous function and* $\alpha : R_+ \to R_+$ *be a measurable function such that*

(11) $\quad F_0(t) > 0$ *for* $0 < t \leqq r$ *and* $\int_0^\infty \lambda(t)\,[\alpha(t)\,\omega(t) + v_+(t)]\,dt < \infty.$

Moreover let

(i') $\quad F(t, u) \geqq F_0(\| u \|)$ *for* $t \in R_+$, $\| u \| \leqq r$,

(ii') $\quad \dfrac{\partial F(t, u)}{\partial t} \leqq \alpha(t)\,F_0(\| u \|),$ *for* $t \in R_+$, $\| u \| \leqq r$,

(iii') $\quad \| Tx \|_t \leqq \alpha(t)\,[\sqrt{\lambda(t)\,F_0(\| x \|_t)} + \| x' \|_t]$

for $t \in R_+$, $x \in C'_{loc}(R_+, H)$, *and* $\| x \|_t + \dfrac{\| x' \|_t^2}{\sqrt{\lambda(t)}} \leqq r.$

Then the solution $x = 0$ *of (1) is* λ*-stable.*

Proof. Without loss of generality suppose that $F_0(t)$ is an increasing function. Let $\varepsilon > 0$ be an arbitrary number. To ε take $0 < \delta < \min\{\varepsilon, \lambda(0)\,\varepsilon\}$ such that

(12) $$\beta \| B^{-1}(0) \| \delta + 2\beta \max\{F(0, u), \| u \| \leqq \sqrt{\delta}\} < \min\left\{\frac{\varepsilon}{2}, 2F_0\left(\sqrt{\frac{\varepsilon}{2}}\right)\right\},$$

where $\beta = 2\exp(8 \int_0^\infty \lambda(t)\,[\alpha(t)\,\omega(t) + v_+(t)]\,dt)$ holds.

Let $x(t)$ be an arbitrary solution of (1) fulfilling an inequality

(13) $$\| x(0) \|^2 + \| x'(0) \|^2 < \delta.$$

We show that this solution does exist on R_+ and that fulfils

(14) $$\| x(t) \|^2 + \frac{\| x'(t) \|^2}{\lambda(t)} < \varepsilon \quad \text{for } t \in R_+.$$

On the contrary suppose, that (14) does not holds. Then there exists $t_0 \in (0, \infty)$ such that

$$\| x(t) \|^2 + \frac{\| x'(t) \|^2}{\lambda(t)} < \varepsilon \quad \text{for } t \in [0, t_0),$$

and

$$\| x(t_0) \|^2 + \frac{\| x'(t_0) \|^2}{\lambda(t_0)} = \varepsilon. \tag{15}$$

Similarly as in the proof of Theorem 1 we obtain from (5) according to assumptions the following estimate

$$\frac{\| x' \|_t^2}{\lambda(t)} + 2F_0(\| x \|_t) \leqq 2C_0 + 8 \int_0^\infty \lambda(s) [\alpha(s) \omega(s) + v_+(s)] \times \tag{16}$$

$$\times \left[\frac{\| x' \|_s^2}{\lambda(s)} + 2F_0(\| x \|_s) \right] ds.$$

On applying of Gronwall inequality to (16) we get

$$\frac{1}{\lambda(t)} \| x'(t) \|^2 + 2F_0(\| x(t) \|) \leqq \beta[\| B^{-1}(0) \| \ \| x'(0) \|^2 + 2F(0,x(0))],$$

$$t \in [0, t_0],$$

from which there holds that

$$\frac{\| x'(t_0) \|^2}{\lambda(t_0)} < \frac{\varepsilon}{2}$$

and simultaneously

$$2F_0(\| x(t_0) \|) \leqq 2F_0\left(\sqrt{\frac{\varepsilon}{2}}\right) \quad \text{or} \quad \| x(t_0) \| < \sqrt{\frac{\varepsilon}{2}}.$$

From the last two inequalities then we have

$$\| x(t_0) \|^2 + \frac{\| x'(t_0) \|^2}{\lambda(t_0)} < \varepsilon$$

which is a contradiction with (15). This proves that the solution $x(t)$ exists on R_+ and (14) holds. Thus, the solution $x = 0$ of (1) is λ-stable.

Next we shall consider a delay differential equation

$$x''(t) + A(t)\, x'(t) + B(t) f_1(t, \sigma(x; h(t))) + f_2(t, \sigma(x; h(t)), \sigma(x'; h(t))) = 0, \tag{17}$$

where A, B have the same meaning as above, $f_1\colon R_+ \times H \to H$ and $\frac{\partial f_1}{\partial t}$ are continuous functions, $f_2\colon R_+ \times H \times H \to H$ fulfils local Carathéodory conditions and σ is an operator defined by

$$\sigma(u; t) = \begin{cases} u(t - h(t)) & \text{for } t \geqq h(t), \\ 0 & \text{for } t < h(t), \end{cases}$$

where $h\colon R_+ \to R_+$ is a measurable function locally bounded on R_+.

Theorem 3. *Let $F_0\colon R_+ \to R_+$ be a continuous function and $\alpha, \mathscr{C}\colon R_+ \to R_+$ be measurable functions such that*

$$\lim_{t\to\infty} F_0(t) = \infty, \qquad \int_0^\infty \lambda(t)\,[\alpha(t)\,\omega(t) + \nu_+(t)]\,dt < \infty, \qquad \int_0^\infty h(t)\,\lambda(t)\,\varphi(t)\,dt < \infty.$$

Further let the function $F(t, u)$ be defined by means of $f_1(t, u)$ as in the Theorem 1 and let fulfil conditions (i), (ii) *of the Theorem 1.*

Suppose that on $R_+ \times H$ there holds

$$\| f_1(t, u) - f_1(t, \bar{u}) \| \leqq \mathscr{C}(t) \| u - \bar{u} \| \tag{18}$$

and on $R_+ \times H \times H$

$$\| f_2(t, u, v) \| \leqq \alpha(t) [\sqrt{\lambda(t)} + \sqrt{\lambda(t) F_0(\| u \|)} + \| v \|]. \tag{19}$$

Then any solution of (17) *is defined on R_+ and is bounded.*

Proof. If we put

$$f(t, u) = f_1(t, u)$$

and

$$T(x)(t) = B(t)\,[f_1(t, \sigma(x; h(t))) - f_1(t, x)] + f_2(t, \sigma(x; h(t)), \sigma(x'; h(t)))$$

then (17) is transformed into (1).

According to assumptions (18) and (19) we obtain

$$\| f_1(t, \sigma(x; h(t))) - f_1(t, x) \| \leqq \varphi(t) \| \sigma(x; h(t)) - x(t) \| \leqq \varphi(t)\, h(t) \| x' \|_t \quad \text{for } x \in C'_{\text{loc}}(R_+, H),$$

and

$$\| Tx \|_t \leqq \lambda(t)\,\varphi(t)\,h(t) \| x' \|_t + \alpha(t) [\sqrt{\lambda(t)} + \sqrt{\lambda(t) F_0(\| x \|_t)} + \| x' \|_t] \leqq$$
$$\leqq \varrho(t) [\sqrt{\lambda(t)} + \sqrt{\lambda(t) F_0(\| x \|_t)} + \| x' \|_t],$$

where $\varrho(t) = \lambda(t)\,\varphi(t)\,h(t) + \alpha(t)$ and $\int_0^\infty \varrho(t)\,dt < \infty$.

Thus, all assumptions of Theorem 1 are fulfilled. Therefore any solution of (17) exists on R_+ and is bounded.

Theorem 4. *Let $F_0 : [0, r] \to R_+$ ($r > 0$), be a continuous function, $\alpha, \varphi : R_+ \to R_+$ be measurable functions such that*

$$F_0(t) > 0 \quad \textit{for } 0 < t \leqq r, \qquad \int_0^\infty \lambda(t)\,[\alpha(t)\,\omega(t) + \nu_+(t)]\,dt < \infty,$$
$$\int_0^\infty h(t)\,\lambda(t)\,\varphi(t)\,dt < \infty.$$

Let the function $F(t, u)$ have the same meaning as in Theorem 3 and let for $t \in R_+$ and $\| u \| \leqq r$ fulfil conditions (i'), (ii') *from Theorem 2.*

Suppose that on the set $\{(t, u), t \in R_+, \| u \| \leqq r\}$ (18) *holds and on the set $\{(t, u, v), t \in R_+, \| u \| \leqq r, \| v \| \leqq r\}$*

$$\| f_2(t, u, v) \| \leqq \alpha(t) [\sqrt{\lambda(t) F_0(\| u \|)} + \| v \|]$$

holds.

Then the solution $x = 0$ *of* (17) *is λ-stable.*

Proof can be carried out in the same way as the of Theorem 2 by using of Theorem 3.

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J. Futák
Department of Mathematics
Transport College
Marxa a Engelsa 25
010 88 Žilina
Czechoslovakia