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ON LIAPUNOV TRANSFORMATIONS OF LINEAR SYSTEMS OF IMPLICIT DIFFERENTIAL EQUATIONS

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Abstract. An asymptotical equivalence of linear systems of implicit differential equations is investigated. Sufficient conditions to transform a linear system of implicit differential equations to a system of special type are given

We shall consider a system of implicit linear differential equations

$$A(t)\dot{x} + B(t)x = 0, \quad t \in R_+ = [0, +\infty] \tag{1}$$

where $A(t)$, $B(t)$ are $n \times n$ matrix functions of a real variable t and x is an n-dimensional vector-function. We shall call an $n \times n$ matrix $X(t)$ a fundamental solution matrix of the system (1) if for any constant k-dimensional vector c the function

$$x(t) = X(t)c \tag{2}$$

is a solution of (1) and any solution $x(t)$ of (1) has the form (2) with some constant vector c. We shall also call the system (1) asymprotically equivalent to the system

$$P(t)\dot{y} + Q(t)y = 0, \quad t \in R_+, \tag{3}$$

if there is an $n \times n$ Liapunov matrix $L(t)$ (see [1]) such that for any solution $y(t)$ of (3) the function $x(t) = L(t)y(t)$ is a solution of (1) and for any solution $x(t)$ of (1) the function $y(t) = L^{-1}(t)x(t)$ is a solution of (3). It follows (see [5]) that the system (1) is asymptotically equivalent to the system (3) if and only if there is a fundamental solution matrix $X(t)$ of (1) such that $X(t) = L(t)Y(t)$ where $L(t)$ is a Liapunov matrix and $Y(t)$ is a fundamental solution matrix of (3).

We shall consider the systems (1) and (3) when the matrices $A(t)$ and $P(t)$ are singular on R_+, i.e., $\det A(t) = \det P(t) = 0$ for all $t \geq 0$.

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Theorem. *Let A and B be absolutely continuous and bounded together with their first derivatives almost everywhere on R_+. Let also there exist a $k \times k$ submatrix $A_0(t)$ of the matrix $A(t)$ such that*

$$\inf_{t\geq 0} |\det A_0(t)| > 0, \tag{4}$$

$$\operatorname{rank} A(t) = k \quad \text{for all } t \geq 0, \tag{5}$$

$$\inf_{t\geq 0} \left| \frac{\partial^k}{\partial \lambda^k} \det(A(t)\lambda + B(t)) \right| > 0. \tag{6}$$

Then the system (1) is asymptotically equivalent to the system (3) where the matrices $P(t)$ and $Q(t)$ have the forms

$$P = \begin{pmatrix} O_1 & O_2 \\ O_3 & E_0 \end{pmatrix}, \quad Q = \begin{pmatrix} E_1 & O_2 \\ O_3 & Q_0 \end{pmatrix}, \tag{7}$$

O_1, O_2, O_3 are zero $(n-k) \times (n-k)$, $(n-k) \times k$, $k \times (n-k)$ matrices, E_1, E_0 are identity $(n-k) \times (n-k)$, $k \times k$ matrices, respectively, and Q_0 is a $k \times k$ matrix summable and bounded a.e. on R_+.

Proof. Without losing generality we may assume that the submatrix A_0 is in the right-down corner of the matrix A. Then (4), (5) imply that the matrix A can be written in the form

$$A = \begin{pmatrix} C_1 A_0 C_3 & C_1 A_0 \\ A_0 C_3 & A_0 \end{pmatrix}$$

where C_1 and C_3 are respectively $(n-k) \times k$ and $k \times (n-k)$ matrices absolutely continuous and bounded together with their first derivatives a.e. on R_+.

Denote

$$T = \begin{pmatrix} E_1 & -C_1 \\ O_3 & A_0^{-1} \end{pmatrix}, \quad S = \begin{pmatrix} E_1 & O_2 \\ -C_3 & E_0 \end{pmatrix}.$$

Multiplying (1) on the left by the nonsingular matrix T and making the change of variables $x = Sz$ we obtain the system

$$P(t)\dot{z} + F(t)z = 0 \tag{8}$$

where P is defined by (7) and

$$F = T(A\dot{S} + BS). \tag{9}$$

It is easy to check that S is a Liapunov matrix and thus (8) is asymptotically equivalent to (1).

We can consider the system (8) as two systems

$$F_1(t)z_1 + F_2(t)z_2 = 0 \tag{10}$$

$$\dot{z}_2 + F_3(t)z_1 + F_0(t)z_2 = 0 \tag{11}$$

where

$$F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_0 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{12}$$

F_1, F_2, F_3, F_0 are $(n-k)\times(n-k)$, $(n-k)\times k$, $k\times(n-k)$, $k\times k$ matrices, z_1, z_2 are $(n-k)$- and k-dimensional vectors respectively.

It is seen from (9) that

$$F = G + TBS \tag{13}$$

where

$$G = PS^{-1}\dot{S} = \begin{pmatrix} O_1 & O_2 \\ -C_3 & O_0 \end{pmatrix}, \tag{14}$$

O_0 is zero $k\times k$ matrix.

From (9), (12), (13) we obtain

$$\det(A\lambda + B) = \det T^{-1}\det(P\lambda + F - G)\det S^{-1} = \det A_0 \det(P\lambda + F - G). \tag{15}$$

The relations (7), (12), (14) imply

$$\det(P\lambda + F - G) = \lambda^k \det F_1 + \sum_{i=0}^{k-1} f_i\lambda^i \tag{16}$$

where f_i, $i = 1, \ldots, k-1$, are functions of real variable t.

Hence taking (5), (15), (16) into account we have

$$\inf_{t\geq 0} |\det F_1(t)| > 0$$

and thus from (10) and (11) we get

$$z_1 = -F_1^{-1}F_2 z_2, \tag{17}$$

$$\dot{z}_2 = (F_3F_1^{-1}F_2 - F_0)z_2. \tag{18}$$

Let z_2 be a fundamental solution matrix of the system (18). Then

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

is a fundamental solution matrix of the system (8) where, in view of (17),

$$Z_1 = -F_1^{-1} F_2 Z_2. \tag{19}$$

Consider the system (3) where P, Q are given by (7) with $Q_0 = F_0 - F_3 F_1^{-1} F_2$ that is summable and bounded a.e. on R_+. If Y is a fundamental solution of (3) then there exists a constant $k \times k$ matrix C_0 such that (see (19))

$$Y = LZC_0 = \begin{pmatrix} E_1 & F_1^{-1}F_2 \\ O_3 & E_0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} C_0.$$

From (9), (13), (14) we have that F_1, F_2 are absolutely continuous and bounded together with their first derivatives a.e. on R_+. Thus L is a Liapunov matrix. Consequently, (3) is asymptotically equivalent to (8) and hence (3) is a asymptotically equivalent to (1). The theorem is proved. □

Corollary. *If A and B satisfy the conditions of Theorem then the system (1) is asymptotically equivalent to a system with piecewise constant coefficients. Moreover, the sequence (t_n), where t_n are points of discontinuity of coefficients, is a subsequence of the sequence (ml), where l is a positive number, $m = 1, 2, 3, \ldots$.*

Proof. By Theorem the system (1) is asymptotically equivalent to the system (3) where P and Q are of the form (7). Let

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where y_1, z_1 are $(n-k)$-dimensional, y_2, z_2 are k-dimensional vectors. Then (see [3]) the system

$$\dot{y}_2 + Q_0 y_2 = 0 \tag{20}$$

is asymptotically equivalent to the system

$$\dot{z}_2 + R_0 z_2 = 0 \tag{21}$$

where R_0 is a piecewise constant $k \times k$ matrix whose points of discontinuity make a subsequence of the sequence (ml). Thus

$$Y_2 = L_0 Z_2 \tag{22}$$

where Y_2, Z_2 are fundamental solution matrices of (20) and (21) respectively and L_0 is a Liapunov matrix. From (22) we have $Y = LZ$, where

$$Y = \begin{pmatrix} O_1 \\ Y_2 \end{pmatrix}, \quad Z = \begin{pmatrix} O_1 \\ Z_2 \end{pmatrix}$$

are fundamental solution matrices of (3) and the system

$$\begin{pmatrix} O_1 & O_2 \\ O_3 & E_0 \end{pmatrix} \dot{z} + \begin{pmatrix} E_1 & O_2 \\ O_3 & R_0 \end{pmatrix} z = 0 \tag{23}$$

respectively, and

$$L = \begin{pmatrix} E_1 & O_2 \\ O_3 & L_0 \end{pmatrix}$$

is a Liapunov matrix. Thus the system (3), and hence the system (1), is asymptotically equivalent to the system (23). Corollary is proved. □

For other results connected with transformations of (1) to systems of special types see [4,5].

We close this article with some examples which show that the conditions of Theorem are only sufficient but not necessary.

Example 1. Consider the system (1) in which

$$A = \begin{pmatrix} r\cos\varphi & \varrho\cos\psi \\ r\sin\varphi & \varrho\sin\psi \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \tag{24}$$

where the functions r, ϱ, φ, ψ, b_{ij}, $i,j = 1,2$, are absolutely continuous and bounded together with their first derivatives a.e. on R_+. The condition $\det A(t) = 0$ for all $t \geq 0$ implies

$$r(t)\varrho(t)\sin(\varphi(t) - \psi(t)) = 0 \text{ for all } t \geq 0. \tag{25}$$

Let rank $A(t) = k = 1$ for all $t \geq 0$, then $R^2(t) = r^2(t) + \varrho^2(t) \neq 0$ for all $t \geq 0$. Assume that

$$\inf_{t\geq 0} R^2(t) > 0 \tag{26}$$

but there is no 1×1 submatrix A_0 for which the condition (4) is fulfilled. Let also the condition (5) and (6) be valid.

Denote $T_r = \{t \in R_+ | r(t) = 0\}$ and $T_\varrho = \{t \in R_+ | \varrho(t) = 0\}$. The relations (25), (26) imply $T_r \cap T_\varrho = \emptyset$ and $\varphi(t) = \psi(t) + k(t)\pi$ for all $t \in R_+ \setminus (T_r \cup T_\varrho)$ where $k(t)$ is a piecewise constant function which admits only integer values. It is not difficult to verify that there are piecewise constant functions $k_1(t)$, $k_2(t)$ and $k_3(t)$ admitting only integer values such that the function

$$\omega(t) = \begin{cases} \varphi(t) & +k_1(t)\pi, \quad t \in T_\varrho, \\ \psi(t) & +k_2(t)\pi, \quad t \in T_r, \\ \varphi(t) & +k_3(t)\pi, \quad t \in R_+ \setminus (T_r \cup T_\varrho) \end{cases}$$

is absolutely continuous and bounded together with its first derivatives a.e. on R_+. Define

$$\alpha(t) = \begin{cases} k_1(t), & t \in T_\varrho, \\ k_3(t), & t \in T_r, \\ a(t), & t \in R_+ \setminus (T_r \cup T_\varrho), \end{cases} \qquad \beta(t) = \begin{cases} b(t), & t \in T_\varrho, \\ k_2(t), & t \in T_r, \\ k_3(t) + k(t), & t \in R_+ \setminus (T_r \cup T_\varrho), \end{cases}$$

where $a(t)$, $b(t)$ are arbitrary functions such that $\alpha(t)$, $\beta(t)$ are continuous on R_+. Denote

$$T = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}, \quad S = \begin{pmatrix} (-1)^\alpha r & (-1)^{\beta-1}\varrho \\ (-1)^\beta \varrho & (-1)^\alpha r \end{pmatrix}.$$

The matrices T and S are absolutely continuous and bounded together with their first derivatives a.e. on R_+. Moreover, $\det T(t) = 1$ and $\det S(t) = r^2(t) + \varrho^2(t) = R^2(t)$ for all $t \in R_+$. From (26) we see that S is a Liapunov matrix and therefore the system (1) with the matrices (24) is asymptotically equivalent to the system

$$TAS\dot{y} + (TA\dot{S} + TBS)y = 0 \tag{27}$$

where

$$TAS = \begin{pmatrix} R^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $TBS = (c_{ij})$, $i, j = 1, 2$. Then $\det(TAS\lambda + TA\dot{S} + TBS) = R^2 c_{22}\lambda + f$ where f is a function of the variable t. On the other hand,

$$R^2 \det(A\lambda + B) \doteq \det T \det(A\lambda + B) \det S = \det(TAS\lambda + TBS) = R^2 c_{22}\lambda + g$$

where g is a function of the variable t. Thus the conditions (6) and (26) imply

$$\inf_{t \geq 0} \left| \frac{\partial}{\partial\lambda} \det(TAS\lambda + TA\dot{S} + TBS) \right| > 0.$$

Therefore, the matrices TAS and $(TA\dot{S} + TBS)$ satisfy the conditions of Theorem. Thus the system (27), and hence the system (1) with the matrices (24), is asymptotically equivalent to a system of the type (3) with P and Q of the form (7).

Example 2. Consider (1) with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\det A_0(t) = 1$ and $\operatorname{rank} A(t) = 2$ for all $t \geq 0$, i.e., the conditions (4) and (5) are fulfilled. But $\det(A\lambda + B) = \lambda + 1$ and hence the condition (6) is not valid. However, since in this case A and B are constant matrices and $\deg \det(A\lambda+B) = 1$ then (see [2, Kapitel 12]) the system (1) is asymptotically equivalent to the system $\operatorname{diag}\{0,0,1\}\dot{z} + \operatorname{diag}\{1,1,1\}z = 0$ which has the type (3) with (7).

Example 3. Now we show that the condition $\deg \det(A\lambda + B) = \text{const}$ (see [2, Kapitel 12]) is not necessary, too. Consider (1) where now

$$A = \begin{pmatrix} \sin t - \cos t & \sin t & 0 \\ \sin t & \sin t - \cos t & \cos t \\ 1 + \cos t - \sin t & -1 - \sin t & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} (\sin t + \cos t)(1 + \cos t) & \cos t(1 + \cos t) & 0 \\ \sin t + 2\cos^2 t & \cos t(2 + \cos t - \sin t) & -\cos t \\ -1 - (\sin t + \cos t)(1 + \cos t) & 1 - \cos t(1 + \cos t) & -1 \end{pmatrix},$$

Then $\det(A(t)\lambda + B(t)) = (1-\lambda)\cos t(1+\cos t)$, however the system (1) in this case has the fundamental solution matrix $X(t) = (0,0,e^t)^T$ and hence (1) is obviously asymptotically equivalent to the system $\operatorname{diag}\{0,0,1\}\dot{z} - \operatorname{diag}\{1,1,1\}z = 0$ which has the type (3) with (7).

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References

[1] Ю.С. Богданов, *Об асимптотически эквивалентных линейных дифференциальных системах*, Дифференц. уравнения, 1965, т. 1, No. 6, с. 707- -716.

[2] F.R. Gantmacher, *Matrizentheorie*, Deutscher Verlag der Wissenschaften, Berlin, 1986.

[3] С.А. Мазаник, *Об асимптотически эквивалентных дифференциальных системах*, Дифференц. уравнения, 1981, т. 17, No. 5, с. 923-926.

[4] С.А. Мазаник, *О линейных системах дифференциальных уравнений, не разрешенных относительно производных*, Доклады АН БССР, 1985, т.29, No.9, с. 784-787.

[5] С.А. Мазаник, *Приводимость систем линейных дифференциальных уравнений, не разрешенных относительно производных*, Актуальные вопросы краевых задач, Рига, 1988, с. 91-98.

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