Archivum Mathematicum

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Archivum Mathematicum, Vol. 26 (1990), No. 1, 55--63

Persistent URL: `http://dml.cz/dmlcz/107369`

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 26, No. 1 (1990), 55–64

ASYMPTOTIC AND OSCILLATORY BEHAVIOR OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENTS

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(Received September 25, 1985)

Abstract. We study the asymptotic behavior of solutions of the differential equation $u^{(n)}(t) + f(t, u(\sigma(t))) = h(t)$ with advanced arguments which extend some earlier results of the authors. We also establish a necessary and sufficient condition that all solutions are oscillatory when n is even and are either oscillatory or strongly monotone when n is odd.

Key words. Ordinary differential equations, advanced arguments, asymptotic behavior, oscillatory criteria.

MS Classification. 34 K 15.

§1 INTRODUCTION

The purpose of this paper is to study the asymptotic and oscillatory behavior of solutions of the non-linear differential equation with advanced argument

$$u^{(n)}(t) + f(t, u(\sigma(t))) = h(t), \tag{1}$$

where $f \in C([0, \infty) \times R, R)$ and satisfies conditions which guarantee the existence of solutions of (1) on $[t_0, \infty)$, $t_0 \geqq 0$, $h \in C([0, \infty), R)$ and $\sigma(t) \geqq t \geqq 0$. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros. Otherwise it is called nonoscillatory. A nonoscillatory solution is said to be strongly monotone if it tends monotonically to zero as $t \to \infty$ together with its first $n - 1$ derivatives.

Recently the authors [1] generalized results obtained earlier by Cohen [3], Tong [8] and Singh [7] for ordinary differential equations to delay differential equations of the form (1) with retarded arguments. Here we present several results some of which further extend our results to advanced arguments.

§2 MAIN RESULTS

We shall need the following two lemmas. The first lemma can be proved easily and the second lemma is due to Kiguradze [5].

Lemma 1. *Let $u(t)$ and $g(t)$ be nonnegative, real-valued continuous functions $[0, \infty)$ such that*

$$u(t) \leqq u_0 + \int_{t_0}^{t} g(s)\, u^{\alpha}(s)\, ds, \qquad 0 < \alpha \leqq 1,$$

for u_0 as a positive constant and $t \geqq t_0$. Then for $t \in [0, \infty)$, $t \geqq t_0$ we have

$$u(t) \leqq [u_0^{1-\alpha} + (1 - \alpha) \int_{t_0}^{t} g(s)\, ds]^{\frac{1}{1-\alpha}}, \qquad 0 < \alpha < 1$$

and

$$u(t) \leqq u_0 \exp\left(\int_{t_0}^{t} g(s)\, ds\right), \qquad \alpha = 1.$$

Lemma 2. *If $u(t), u'(t), \ldots, u^{(n-1)}(t)$ are absolutely continuous and constant* sign *on the interval $[t_0, \infty)$ and $u^{(n)}(t)\, u(t) \leqq 0$, then there exists an integer l, $0 \leqq l \leqq \leqq n - 1$ which is even if n is odd and odd if n is even such that*

$$| u(t) | \geqq \frac{(t - t_0)^{n-1}}{(n - 1) \ldots (n - l)} | u^{(n-1)}(2^{n-l-1} t) |, \qquad t \geqq t_0.$$

Theorem 1. *Assume that the following hold:*

(i) *$p(t)$ is a continuous and nonnegative function on $[0, \infty)$ and $p(t) > 0$ for $t > 0$,*

(ii) $\int_{1}^{\infty} (\sigma(s))^{\alpha(n-1)} p(s)\, ds < \infty$, $0 < \alpha \leqq 1$,

(iii) $| f(t, u(\sigma(t)) | \leqq p(t) \, | u(\sigma(t)) |^{\alpha}$, $0 < \alpha \leqq 1$,

(iv) $\int^{\infty} | h(s) | \, ds < \infty$.

Then equation (1) has

(a) *solutions which are asymptotic to the solutions of $u^{(n)}(t) = 0$ as $t \to \infty$,*

(b) *solutions which are also asymptotic to γt^{n-1}, $\gamma \neq 0$ provided $\alpha = 1$.*

Proof (a). Applying Taylor's theorem for $t \geqq 1$, we have

$$u(t) = \sum_{j=0}^{n-1} \frac{u^{(j)}(1)}{j!} (t - 1)^j + \frac{1}{(n - 1)!} \int_{1}^{t} (t - s)^{n-1} u^{(n)}(s)\, ds.$$

With appropriate choice of constants $c_0, c_1, \ldots, c_{n-1}$ and $t > 1$, we get

$$|u(t)| \leqq \left(\sum_{j=0}^{n-1} |c_j|\right) t^{n-1} + \frac{t^{n-1}}{(n-1)!} \int_1^t |u^{(n)}(s)|\, ds \leqq \tag{2}$$

$$\leqq ct^{n-1} + \frac{t^{n-1}}{(n-1)!} \int_1^t |h(s)|\, ds + \frac{t^{n-1}}{(n-1)!} \int_1^t p(s)\, |u(\sigma(s))|^\alpha\, ds,$$

where $\sum_{j=0}^{n-1} |c_j| = c$, $0 < \alpha \leqq 1$.

Now replacing t by $\sigma(t)$, it follows that

$$|u(\sigma(t))| \leqq c(\sigma(t))^{n-1} + \frac{(\sigma(t))^{n-1}}{(n-1)!} \int_1^{\sigma(t)} |h(s)|\, ds +$$

$$+ \frac{(\sigma(t))^{n-1}}{(n-1)!} \int_1^{\sigma(t)} p(s)\, |u(\sigma(s))|^\alpha\, ds.$$

From the above inequality, we have

$$\frac{|u(\sigma(t))|}{(\sigma(t))^{n-1}} \leqq c + \frac{1}{(n-1)!} \int_1^{\sigma(t)} |h(s)|\, ds + \frac{1}{(n-1)!} \int_1^{\sigma(t)} p(s)\, |u(\sigma(s))|^\alpha\, ds \leqq$$

$$\leqq k + \frac{1}{(n-1)!} \int_1^{\sigma(t)} p(s)\, |u(\sigma(s))|^\alpha\, ds \text{ (using (iv))}$$

$$\leqq k + \int_1^{\sigma(t)} p(s)\, (\sigma(s))^{\alpha(n-1)} \frac{|u(\sigma(s))|^\alpha}{(\sigma(s))^{\alpha(n-1)}}\, ds,$$

where $k = c + \frac{1}{(n-1)!} \int_1^\infty |h(s)|\, ds$.

Applying Lemma 1, we get

$$\frac{|u(\sigma(t))|}{(\sigma(t))^{n-1}} \leqq \left[k^{1-\alpha} + \frac{(1-\alpha)}{(n-1)!} \int_1^{\sigma(t)} (\sigma(s))^{\alpha(n-1)}\, p(s)\, ds\right]^{\frac{1}{1-\alpha}}, \tag{3}$$

and hence

$$\frac{|u(\sigma(t))|}{(\sigma(t))^{n-1}} \leqq M \tag{4}$$

n view of (ii), for all $t \geqq 1$ and $0 < \alpha \leqq 1$.

Furthermore

$$\int_1^\infty |f(s, u(\sigma(s)))|\, ds \leqq \int_1^\infty p(s)\, |u(\sigma(s))|^\alpha\, ds \leqq$$

$$\leqq M^\alpha \int_1^\infty (\sigma(s))^{\alpha(n-1)}\, p(s)\, ds < \infty.$$

Now integrating (1) from 1 to t, we get

$$u^{(n-1)}(t) = u^{(n-1)}(1) - \int_1^t f(s, u(\sigma(s)))\,ds + \int_1^t h(s)\,ds.$$

Set $u^{(n-1)}(1) + \int_1^\infty h(s)\,ds = c_2$ and choose t_0 large enough so that

$$M^\alpha \int_{t_0}^\infty p(s)\,(\sigma(s))^{(n-1)\alpha} ds < c_2, \qquad \text{then } \lim_{t\to\infty} u^{(n-1)}(t) \neq 0.$$

(b) Now for $\alpha = 1$, it follows from (2)

(5) $$\frac{|u(t)|}{t^{n-1}} \leqq k + \frac{1}{(n-1)!} \int_1^\infty p(s)\,|u(\sigma(s))|\,ds \leqq$$

$$\leqq k + \frac{M}{(n-1)!} \int_1^\infty (\sigma(s))^{n-1}\, p(s)\,ds \leqq k_1 \qquad \text{in view of (ii) for some } k_1 > 0.$$

Integrating (1) from t_1 to t with $t_1 > 1$, it follows

$$u^{(n-1)}(t) \leqq u^{(n-1)}(t_1) + \int_{t_1}^t M p(s)\,(\sigma(s))^{n-1}\,ds + \int_{t_1}^t |h(s)|\,ds$$

and as $t \to \infty$,

$$u^{(n-1)}(t) \leqq u^{(n-1)}(t_1) + M \int_{t_1}^\infty p(s)\,(\sigma(s))^{n-1}\,ds + \int_{t_1}^\infty |h(s)|\,ds.$$

For some $k_2 > 0$, set $u^{(n-1)}(t_1) + \int_{t_1}^\infty |h(s)|\,ds = \frac{k_2}{2}$ and choose t_1 large enough so that $M \int_{t_1}^\infty p(s)\,(\sigma(s))^{n-1}\,ds \leqq \frac{k_2}{2}$, then $u^{(n-1)}(t) \leqq k_2$. Hence $\lim_{t\to\infty} u^{(n-1)}(t)$ exists and ia a nonzero constant. Moreover, $|u(t)| \leqq k_1 t^{n-1}$ will make $u(t)$ asymptotic to γt^{n-1}, $\gamma \neq 0$.

Example 1. Consider the third order equation

(6) $$u'''(t) + t^{-5} u^{1/2}(t + \pi) = t^{-4}, \qquad t > 0.$$

Now $f(t, u(\sigma(t))) = t^{-5} u^{1/2}(t + \pi)$, so that $p(t) = t^{-5}$, $\sigma(t) = t + \pi$, $h(t) = t^{-4}$ and $\alpha = \frac{1}{2}$. The hypothesis of Theorem 1 are satisfied with $\int_1^\infty h(t)\,dt < \infty$. The conclusion of Theorem 1 (a) therefore holds. A solution of the given equation is given by $u(t) = (t - \pi)^2$.

Example 2. Consider the fourth-order equation

(7) $$u^{iv}(t) + e^{-t}(t + \pi)^{-3}\, u(t + \pi) = e^{-t}, \qquad t \geqq 0.$$

$$|f(t, u(\sigma(t)))| = \left|\frac{e^{-t}}{(t+\pi)^{3}} u(t+\pi)\right| = \frac{e^{-t}}{(t+\pi)^3} |u(t+\pi)|,$$

$$p(t) = \frac{e^{-t}}{(t+\pi)^3}, \qquad \sigma(t) = t + \pi, \qquad h(t) = e^{-t} \qquad \text{and} \qquad \alpha = 1.$$

Again the hypothesis of Theorem 1 are satisfied and the conclusion (b) of Theorem 1 holds. A solution of the equation is given by $u(t) = t^3$.

Example 3. Consider the n-th order equation

$$u^{(n)}(t) + t^{-(n+2)} u^{1/2}(t+\pi) = e^{-t}, \tag{8}$$

$$|f(t, u(\sigma(t)))| < t^{-(n+2)} |u(t+\pi)|^{1/2},$$

so that

$$p(t) = t^{-(n+2)}, \qquad \sigma(t) = t + \pi, \qquad \alpha = \frac{1}{2} \qquad \text{and} \qquad h(t) = e^{-t}.$$

The hypothesis of Theorem 1 are satisfied and the conclusion therefore implies that there exist solutions which are asymptotic to the solutions of $u^{(n)}(t) = 0$ as $t \to \infty$.

Theorem 2. *Assume that $\varphi(t)$ is a nonnegative continuous function on $[0, \infty)$ and $g(u) > 0$ is continuous for $u > 0$ and nondecreasing on $[0, \infty)$ such that the following hold:*

(v) $\int^{\infty} \varphi(s)\,ds < \infty$,

(vi) $\int^{\infty} |h(s)|\,ds < \infty$,

(vii) $|f(t, u(\sigma(t))| \leqq \varphi(t)\, g\left(\frac{|u(\sigma(t))|}{(\sigma(t))^{n-1}}\right)$.

Then the conclusion of Theorem 1(a) *holds.*

Proof. Following the proof of Theorem 1 and using the hypothesis, we obtain

$$\frac{|u(\sigma(t))|}{(\sigma(t))^{n-1}} \leqq k + \int_1^{\sigma(t)} \varphi(s)\, g\left(\frac{|u(\sigma(s))|}{(\sigma(s))^{n-1}}\right) ds.$$

Applying Bihari's lemma [2], we get

$$\frac{|u(\sigma(t))|}{(\sigma(t))^{n-1}} \leqq G^{-1}[G(k) + \int_1^{\sigma(t)} \varphi(s)\,ds],$$

where $G(\omega) = \int_1^{\omega} \frac{ds}{g(s)}$ and G^{-1} is the inverse of G. Now using hypothesis (v), we see that

$$\frac{|u(\sigma(t))|}{(\sigma(t))^{n-1}} \leqq M \qquad \text{for some } M > 0 \text{ and all } t \geqq 1$$

and hence

$$\int_1^{\infty} | f(s, u(\sigma(s)) | \mathrm{d}s < \infty.$$

The remaining proof is similar to that of Theorem 1.

Remark. In Theorem 2, the choice $g(u) = | u |^{\alpha}$, where α is any positive number, is permitted. In particular, if we choose $g(u) = | u |^{\alpha}$ where $\alpha > 1$, then we still have the same conclusion provided the equation (1) has solutions that exist on $[T, \infty)$ for any $T > 0$.

The proof in the following theorem is similar to the method by Sevelo and Vareh [6] for even order linear delay equations.

Theorem 3. *Suppose there exists a continuous function $p(t)$ on $[0, \infty)$ and $p(t) > 0$ for $t > 0$, $\beta < 1$ such that $f(t, u) > 0$, if $u > 0$, $f(t, u) < 0$, if $u < 0$,*

$$| f(t, u) | \geqq p(t) | u |^{\beta}, \qquad (t, u) \in [0, \infty) \times R,$$

and there is a function $\varrho(t)$ such that

$$\varrho^{(n)}(t) = h(t) \qquad \text{with} \qquad \lim_{t \to \infty} \varrho^{(i)}(t) = 0 \qquad \text{for } 0 \leqq i \leqq n - 1.$$

If

$$\int^{\infty} t^{\beta(n-1)} p(t) \, \mathrm{d}t = \infty,$$

then every solution of (1) is oscillatory in the case n is even and is either oscillatory or strongly monotone in the case n is odd.

Proof. Let n be even and $u(t)$ be a nonoscillatory solution of (1). We assume that $u(t) > 0$ for large t. Set $u(t) = y(t) + \varrho(t)$, then $u(\sigma(t)) = y(\sigma(t)) + \varrho(\sigma(t))$ and

$$y^{(n)}(t) = -f(t, u(\sigma(t))).$$

Now $y^{(n)}(t) < 0$ for large t due to a condition in the theorem. Hence $y^{(n-1)}(t)$ is decreasing and so the derivatives of $y(t)$ of orders up to $(n - 1)$ are eventually of constant sign, the odd order derivatives being eventually positive. Hence

$$y'(t) > 0 \qquad \text{and} \qquad y(t) \qquad \text{is increasing for large } t.$$

Using Kiguradze's Lemma,

$$y(t) \geqq y(2^{(l-n+1)}t) \geqq \frac{2^{(l-n+1)(n-1)}}{(n-1) \dots (n-l)} (t - t_0)^{n-1} y^{(n-1)}(t)$$

for $t \geq t_0$ provided t_0 is sufficiently large. Hence if

$$k = \frac{2^{(l-n+1)(n-1)}}{(n-1)\dots(n-l)},$$

then

$$y(t) \geqq kt^{n-1}y^{(n-1)}(t), \qquad t \geqq 2t_0.$$

Since $\sigma(t) \geqq t$ and $y(t)$ is increasing for large t, there exists t_1 such that

$$y(\sigma(t)) \geqq y(t) \geqq kt^{n-1}y^{(n-1)}(t) \qquad \text{for } t \geqq t_1.$$

Moreover, since $\lim_{t\to\infty} \varrho^{(i)}(t) = 0$ for $0 \leqq i \leqq n-1$ and $u(t) = y(t) + \varrho(t)$, for large t, $u^{(n-1)}(t) \geqq y^{(n-1)}(t)$, so

$$y^{(n)}(t) + k^{\beta}t^{\beta(n-1)}p(t)\left[y^{(n-1)}(t)\right]^{\beta} \leqq y^{(n)}(t) + p(t)\left[y(\sigma(t))\right]^{\beta} \leqq$$
$$\leqq y^{(n)}(t) + p(t)\left[u(\sigma(t))\right]^{\beta} \leqq y^{(n)}(t) + f(t, u(\sigma(t))) = 0.$$

Dividing the inequality by $\left[y^{(n-1)}(t)\right]^{\beta}$ and integrating from t_1 to t, we obtain

$$\left.\frac{\left[y^{(n-1)}(s)\right]^{1-\beta}}{1-\beta}\right|_{t_1}^{t} + k^{\beta}\int_{t_1}^{t} t^{\beta(n-1)}p(t)\,\mathrm{d}t \leqq 0.$$

For large enough t, we see that

$$\int_{t_1}^{\infty} t^{\beta(n-1)}p(t)\,\mathrm{d}t < \infty \qquad \text{which is a contradiction.}$$

Now let n be odd and assume the existence of a nonoscillatory solution $u(t)$. If $u(t)$ does not approach zero as $t \to \infty$, then $y(t)$ does not approach zero as $t \to \infty$, since $u(t) = y(t) + \varrho(t)$.

Now

$$|y(t)| = \left|\frac{y(t)}{y(2^{l-n+1}t)}\right| \cdot |y(2^{l-n+1}t)|$$

and an application of Kiguradze's Lemma to $|y(2^{l-n+1})t)|$ yields with the increasing property of $y(t)$,

$$|y(\sigma(t))| \geq |y(t)| \geq mkt^{n-1}|y^{(n-1)}(t)|,$$

where

$$m = \inf_{t \geq t_0}\left|\frac{y(t)}{y(2^{l-n+1}t)}\right|.$$

The proof now follows in the same way as for n even. It follows that if a nonoscillatory solution exists then it approaches zero as $t \to \infty$. Hence $\lim_{t\to\infty} u^{(i)}(t) = 0$,

$i = 1, 2, \ldots, n-1$ monotonically. If $u(t) < 0$ then the proof can be constructed similarly.

Theorem 4. *Suppose there exists a continuous function $p(t)$ on $[0, \infty)$, $p(t) > 0$, $\gamma < 1$ and $f(t, u)$, $h(s)$ satisfy conditions of Theorem 3 such that*

(i) $| f(t, u) | \leqq p(t) | u |^{\gamma}$,

(ii) $\int^{\infty} | h(s) | \mathrm{d}s < \infty$.

Then a necessary and sufficient condition that every solution of (1) *be oscillatory if n is even and be either oscillatory or strongly monotone if n is odd is that*

$$\int^{\infty} [\sigma(t)]^{\gamma(n-1)} p(t) \, \mathrm{d}t = \infty.$$

Proof. Suppose (1) is oscillatory if n is even and is either oscillatory or strongly monotone if n is odd and

$$\int^{\infty} [\sigma(t)]^{\gamma(n-1)} p(t) \, \mathrm{d}t = \infty \qquad \text{does not hold,}$$

then by Theorem 1, equation (1) has a nonoscillatory solution $u(t)$ which are asymptotic to the solutions of $u^{(n)}(t) = 0$ as $t \to \infty$. Hence (1) is not oscillatory, and also not strongly monotone.

Conversely suppose

$$\int^{\infty} [\sigma(t)]^{\gamma(n-1)} p(t) \, \mathrm{d}t = \infty,$$

then by Theorem 3, every solution of (1) is oscillatory if n is even and is either oscillatory or strongly monotone if n is odd. The proof is complete.

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