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Archivum Mathematicum, Vol. 27 (1991), No. 3-4, 199--200

Persistent URL: http://dml.cz/dmlcz/107421

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 27 (1991), 199 – 200

A SHORT PROOF OF KY FAN'S INEQUALITY

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(Received April, 1990)

ABSTRACT. We provide a new proof of Ky Fan's inequality by presenting an identity from which the inequality immediately follows.

The celebrated Ky Fan inequality states:
If A_n and G_n (respectively A'_n and G'_n) denote the weighted arithmetic and geometric means of $x_1, \ldots, x_n$ (respectively $1 - x_1, \ldots, 1 - x_n$) with $x_i \in (0, 1/2]$, $i = 1, \ldots, n$, i.e.

$$A_n = \sum_{i=1}^{n} p_i x_i \quad \text{and} \quad G_n = \prod_{i=1}^{n} x_i^{p_i}$$

(respectively

$$A'_n = \sum_{i=1}^{n} p_i(1 - x_i) \quad \text{and} \quad G'_n = \prod_{i=1}^{n} (1 - x_i)^{p_i})$$

with $\sum_{i=1}^{n} p_i = 1$ and positive weights $p_1, \ldots, p_n$, then

$$G_n/G'_n \le A_n/A'_n \tag{1}$$

where the sign of equality holds if and only if $x_1 = \cdots = x_n$. Inequality (1) is due to Ky Fan who established (1) for the special case $p_1 = \cdots = p_n = 1/n$ by using Cauchy's method of forward and backward induction. Since its publication in 1961 in the well-known book "Inequalities" by E.F. Beckenbach and R. Bellman [1, p.5] Fan's result has evoked a considerable interest and several proofs as well as noteworthy sharpenings, extensions and inversions were given. We refer to the monograph [3, Chapter IV, §8.3] and the references therein.

"One idea in the theory of inequalities is that every inequality is a consequence of an equality" [2]. However, among the different proofs for Fan's inequality we could not localize one where an equality is given which implies (1). Inspired by a

1991 *Mathematics Subject Classification*: 26D15.
Key words and phrases: arithmetic and geometric means.

paper of A. Dinghas [4] we will provide an identity from which inequality (1) can be deduced immediately.

In order to establish the arithmetic mean-geometric mean inequality Dinghas presented a remarkable short and simple proof for

$$A_n/G_n = \exp \sum_{i=1}^{n} p_i (x_i - A_n)^2 J(x_i, A_n) \tag{2}$$

where

$$J(x,y) = \int_0^\infty \frac{u\,du}{(1+u)(x+yu)^2} \,.$$

Because of

$$A_n + A'_n = 1$$

we obtain from (2):

$$\begin{aligned}\frac{A_n}{G_n}\frac{G'_n}{A'_n} &= \exp \sum_{i=1}^{n} p_i (x_i - A_n)^2 \left[J(x_i, A_n) - J(1-x_i, 1-A_n)\right] \\ &= \exp \sum_{i=1}^{n} p_i (x_i - A_n)^2 K(x_i, A_n) \end{aligned} \tag{3}$$

where

$$K(x,y) = \int_0^\infty \frac{(1-2y)u^2 + (1-x-y)2u + 1 - 2x}{[(x+yu)(1-x+(1-y)u)]^2} \frac{u}{1+u}\, du \,.$$

Since $0 < x_i \le 1/2$, $i = 1, \ldots, n$, and $0 < A_n \le 1/2$ we conclude

$$K(x_i, A_n) \ge 0 \quad \text{for} \quad i = 1, \ldots, n \,.$$

Hence we obtain from (3)

$$\frac{A_n}{G_n}\frac{G'_n}{A'_n} \ge 1$$

where equality holds if and only if $x_1 = \cdots = x_n$.

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