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Archivum Mathematicum, Vol. 25 (1989), No. 1-2, 95--102

Persistent URL: http://dml.cz/dmlcz/107344

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 25, No. 1–2 (1989), 95–102

COMPLETE PERMUTABILITY OF PARTITIONS IN A SET

Part I

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(Received May 16, 1988)

Dedicated to the memory of Milan Sekanina

Abstract. J. Hashimoto introduced in [7] a notion of a complete permutability of partitions on a set as a generalization of commutativity of two partitions. By H. Draškovičová [5] this is transferred unchanged to partitions in a set. Unfortunately, her notion fails to generalize the commutativity. In the present paper, this disadvantage is removed by a little modification of the definition. This definition reproduces that of Hashimoto in the case of partitions "on". The relations of the modified notion to that of the associability, which represents another generalization of the commutativity, are found. Properties of the introduced complete permutability are discussed in Theorems 1.9, 1.10 and 1.11. An analogous definition is given for congruence relations in an algebra. Some sufficient conditions are found that guarantee the possibility of the preceding results to be applied for congruence relations. In particular, a characterization of complete permutability for Ω-groups is derived.

Key words. Partition "on" and "in", ST-relation, congruence relation, complete permutability, associability, commutativity.

MS Classification. 06 F 15

A partition *in* a set G is a system A (possibly empty) of nonempty mutually disjoint subsets of G [2, 3, 4]. The elements of A are called *blocks* of the partition A. The union $\bigcup A$ of all blocks of A is said to be a *domain* of A. If the domain of A is equal to G, $\bigcup A = G$, the partition A is called a partition *on* G.

It is a well-known fact, that there exists an one-one correspondence between all partitions on a set and all equivalence relations in the same set, and analogously, an one-one correspondence between all partitions in the set G and all symmetric and transitive relations (ST-relations) in G. We shall find it useful to hold, if need be, the partitions in G (on G) for ST-relations (equivalence relations) in G and vice versa. If A is an ST-relation in G, the corresponding partition in G is $\bigcup A/A$, where in this case $\bigcup A = \{x \in G: xAx\}$.

Let (G, Ω) be a universal algebra with the system of operations Ω and let A be an ST-relation (a partition) in the set G. We say, that A is a *congruence relation* *in* the algebra (G, Ω) if A preserves the operations of Ω (of arity ≥ 1). Congruence relations A in (G, Ω) with $\bigcup A = G$ are said to be *congruence relations on* the algebra (G, Ω). We denote by

$P(G)$ the system of all partitions (ST-relations) in the set G,
$\pi(G)$ the system of all partitions (equivalence relations) on the set G,
$\mathscr{K}(G)$ the system of all congruence relations in the algebra (G, Ω)
(the system of the corresponding ST-relations in the set G),
$\mathscr{C}(G)$ the system of all congruence relations on the algebra (G, Ω),
(the system of the corresponding equivalence relations in the set G).

Some known and relevant facts related to these notions are summarized in the following

Theorem. *The set* $\pi(G)$ *is a complete, semimodular and relatively complemented lattice* [13], [11] Th. 67. *The lattice* $P(G)$ *is complete, semimodular and Brouwerian, it is not relatively complemented* [4] Ths. 4.5, 4.1 and 5.3, $\pi(G)$ *is a closed sublattice of* $P(G)$. *If* (G, Ω) *is an algebra, then the set* $\mathscr{C}(G)$ *is a closed sublattice of* $\pi(G)$ (see e.g. [11] Th. 84; [1] VI § 4 Th. 8). *The lattice* $\mathscr{C}(G)$, *where* (G, Ω) *is an* Ω*-group, is modular* (see e.g. [8] IV, 2.2). *If* G *is a lattice or an l-group, then* $\mathscr{C}(G)$ *is a distributive lattice* (see e.g. [11] Th. 90, [1] XIII § 9 Th 16). *All the lattices* $P(G)$, $\pi(G)$, $\mathscr{K}(G)$, $\mathscr{C}(G)$ *are algebraic* [12] 1.6; [1] XII § 9 Th. 16, VI § 4 Th. 9; [10] § 5.

The domain $\bigcup A$ *of a congruence relation* $A \in \mathscr{K}(G)$ *is a subalgebra of the algebra* (G, Ω). *The nullblock* $A(0) = \{x \in G: xA0\}$ *of a congruence relation* A *in an* Ω*-group* (G, Ω) *is an ideal in* $\bigcup A$ *and there holds* $A = \bigcup A / A(0)$ [9] I 1.3 and 1.4. □

1. The notion of the complete permutability of partitions was introduced in [7], p. 90 for partitions *on* a set and in [5], Def. 1.3, unchanged transferred to partitions *in* a set. The mentioned definition in [7] reads

(*) *A system* $\{A_\iota : \iota \in \Gamma\}$ *of partitions on a set* G *is called* completely permutable *if for arbitrary subsets* $\emptyset \neq \Lambda \subseteq \Gamma$ *and* $\{x^\iota : \iota \in \Lambda\} \subseteq G$ *it holds*

whenever $x^\mu (C_\mu \vee C_\nu) x^\nu$, $\mu, \nu \in \Lambda$, *where* $C_\alpha = \bigwedge_{\substack{\iota \in \Lambda \\ \iota \neq \alpha}} A_\iota$, $\alpha \in \Lambda$, *then there exists* $x^\iota \in G$ *such that* $x^\iota A_\iota x$, $\iota \in \Lambda$.

Note that the definition (*) was introduced in [7] for congruence relations on a universal algebra (G, Ω). Since the lattice $\mathscr{C}(G)$ of all congruence relations on (G, Ω) is a closed sublattice of the lattice of all partitions $\pi(G)$ on the set G, the replacement of congruence relations by partitions is unessential (viz. $\vee_{\mathscr{C}} \equiv \vee_P$)

For a reason, which will be explained in the sequel, it is suitable to modify the definition (*) for partitions in a set as follows

1.1. Definition. *A system of partitions in a set* G $\{A_\iota : \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *is called* completely permutable [finitely permutable] *if for an arbitrary* [*finite*] *subset* $\Lambda \subseteq \Gamma$ *with* card $\Lambda \geq 2$ *and for an arbitrary system of elements* $\mathfrak{A} =$

$= \{x^\iota : \iota \in \Lambda\} \subseteq G$ *fulfilling* $x^\mu(C_\mu \vee C_\nu)\, x^\nu$, μ, $\nu \in \Lambda$, $\mu \neq \nu$, *where* $\alpha \in \bigwedge C_\alpha = \bigwedge_{\substack{\iota \in \Lambda \\ \iota \neq \alpha}} A_\iota$, *the condition* (1.1Z) *or* (1.2Z) *is valid*

(1.1Z) $x \in G$ *exists such that* $x^\iota A_\iota x$, $\iota \in \Lambda$, *or*

(1.2Z) *there exist* $\alpha \in \Lambda$ *and* $A_\alpha^1 \in \bigcup A_\alpha / A_\alpha$ *which satisfy* a) $\mathfrak{A} \subseteq A_\alpha^1$ and b) $A_\alpha^1 \cap \bigcup A_\iota \neq \emptyset$ *for some* $\iota \in \Lambda$ *implies* $A_\alpha^1 \in \bigcup A_\iota / A_\iota$. (By $\vee$ there is meant the supremum $\vee_P$ in $P(G)$.)

Note. If we should admit card $\Lambda = 1$ in the Definition 1.1, then it would hold $\bigcup_{\iota \in \Gamma} \cup A^\iota = G$.

Indeed $\iota \in \Gamma$, choose $y \in G$ and $\Lambda = \{\iota_0\}$ for some $\iota_0 \in \Gamma$. Then the requirement for the singleton $\mathfrak{A} = \{y\} \subseteq G\,(y = x^{\iota_0})$ is satified trivially and thus (1.1Z) or (1.2Z) do not be fulfilled unless it holds $y \in \bigcup A_{\iota_0}$, i.e. $\bigcup_{\iota \in \Gamma} \bigcup A_\iota = G$. □

1.2. Definition ([9] IV Def. 4.1; [5] Def. 1.2) *A system* $\{A_\iota : \iota \in \Gamma\}$ *of partitions in a set G is called* associable *if it satisfies; For any system* $\mathfrak{A} = \{x^\iota : \iota \in \Gamma\}$ *of elements of the set G fulfilling* $x^\mu(\bigvee_{\iota \in \Gamma} A_\iota)\, x^\nu$, μ, $\nu \in \Gamma$, *one of the following conditions holds*

(1.1A) $x \in G$ *exists such that* $x^\iota A_\iota x$, $\iota \in \Gamma$, *or*

(1.2A) $\alpha \in \Gamma$ *and* $A_\alpha^1 \in \bigcup A_\alpha / A_\alpha$ *exist such that* a) $\mathfrak{A} \subseteq A_\alpha^1$ and b) if $A_\alpha^1 \cap \bigcup A_\beta \neq \emptyset$ *for some* $\beta \in \Gamma$, $A_\alpha^1 \in \bigcup A_\beta / A_\beta$.

Any nonempty subset of an associable system is associable [5] and [9] IV 4.5. Then the following two Propositions are evidently true.

1.3. Proposition. *Any associable system of (at least two) partitions in a set is completely permutable.* □

1.4. Proposition. *If* $\{A_\iota : \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *is a completely permutable [finitely permutable] system of partitions in a set G,* $K \subseteq \Gamma$ *with* card $K \geq 2$, *then the system* $\{A_\iota : \iota \in K\}$ *is completely permutable [finitely permutable] as well.* □

1.5. Proposition. *Let* $\{A_1, A_2\}$ *be a system of two partitions in a set G. Then the following are equivalent.*

a) *The system* $\{A_1, A_2\}$ *is completely permutable;*

b) *The system* $\{A_1, A_2\}$ *is associable;*

c) *The partitions* A_1 *and* A_2 *commute.*

Proof. c implies b. Let the partitions A_1 and A_2 commute and let $x^1(A_1 \vee A_2)\, x^2$. By [9] III 3.1.1(1), $A_1 \vee A_2 = A_1A_2 \cup A_1 \cup A_2$, thus $x^1A_1A_2x^2$ or $x^1A_1x^2$ or $x^1A_2x^2$. In the first case $x \in G$ exists such that $x^1A_1xA_2x^2$, consequently (1.1A) holds. In the second case x^1, $x^2 \in A_1^1$ ($\alpha = 1$) for some block $A_1^1 \in \bigcup A_1/A_1$. If for $i = 2$ there exists an element $x \in A_1^1 \cap \bigcup A_2$, then $x^2A_1A_2y$ for some $y \in G$. From the commutativity of A_1 and A_2 it follows $x^2A_2A_1y$, thus $x^2A_2x^2$. Hence

$x^1 A_1 x^2 A_2 x^2$ and therefore the condition (1.1A) is satisfied. The last case $x^1 A_2 x^2$ is symmetric to the preceding one.

b implies a is evident.

a implies c. Let the system $\{A_1, A_2\}$ be completely permutable, $\mathfrak{A} = \{x^1, x^2\} \subseteq G$ and $x^1 A_2 A_1 x^2$. Then $x^1(A_1 \vee A_2) x^2$. If the condition (1.1Z) is satisfied, then $x \in G$ exists such that $x^1 A_1 x A_2 x^2$, hence $x^1 A_1 A_2 x^2$. Let (1.2Z) hold, let $\alpha \in \{1, 2\}$ and a block $A_\alpha^1 \in \bigcup A_\alpha / A_\alpha$ exist such that $\mathfrak{A} \subseteq A_\alpha^1$. For any $\iota \in \{1, 2\}$ there holds x^1 or $x^2 \in A_\alpha^1 \cap \bigcup A_\iota$. Thus $A_\alpha^1 \in \bigcup A_\iota / A_\iota$ ($\iota = 1, 2$), i.e. $(x^1, x^2) \in A_1 \cap A_2 \subseteq A_1 A_2$. The reverse inclusion can be proved symmetrically. Hence $A_1 A_2 = A_2 A_1$, the desired commutativity. □

1.6. Corollary. *Any two partitions belonging to a completely permutable [finitely permutable] system of partitions in a set commute.*

Proof follows from 1.4 and 1.5. □

1.7. Remark. For partitions on a set the definition 1.1 of the complete permutability is identical with the definition (*) (appart from the requirement card $\Gamma \geq 2$ and card $\Lambda \geq 2$).

Proof. For partitions $\{A_\iota: \iota \in \Gamma\}$ on a set it holds $x^\mu(C_\mu \vee C_\nu) x^\nu$, $\mu \neq \nu$, $\mu, \nu \in \Lambda \Leftrightarrow x^\mu(C_\mu \vee C_\nu) x^\nu$, $\mu, \nu \in \Lambda$. Further the condition (1.2Z) for partitions "on" mplies the condition (1.1Z) (with an arbitrary $x \in A_\alpha^1$). □

1.8. The principal disadvantage of the definition 1.3 [5] for the complete permutability of partitions in a set is that it is not equivalent to the commutativity in case of two partitions. The 1.3 [5] version of the complete permutability of two partitions A_1, A_2 implies the relation $A_1 \vee A_2 = A_1 A_2$, while their commutativity implies $A_1 \vee A_2 = A_1 A_2 \cup A_1 \cup A_2$ by [9] 3.1.1(1). Thus the complete permutability by [5] implies the commutativity ([9] III 3.1.1(3)), the converse does not hold in general ([9] III 3.1.1(1)).

As we see, the notions of associability and complete permutability generalize the notion of commutativity. It is usefull when two generalizations are comparable. Our definition 1.1 admits such a comparison. This is given by Proposition 1.3. Such a comparison is not true for the complete permutability version [5]. Namely, all the partitions of a system $\{A_\iota: \iota \in \Gamma\}$, which is completely permutable in the sense of [5] have the same domain. Indeed, in this case $A_\mu \vee A_\nu = A_\mu A_\nu$, $\mu, \nu \in \Gamma$, so that $\bigcup A_\mu = \bigcup A_\nu$ by [9] III 3.1.1(6).

Now, we give some properties of complete permutability. To this end let us recall two notions..

A subset H of a set G is said to *respect* a partition A in G if H contains each block of A, which it intersects ([9] IV Def. 4.8).

Let $\mathbf{A} = \{A_\iota: \iota \in \Gamma\}$ be a system of partitions in a set G and $\emptyset \neq H \subseteq G$. Under $\mathbf{A} \sqcap H$ we understand the system $\{A_\iota \sqcap H: \iota \in \Gamma\}$ ([9] IV Def. 4.8.2). As for the symbol $A_\iota \sqcap H$, see [3] I 2.3: $A_\iota \sqcap H = \{A^1 \cap H: A^1 \in \bigcup A/A, A^1 \cap H \neq \emptyset\}$.

1.9. Theorem. *If a system* $\mathbf{A} = \{A_\iota : \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *of partitions in a set G is completely permutable [finitely permutable], then the following conditions are true.*

(A.1Z) *For any* $K \subseteq \Gamma$ *with* card $K \geq 2$ *and* $G_K = \bigcap_{\iota \in K} \bigcup A_\iota \neq \emptyset$ *it holds that* $\{A_\iota \sqcap G_K : \iota \in K\}$ *is a completely permutable [finitely permutable] system of partitions on* G_K;

(A.2Z) $\bigcup A_\alpha$ *respects the partitions* A_β, $\alpha, \beta \in \Gamma$.

Note. Let $\overline{\bigvee}$ be the symbol for supremum in the lattice $P(G_K)$. Then for $\emptyset \neq \Lambda \subseteq K$ there holds $G_\Lambda \sqcap \bigvee_{\iota \in \Lambda} A_\iota = \overline{\bigvee}_{\iota \in \Lambda} (G_\Lambda \sqcap A_\iota)$. Analogously for infimum.

The result follows after some easy manipulation.

Proof of Theorem will be-carried out for the complete permutability. The case of finite permutability is analogous.

First, from 1.6 it follows that A_α and A_β commute ($\alpha, \beta \in \Gamma$). Suppose that $\bigcup A_\beta$ does not respect A_α, i.e. that for some $x, y \in G$ it holds $x, y \in A_\alpha^1 \in \bigcup A_\alpha / A_\alpha$, $x \in \bigcup A_\beta$, $y \bar{\in} \bigcup A_\beta$. Then $aA_\beta x A_\alpha y$ for some $a \in G$ and thus $aA_\alpha A_\beta y$, which means that $y \in \cup A_\beta$ – a contradiction. Thus (A.2Z) holds.

To prove (A.1Z), let us choose $\Lambda \subseteq K$ with card $\Lambda \geq 2$ and denote $A_\iota \sqcap G_K = \bar{A}_\iota (\iota \in K)$, $\bar{C}_\alpha = \bigwedge_{\substack{\iota \in \Lambda \\ \iota \neq \alpha}} \bar{A}_\iota (\alpha \in \Lambda)$. Consider $\mathfrak{A} = \{x^\iota : \iota \in \Lambda\} \subseteq G_K$ with $x^\mu (\bar{C}_\mu \overline{\vee} \bar{C}_\nu) x^\nu$, $\mu \neq \nu$, $\mu, \nu \in \Lambda$. Then $x^\mu (C_\mu \vee C_\nu) x^\nu$, $\mu \neq \nu$, $\mu, \nu \in \Lambda$, where $C_\alpha = \bigwedge_{\substack{\iota \in \Lambda \\ \iota \neq \alpha}} A_\iota$. Consequently (1.1Z) or (1.2Z) hold for the completely permutable system $\{A_\iota : \iota \in \Gamma\}$ of partitions in G. Now, (1.1Z) reads that $x \in G$ exists with $x^\iota A_\iota x$, $\iota \in \Lambda$. We shall prove that $x \in G_K$ and so it will be showed $x^\iota \bar{A}_\iota x$, $\iota \in \Lambda$. On the one hand it holds $x, x^\iota \in A_\iota^1$ for some $A_\iota^1 \in \bigcup A_\iota / A_\iota$ ($\iota \in \Lambda$). On the other hand $\Lambda \subseteq G_K$ implies $x^\iota \in \bigcup A_\alpha$, $\alpha \in K$. Since $\bigcup A_\alpha$ respects A^ι and $x^\iota \in \bigcup A_\alpha$ we have $x \in A_\iota^1 \subseteq \bigcup A_\alpha$, $\alpha \in K$, so that $x \in G_K$, whence $x^\iota \bar{A} x_\iota$, $\iota \in \Lambda$.

From the condition (1.2Z) we get an anlogous condition for the system $\{\bar{A}_\iota : \iota \in \Lambda\}$ of partitions on G_K, which by 1.7 implies that (1.1Z) for the system $\{\bar{A}_\iota : \iota \in \Lambda\}$ is true, completing the proof. □

1.10. Theorem. *A system* $\mathbf{A} = \{A_\iota : \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *of partitions in a set G is completely permutable [finitely permutable] iff the following conditions* (B1) *and* (B2) *are true.*

(B1) *Any two partitions of the system* $\mathbf{A}$ *commute*;

(B2) *If* $\Lambda \subseteq \Gamma$, card $\Lambda \geq 3$ [$\aleph_0 >$ card $\Lambda \geq 3$], $\mathfrak{A} = \{x^\iota : \iota \in \Lambda\} \subseteq G$, $x^\mu (C_\mu \vee C_\nu) x^\nu$, $\mu, \nu \in \Lambda$, $\mu \neq \nu$, *then there exists* $x \in G$ *such that* $x^\iota A_\iota x$, $\iota \in \Lambda$.

Proof will be carried out for the complete permutability. The case of finite permutability is analogous.

(B1) and (B2) imply evidently the complete permutability of $\mathbf{A}$.

Conversely, let $\boldsymbol{A}$ be completely permutable. By 1.6, (B1) holds. To prove (B2), let $\Lambda \subseteq \Gamma$ with card $\Lambda \geq 3$ and $\mathfrak{A} = \{x^\iota: \iota \in \Lambda\} \subseteq G$ with $x^\mu(C_\mu \vee C_\nu)\, x^\nu$, $\mu, \nu \in \Lambda$, $\mu \neq \nu$. If (1.1Z) is true, then $x \in G$ exists with $x^\iota A_\iota x$, $\iota \in \Lambda$. Let (1.2Z) be satisfied. We shall prove that $A^1_\alpha \cap \bigcup A_\iota \neq \emptyset$ for all $\iota \in \Lambda$ and thus that A^1_α is a block of every A_ι ($\iota \in \Lambda$). From this one derives that an arbitrary $x \in A^1_\alpha$ fulfils $x^\iota A_\iota x$, $\iota \in \Lambda$. Choose $\iota \in \Lambda$. There are $\mu, \nu \in \Lambda$ such that μ, ν and ι are different (recall that card $\Lambda \geq 3$). From $x^\mu(C_\mu \vee C_\nu)\, x^\nu$ we get $x^\mu A_\iota x^\nu$. It follows that e.g. $x^\mu \in \bigcup A_\iota$. Since $x^\mu \in A^1_\alpha$ we get then $x^\mu \in A^1_\alpha \cap \bigcup A_\iota$ which was to be proved. □

1.11. Theorem. *A system $\{A_\iota: \iota \in \Gamma\}$ with* card $\Gamma \geq 2$ *of partitions in a set G is completely permutable [finitely permutable] iff the following condition is satisfied.*

If $\Lambda \subseteq \Gamma$ with card $\Lambda \geq 2[\aleph_0 >$ card $\Lambda \geq 2]$ *and $\mathfrak{A} = \{x^\iota: \iota \in \Lambda\} \subseteq G$ with $x^\mu(C_\mu \vee C_\nu)\, x^\nu$, $\mu \neq \nu$, $\mu, \nu \in \Lambda$ are given, where $C_\alpha = \bigwedge_{\substack{\iota \in \Lambda \\ \iota \neq \alpha}} A_\iota$, then one of the following conditions* is *fulfilled*

(C1) *$x \in G$ exists such that $x^\iota A_\iota x$, $\iota \in \Lambda$, or*

(C2) card $\Lambda = 2$ *and there exist $\alpha \in \Lambda$ and $A^1_\alpha \in \bigcup A_\alpha / A_\alpha$ such that $\mathfrak{A} \subseteq A^1_\alpha$ and $A^1_\alpha \cap \bigcup A_\beta = \emptyset$ for $\beta \in \Lambda$, $\beta \neq \alpha$.*

Proof. Obviously the condition of the Theorem implies that (1.1Z) and (1.2Z) are fulfilled. It remains to prove that the complete permutability [finite permutability] implies that the condition of the Theorem is fulfilled.

Suppose that $\Lambda \subseteq \Gamma$ with card $\Lambda \geq 2$ and $\mathfrak{A} = \{x^\iota: \iota \in \Lambda\} \subseteq G$ with $x^\mu(C_\mu \vee C_\nu)\, x^\nu$, $\mu \neq \nu$, $\mu, \nu \in \Lambda$ are given. Provided card $\Lambda \geq 3$ holds, (C1) is true by Theorem 1.10. Suppose card $\Lambda = 2$ and denote $\Lambda = \{1, 2\}$. Then $x^1(C_1 \vee C_2)\, x^2$ means that $x^1(A_2 \vee A_1)\, x^2$. By 1.10, A_1 and A_2 commute and by [9] III 3.1.1(1) $A_1 \vee A_2 = A_1 A_2 \cup A_1 \cup A_2$. Then $x^1 A_1 A_2 x^2$ or $x^1 A_1 x^2$ or $x^1 A_2 x^2$. In the first case $x^1 A_1 x A_2 x^2$ for some $x \in G$, thus we have (C1). The second case leads to the relation $x^1, x^2 \in A^1_1$ for some block $A^1_1 \in \bigcup A_1 / A_1$. If $A^1_1 \cap \bigcup A_2 \neq \emptyset$, then since $\bigcup A_1$ respects A_1 we have $A^1_1 \subseteq \bigcup A_2$ and the condition (C1) is fulfilled for an arbitrary $x \in \{x^1, x^2\}$. Analogously for the third case. □

1.11. Example of an associable (and therefore completely permutable) system $\{A_\iota: \iota \in \Gamma\}$ of partitions in the set $G = \{1, 2, 3, \ldots, 11\}$, $\Gamma = \{1, 2, 3, 4\}$ (for which, in addition, $G_\Lambda \neq G_K$ whenever $\Lambda \neq K$, $\Lambda, K \subseteq \Gamma$, is satisfied).

All blocks of any partition A_1 to A_4 are singletons:

A_1:	1	2	3	4		6	7	8			
A_2:	1	2	3		5	6			9	10	
A_3:	1	2		4	5		7		9		11
A_4:	1		3	4	5			8		10	11

$G_{\{1,2,3,4\}} = \{1\}$, $G_{\{1,2,3\}} = \{1, 2\}$, $G_{\{1,2,4\}} = \{1, 3\}$, $G_{\{1,3,4\}} = \{1, 4\}$, $G_{\{2,3,4\}} = \{1, 5\}$, $G_{\{1,2\}} = \{1, 2, 3, 6\}$, $G_{\{1,3\}} = \{1, 2, 4, 7\}$, $G_{\{1,4\}} = \{1, 3, 4, 8\}$, $G_{\{2,3\}} = \{1, 2, 5, 9\}$, $G_{\{2,4\}} = \{1, 3, 5, 10\}$, $G_{\{3,4\}} = \{1, 4, 5, 11\}$.

Evidently for card $\Lambda = 1$ the sets G_Λ differ from one another and from the preceding ones, as well.

2. The definition of a completely permutable system of congruence relations in an algebra (G, Ω) can be formulated analogously as in Definition 1.1 with the distinction that in the join $C_\mu \vee C_\nu$ under $\vee$ there is to understand the supremum $\vee_{\mathscr{K}}$ in the lattice $\mathscr{K}(G)$. Unfortunately, suprema in $\mathscr{K}(G)$ do not coincide with those in $P(G)$ hence the lattice $\mathscr{K}(G)$ is not a sublattice of the lattice $P(G)$. Consequently, the theory concerning partitions cannot be applied directly to congruence relations. Thus it is useful to study conditions which assure "the closedness" of some subsets of $\mathscr{K}(G)$ in $P(G)$. In the following we point out some cases.

2.1. Definition. *Let* $\{A_\iota\colon \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *be a system of congruence relations in a universal algebra* (G, Ω). *This system is called* completely permutable [finitely permutable] *if the system of corresponding partitions in the set G is completely permutable [finitely permutable] according to the lattice* $\mathscr{K}(G)$ *of congruence relations in the algebra* (G, Ω). (Now, in the Definition 1.1 under $\vee$ supremum $\vee_{\mathscr{K}}$ in $\mathscr{K}(G)$ is meant.)

Under certain conditions the permutability of congruence relations can be related to the lattice $P(G)$. Later, one of these will be given.

2.2. Proposition ([9] I 1.2) *Let* (G, Ω) *be an algebra and* $\{A_\iota\colon \iota \in \Gamma\} \subseteq \mathscr{K}(G)$. *Then* $\bigvee_{\mathscr{K}\, \iota \in \Lambda} A_\iota = \bigvee_{P\, \gamma} B_\gamma$, *where by* B_γ *there is meant the congruence relation* $A_{\iota_1} \vee_{\mathscr{K}} \dots \vee_{\mathscr{K}} A_{\iota_n}$ *for an arbitrary finite choice* $A_{\iota_1}, \dots, A_{\iota_n}$ *in* $\{A_\iota\colon \iota \in \Lambda\}$. □

2.3. Theorem ([9] I 1.2.0) *Let* (G, Ω) *be an algebra and* $\{A_\iota\colon \iota \in \Lambda\}$ *an up-directed subset of the lattice* $\mathscr{K}(G)$. *Then* $\bigvee_{\mathscr{K}\, \iota \in \Lambda} A_\iota = \bigvee_{P\, \iota \in \Lambda} A_\iota$. □

2.4. Corollary. *Let* (G, Ω) *be an algebra and* $\mathbf{A} = \{A_\iota\colon \iota \in \Gamma\} \subseteq \mathscr{K}(G)$. *Let every finite subset* Λ *of* $\mathbf{A}$ *be up-directed. Then* $\bigvee_{\mathscr{K}\, \iota \in \Lambda} A_\iota = \bigvee_{P\, \iota \in \Lambda} A_\iota$.

Proof follows from 2.3 and 2.2. □

2.5. Proposition ([9] IV 4.8.1 (b)) *If A is a congruence relation in an* Ω*-group G and H a subgroup of the additive group G, then H respects the partition A iff* $A(0) \subseteq H$ *(where* $A(0)$ *is the block of the partition A containing the neutral element* 0 *of the group G).* □

2.6. Theorem. *Let a system* $\{A_\iota\colon \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *of congruence relations in an* Ω*-group* (G, Ω) *be completely permutable [finitely permutable]. Then the following conditions* a) *and* b) *are true*

a) $A_\alpha(0) \subseteq \bigcup A_\beta$, $\alpha, \beta \in \Gamma$;

b) *For every* $K \subseteq \Gamma$ *with* card $K \geq 2$ [$\aleph_0 >$ card $K \geq 2$] *the system* $\{G_K/(A_\iota(0))\colon \iota \in K\}$ *of congruence relations on* $G_K = \bigcap_{\iota \in K} \bigcup A_\iota$ *is completely permutable [finitely permutable].*

Proof. The condition a) expresses the requirement (A.2Z), Theorem 1.9 (see also 2.5). As for the condition b), there holds $G_K/(A_\iota(0)) = A_\iota \sqcap G_K$. Reference to a) shows that $A_\iota(0) \subseteq G_K$. Thus, our conditions a) and b) express the conditions (1.1Z) and (1.2Z) of Theorem 1.9 provided G is an Ω-group. Finally, for the congruence relations "on" (on G_K) the equality $\vee_P = \vee_{\mathscr{C}}$ is valid. □

Part II of the present paper will contain applications of the results discussed in this Part I.

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