## Archivum Mathematicum

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Archivum Mathematicum, Vol. 30 (1994), No. 1, 45--57
Persistent URL: http://dml.cz/dmlcz/107494

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# CURVATURE TENSORS IN DIMENSION FOUR WHICH DO NOT BELONG TO ANY CURVATURE HOMOGENEOUS SPACE 

Oldritch Kowalski and Friedbert Prüfer


#### Abstract

A six-parameter family is constructed of (algebraic) Riemannian curvature tensors in dimension four which do not belong to any curvature homogeneous space. Also a general method is given for a possible extension of this result.


## 1. Introduction

According to I.M. Singer [SI], a Riemannian manifold is said to be curvature homogeneous if, for every two points $p, q \in M$, there is a linear isometry $F: T_{p} M \rightarrow$ $T_{q} M$ between the corresponding tangent spaces such that $F^{*} R_{q}=R_{p}$ (where $R$ denotes the curvature tensor of type $(0,4)$ ). Note that a (locally) homogeneous Riemannian manifold is automatically curvature homogeneous. Explicit locally nonhomogeneous examples have been constructed by many authors ([SE],[T],[YA],[K-T-V1] - [K-T-V3],[K1] - [K3]; see especially [K-T-V2] and [K-T-V3] for more complete references).

Let $K$ be a curvature-like tensor of Riemannian type (shortly: curvature tensor) on a vector space $\mathbf{V}$ with a positive scalar product $<,>$. I. e., we assume that

$$
\begin{align*}
& K(X, Y, U, V)=-K(Y, X, U, V)=K(U, V, X, Y)  \tag{1}\\
& K(X, Y, Z, U)+K(Y, Z, X, U)+K(Z, X, Y, U)=0
\end{align*}
$$

A natural problem arises whether there is always a curvature homogeneous space $(M, g)$ such that, under a linear isometry between $\mathbf{V}$ and a tangent space $T_{o} M$, $K$ coincides with the curvature tensor $R_{o}$ of $(M, g)$ at the point $o \in M$. In such a case we say that the tensor $K$ belongs to the curvature homogeneous space ( $M, g$ ).

In dimension $n=3$, the answer to our problem is always positive, as it was proved quite recently (see [K2],[S-T],[K-P]): every curvature tensor $R$ in dimension

[^0]three belongs to some curvature homogeneous space. Moreover, if $R$ is not of the type of constant curvature, one can always find an example which is not locally homogeneous. (On the other hand, it is not always possible to construct a homogeneous Riemannian space with a prescribed curvature tensor (see [M],[ST],[K3]).)

In dimension $n=4$, the situation is essentially different. In the paper $[\mathrm{S}-\mathrm{T}]$ the following example is given: let $R_{S^{4}}$ and $R_{C P^{2}}$ denote the typical curvature tensors of a four-dimensional sphere and a complex projective plane, respectively and let $a, b$ be nonzero real numbers. Then the curvature tensor $a R_{S^{4}}+b R_{C P^{2}}$ does not belong to any curvature homogeneous space. The proof requires nontrivial results from the almost Hermitian geometry (see [T-V]). In this paper we study the problem more systematically, by elementary methods, and we give a new and broader family of curvature tensors with the above property. For our purposes we introduce the notion of $\sigma$-rank (cyclic rank) of a curvature tensor, which may be of some interest by itself.

The authors are obliged to F. Tricerri and K. Voss for valuable informations.

## 2. Generic curvature tensors and the chern bases

Let $\mathbf{V}$ be a vector space with a (positive) scalar product $<,>$ and let $\mathcal{R}$ denote the space of all curvature tensors on $\mathbf{V}$ (satisfying the identities (1)). The orthogonal group $O(\mathbf{V})$ acts on $\mathcal{R}$ in a natural way. A tensor $R \in \mathcal{R}$ is said to be generic if the subgroup $H=\{A \in O(\mathbf{V}) \mid A(R)=R\}$ is finite. Another characterization of a generic tensor $R \in \mathcal{R}$ is that its orbit in $\mathcal{R}$ under $O(\mathbf{V})$ has the maximal dimension (equal to $n(n-1) / 2$ ). Finally, $R \in \mathcal{R}$ is generic if and only if there is no nonzero skew-symmetric endomorhpism $P$ of $\mathbf{V}$ (acting as a derivation on the tensor algebra of $\mathbf{V}$ ) such that $P \cdot R=0$. Indeed, the last identity means $(\exp t P)(R)=R, t \in(-\infty,+\infty)$.

Let us consider now the four-dimensional case.
According to S.S. Chern [C], or R. Klinger [KL], $\operatorname{dim} \mathcal{R}=20$, and for every $R \in \mathcal{R}$ there exists an orthonormal basis $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ of $\mathbf{V}$ (called a Chern basis) such that the components $R_{i j k l}$ satisfy

$$
\begin{equation*}
R_{1213}=R_{1214}=R_{1223}=R_{1224}=0, R_{1314}=R_{1323}=0 \tag{2}
\end{equation*}
$$

We have also the following observations:
a) If $R \in \mathcal{R}$ is generic, then its Chern basis is uniquely determined up to a finite group of reflections.
b) If $R \in \mathcal{R}$ is an Einsteinian curvature tensor, then its Singer-Thorpe basis $\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\}$ (see [SI-TH]) is always a Chern basis.
c) For the curvature tensor $R=c g \wedge g$ of a space form, each orthonormal basis is a Chern basis.

For the purpose of this paper, we shall introduce the following notation w.r. to a

Chern basis:

$$
\begin{array}{ll}
A_{1}=R_{1212}, & A_{2}=R_{1234}, \\
B_{1}=R_{1313}, & B_{2}=R_{1324}, \quad B_{3}=R_{1334}, \\
C_{1}=R_{1414}, & C_{2}=R_{1424}, \quad C_{3}=R_{1434},  \tag{3}\\
D_{1}=R_{2323}, & D_{2}=R_{2324}, \quad D_{3}=R_{2334}, \\
E_{1}=R_{2424}, & E_{2}=R_{2434}, \\
F_{1}=R_{3434} .
\end{array}
$$

All curvature components in (3) are independent, in general, and they uniquely determine the tensor $R$ w.r. to a given basis.

Let us recall that an Einstein curvature tensor is characterized, w.r. to a SingerThorpe basis, by the equalities (2) and the equalities

$$
A_{1}=F_{1}, B_{1}=E_{1}, C_{1}=D_{1}, C_{2}=D_{2}=E_{2}=0, B_{3}=C_{3}=D_{3}=0
$$

We give now some more examples of generic curvature tensors:
d) A curvature tensor $R \in \mathcal{R}$ with four distinct Ricci eigenvalues is always generic. In particular, assume that, w.r. to a Chern basis, the components $A_{1}, B_{1}, C_{1}, D_{1}, E_{1}, F_{1}$ are the only (possible) nonzero components. It can be easily seen that the Ricci eigenvalues are all distinct if and only if the numbers $\left|A_{1}-F_{1}\right|,\left|B_{1}-E_{1}\right|,\left|C_{1}-D_{1}\right|$ are all distinct.
In particular, if $D_{1}=E_{1}=F_{1}=0$ holds but $\left|A_{1}\right|,\left|B_{1}\right|,\left|C_{1}\right|$ are all distinct, the tensor $R$ is still generic.
e) If $A_{1}=B_{1}=E_{1} \neq C_{1}=D_{1}=F_{1}$, and all other components are zero, then there are only two different Ricci eigenvalues but the curvature tensor $R$ is still generic.
To see the last result, one can use the following criterion of genericity w.r. to a Chern basis $\left\{e_{1}, \ldots, e_{4}\right\}$ :

Let $E_{i j}$ denote the elementary skew-symetric endomorphism of $\mathbf{V}$, i.e. those defined by

$$
\begin{equation*}
E_{i j}\left(e_{i}\right)=-e_{j}, E_{i j}\left(e_{j}\right)=e_{i}, E_{i j}\left(e_{k}\right)=0 \text { for } k \neq i, j, 1 \leq i<j \leq 4 \tag{4}
\end{equation*}
$$

Denote

$$
\begin{equation*}
P=\sum_{1 \leq i<j \leq 4} \alpha_{i j} E_{i j}, \tag{5}
\end{equation*}
$$

and let $P$ act as a derivation on the tensor algebra $\mathcal{T}(\mathbf{V})$. Then a tensor $R \in \mathcal{R}$ is generic if and only if $P \cdot R=0$ implies $\alpha_{i j}=0$ for all $i, j, 1 \leq i<j \leq 4$.

We conclude this Section with the following example:
f) Suppose that $B_{1}=D_{1}, C_{1}=E_{1}, A_{2}=2 B_{2}, A_{1}, \quad F_{1}$ are arbitrary, $C_{2}=D_{2}=E_{2}=0, B_{3}=C_{3}=D_{3}=0$. Then $A_{12} \cdot R=0$, and hence the corresponding tensor $R$ is not generic.

## 3. General properties of curvature homogeneous spaces

According to $[\mathrm{SI}]$ (see also $[\mathrm{N}-\mathrm{T}]$ and $[\mathrm{S}-\mathrm{T}]$ ) we have the following
Theorem A. Any curvature homogeneous space ( $M, g$ ) admits an open covering $\left\{V_{\alpha}\right\}_{\alpha \in J}$ such that, in each $V_{\alpha}$, an orthonormal moving frame $\left\{E_{1}^{\alpha}, \ldots, E_{n}^{\alpha}\right\}$ exists for which all the components of the Riemannian curvature tensor field $R$ are constant.

Hence we obtain the first of the following corollaries:
Corollary 3.1. In each $V_{\alpha}$ there exists a flat connection $\tilde{\nabla}^{\alpha}$ (with torsion) such that $\widetilde{\nabla}^{\alpha} g=\widetilde{\nabla}^{\alpha} R=0$. Here $\widetilde{\nabla}^{\alpha}$ is defined as the (unique) connection for which the vector fields $E_{1}^{\alpha}, \ldots, E_{n}^{\alpha}$ define an absolute parallelism.
Corollary 3.2. Suppose that $(M, g)$ is curvature homogeneous and the typical curvature tensor $R$ is generic. Then there is a unique global affine connection $\tilde{\nabla}$ on $(M, g)$ such that $\tilde{\nabla} g=\widetilde{\nabla} R=0$. The connection $\tilde{\nabla}$ is locally flat.

Proof. In each neighborhood $V_{\alpha}$ we choose a point $p_{\alpha}$ and a "generalized Chern basis" $\left\{e_{1}^{\alpha}, \ldots, e_{n}^{\alpha}\right\}$ at $p_{\alpha}$ in the sense of R. Klinger [KL]. Then we can extend this basis to a moving frame $\left\{E_{1}^{\alpha}, \ldots, E_{n}^{\alpha}\right\}$ in $V_{\alpha}$ for which all the components $R_{i j k l}$ are constant and each basis is a generalized Chern basis. Because $R$ is generic, such a moving frame is uniquely determined up to a finite group of reflections. Hence we see $\nabla^{\alpha}=\widetilde{\nabla}^{\beta}$ on each intersection $V_{\alpha} \cap V_{\beta}$, and we get a globally defined connection $\tilde{\nabla}$ on $(M, g)$, which is locally flat. For the uniqueness part, consider two connections $\tilde{\nabla}_{1}, \tilde{\nabla}_{2}$ on $(M, g)$ satisfying $\tilde{\nabla}_{i} g=\tilde{\nabla}_{i} R=0$. Then their difference tensor field $D$ satisfies $D_{X} \cdot g=D_{X} \cdot R=0$ for each tangent vector $X \in T M$. Suppose that $D_{X} \neq 0$ for some vector $X \in T_{m} M$. Because $D_{X}$ is a skew-symmetric endomorphism of $T_{m} M$ such that $D_{X} \cdot R=0$, we obtain a contradiction to the genericity of $R$. Hence $D=0$ and $\widetilde{\nabla}_{1}=\widetilde{\nabla}_{2}$ on $M$.

The following simple result is basic for our further investigations:
Proposition 3.3. Let $(M, g)$ be a curvature homogeneous space. Then, in a neighborhood $U_{p}$ of each point $p \in M$, there exists a tensor field $S$ of type $(1,2)$ such that

$$
\begin{align*}
S_{X} \cdot g & =0 \text { for every } X \in T_{m} M, m \in U_{p}  \tag{6}\\
\mathfrak{S}_{X, Y, Z}\left(S_{X} \cdot R\right)(Y, Z, U, V) & =0 \text { for every } X, Y, Z, U, V \in T_{m} M, m \in U_{p}
\end{align*}
$$

where $\mathfrak{S}$ denotes the cyclic sum.
Proof. Put $S_{X}=\nabla_{X}-\tilde{\nabla}_{X}$, where $\tilde{\nabla}$ is a flat connection as in Corollary 2.1 and $\nabla$ is the Levi-Civita connection. Then (6) is trivial and (7) follows from the second Bianchi identity for $\nabla R$.

## 4. The cyclic rank of a curvature tensor in dimension four

We shall go back to the four dimensional case. Consider a curvature homogeneous space ( $M, g$ ) $\operatorname{dim} M=4$, and an orthonormal moving frame $\left\{E_{1}, \ldots, E_{4}\right\}$ on some open subset $U \subset M$ for which all the curvature components are constant. We can
again assume that $\left\{E_{1}, \ldots, E_{4}\right\}$ consists of Chern bases: we choose any orthonormal moving frame $\left\{F_{1}, \ldots, F_{4}\right\}$ for which all components $R_{i j k l}$ are constant and then transform $\left\{F_{1}, \ldots, F_{4}\right\}$ in $\left\{E_{1}, \ldots, E_{4}\right\}$ by a constant orthogonal matrix. (If the curvature tensor $R$ is not generic, then the choice of the Chern adapted frame $\left\{E_{1}, \ldots, E_{4}\right\}$ is far from being unique.)

Let $S$ be the tensor field defined on $U$ by means of the moving frame $\left\{E_{1}, \ldots, E_{4}\right\}$. Here the vector fields $E_{i}$ are parallel with respect to the corresponding flat connection $\tilde{\nabla}$ and we have

$$
\begin{equation*}
\nabla_{E_{i}} E_{j}=S_{E_{i}} E_{j} \quad(i, j=1, \ldots, 4) \tag{8}
\end{equation*}
$$

Let us introduce the notation

$$
\begin{equation*}
S_{E_{i}} E_{j}=\sum_{k=1}^{4} S_{i j}^{k} E_{k} \tag{9}
\end{equation*}
$$

Obviously, $S_{X} \cdot g=0$ implies

$$
\begin{equation*}
S_{i j}^{k}+S_{i k}^{j}=0 \quad(i, j, k=1, \ldots, 4) \tag{10}
\end{equation*}
$$

Thus we have 24 independent functions $S_{i j}^{k}, 1 \leq i \leq 4,1 \leq j<k \leq 4$, which have to satisfy (7), i.e., in the classical notation

$$
\begin{equation*}
\mathfrak{S}_{i, j, k}^{\mathfrak{S}}\left(S_{i} \cdot R\right)_{j k l u}=0 \tag{11}
\end{equation*}
$$

In detail, we obtain

$$
\begin{align*}
&\left(S_{i j}^{p}-S_{j i}^{p}\right) R_{p k l u}+\left(S_{k i}^{p}-S_{i k}^{p}\right) R_{p j l u}+\left(S_{j k}^{p}-S_{k j}^{p}\right) R_{p i l u}  \tag{12}\\
&+S_{i l}^{p} R_{j k p u}+S_{j l}^{p} R_{k i p u}+S_{k l}^{p} R_{i j p u}+S_{i u}^{p} R_{j k l p} \\
&+S_{j u}^{p} R_{k i l p}+S_{k u}^{p} R_{i j l p}=0
\end{align*}
$$

( $1 \leq i<j<k \leq 4,1 \leq l<u \leq 4$ arbitrary, $p$ is a summation index).
This gives a system of 24 linear equations for the 24 unknown functions $S_{i j}^{k}$.
We shall write down all these equations explicitly using our Chern basis and thus the conditions (2) and the notation (3). We shall first distribute these equations in four groups putting $(i, j, k)=(1,2,3),(1,2,4),(1,3,4),(2,3,4)$, respectively. Then, in each group separately, we put $(l, u)=(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)$,
in this order. We obtain finally the following system:

$$
\begin{align*}
& \left(A_{1}-D_{1}\right) S_{11}^{3}-D_{2} S_{11}^{4}+\left(B_{2}-2 A_{2}\right) S_{12}^{4}  \tag{E1}\\
& \quad+\left(A_{2}+B_{2}\right) S_{21}^{4}+\left(A_{1}-B_{1}\right) S_{22}^{3}=0 \tag{E2}
\end{align*}
$$

$$
\begin{equation*}
\left(E_{1}-C_{1}\right) S_{11}^{2}+E_{2} S_{11}^{3}+C_{3} S_{12}^{3}+\left(A_{2}-2 B_{2}\right) S_{13}^{4} \tag{E9}
\end{equation*}
$$

$$
-2 C_{2} S_{21}^{2}-2 C_{3} S_{21}^{3}+\left(2 A_{2}-B_{2}\right) S_{41}^{3}
$$

$$
-C_{2} S_{41}^{4}+\left(C_{1}-A_{1}\right) S_{42}^{4}=0
$$

$$
\begin{equation*}
\left(A_{2}-2 B_{2}\right) S_{11}^{2}+D_{3} S_{12}^{3}-E_{2} S_{12}^{4} \tag{E10}
\end{equation*}
$$

$$
+\left(E_{1}-D_{1}\right) S_{13}^{4}-D_{2} S_{21}^{2}-D_{3} S_{21}^{3}+C_{3} S_{22}^{4}
$$

$$
-C_{2} S_{23}^{4}+\left(A_{1}-D_{1}\right) S_{41}^{3}-D_{2} S_{41}^{4}+\left(B_{2}-2 A_{2}\right) S_{42}^{4}=0
$$

$$
-E_{2} S_{21}^{3}-C_{3} S_{22}^{3}+\left(2 B_{2}-A_{2}\right) S_{23}^{4}-D_{2} S_{41}^{3}
$$

$$
+\left(A_{1}-E_{1}\right) S_{41}^{4}+\left(A_{2}+B_{2}\right) S_{42}^{3}+C_{2} S_{42}^{4}=0
$$

$$
\begin{align*}
&\left(A_{2}-2 B_{2}\right) S_{11}^{2}+D_{3} S_{12}^{3}-E_{2} S_{12}^{4}+\left(E_{1}-D_{1}\right) S_{13}^{4}  \tag{E5}\\
&-D_{2} S_{21}^{2}+E_{2} S_{21}^{4}-B_{3} S_{22}^{3}-C_{2} S_{23}^{4}-D_{2} S_{31}^{3} \\
&+\left(A_{1}-E_{1}\right) S_{31}^{4}+\left(A_{2}+B_{2}\right) S_{32}^{3}+C_{2} S_{32}^{4}= 0 \\
&-B_{3} S_{11}^{2}+\left(D_{1}-F_{1}\right) S_{12}^{4}+\left(2 A_{2}-B_{2}\right) S_{11}^{3}-D_{2} S_{12}^{3}  \tag{E6}\\
&+E_{2} S_{13}^{4}-D_{3} S_{21}^{2}+\left(F_{1}-B_{1}\right) S_{21}^{4}+\left(A_{2}+B_{2}\right) S_{22}^{3} \\
&-C_{3} S_{23}^{4}-D_{3} S_{31}^{3}-E_{2} S_{31}^{4}+B_{3} S_{32}^{3}+C_{3} S_{32}^{4}=0 \\
&-D_{2} S_{11}^{3}+\left(A_{1}-E_{1}\right) S_{11}^{4}+\left(A_{2}+B_{2}\right) S_{12}^{3}+C_{2} S_{12}^{4}  \tag{E7}\\
&+\left(B_{2}-2 A_{2}\right) S_{21}^{3}+C_{2} S_{21}^{4}+\left(A_{1}-C_{1}\right) S_{22}^{4}=0 \\
& D_{2} S_{11}^{2}-E_{2} S_{11}^{4}+B_{3} S_{12}^{3}+C_{2} S_{13}^{4}  \tag{E8}\\
&+\left(A_{2}-2 B_{2}\right) S_{21}^{2}-B_{3} S_{21}^{3}+C_{3} S_{21}^{4}+\left(B_{1}-C_{1}\right) S_{23}^{4} \\
&-\left(A_{2}+B_{2}\right) S_{41}^{4}+\left(B_{1}-A_{1}\right) S_{42}^{3}=0
\end{align*}
$$

$$
\begin{equation*}
-2 C_{2} S_{11}^{2}+2 E_{2} S_{12}^{3}-2 D_{2} S_{13}^{4}+\left(C_{1}-E_{1}\right) S_{21}^{2} \tag{E11}
\end{equation*}
$$

$$
\begin{equation*}
-C_{3} S_{11}^{2}-C_{2} S_{11}^{3}+\left(F_{1}-E_{1}\right) S_{12}^{3}+\left(A_{2}+B_{2}\right) S_{11}^{4} \tag{E12}
\end{equation*}
$$

$$
+D_{2} S_{12}^{4}-D_{3} S_{13}^{4}-E_{2} S_{21}^{2}+\left(C_{1}-F_{1}\right) S_{21}^{3}
$$

$$
+C_{2} S_{22}^{3}+\left(2 A_{2}-B_{2}\right) S_{22}^{4}+B_{3} S_{23}^{4}
$$

$$
-D_{3} S_{41}^{3}-E_{2} S_{41}^{4}+B_{3} S_{42}^{3}+C_{3} S_{42}^{4}=0
$$

$$
\begin{aligned}
-D_{3} S_{11}^{3}-E_{2} S_{11}^{4}+B_{3} S_{12}^{3}+C_{3} S_{12}^{4} \\
+\left(B_{2}-2 A_{2}\right) S_{31}^{3}+C_{2} S_{31}^{4}+\left(A_{1}-C_{1}\right) S_{32}^{4} \\
-\left(A_{2}+B_{2}\right) S_{41}^{4}+\left(B_{1}-A_{1}\right) S_{42}^{3}=0
\end{aligned}
$$

$$
D_{3} S_{11}^{2}+\left(B_{1}-F_{1}\right) S_{11}^{4}-\left(A_{2}+B_{2}\right) S_{12}^{3}
$$

$$
+C_{3} S_{13}^{4}+\left(A_{2}-2 B_{2}\right) S_{31}^{2}-B_{3} S_{31}^{3}
$$

$$
+C_{3} S_{31}^{4}+\left(B_{1}-C_{1}\right) S_{33}^{4}-2 B_{3} S_{41}^{4}=0
$$

$$
E_{2} S_{11}^{2}+\left(F_{1}-C_{1}\right) S_{11}^{3}-C_{2} S_{12}^{3}+\left(B_{2}-2 A_{2}\right) S_{12}^{4}
$$

$$
-B_{3} S_{13}^{4}-2 C_{2} S_{31}^{2}-2 C_{3} S_{31}^{3}
$$

$$
+\left(2 B_{2}-A_{2}\right) S_{41}^{2}+B_{3} S_{41}^{3}-C_{3} S_{41}^{4}+\left(C_{1}-B_{1}\right) S_{43}^{4}=0
$$

$$
-B_{3} S_{11}^{2}+\left(2 A_{2}-B_{2}\right) S_{11}^{3}-D_{2} S_{12}^{3}+\left(D_{1}-F_{1}\right) S_{12}^{4}
$$

$$
+E_{2} S_{13}^{4}-D_{2} S_{31}^{2}-D_{3} S_{31}^{3}+C_{3} S_{32}^{4}-C_{2} S_{33}^{4}
$$

$$
+\left(D_{1}-B_{1}\right) S_{41}^{2}-D_{3} S_{41}^{4}-B_{3} S_{42}^{4}+\left(2 B_{2}-A_{2}\right) S_{43}^{4}=0
$$

$$
-C_{3} S_{11}^{2}-C_{2} S_{11}^{3}+\left(A_{2}+B_{2}\right) S_{11}^{4}
$$

$$
+\left(F_{1}-E_{1}\right) S_{12}^{3}+D_{2} S_{12}^{4}-D_{3} S_{13}^{4}+\left(C_{1}-E_{1}\right) S_{31}^{2}
$$

$$
-E_{2} S_{31}^{3}-C_{3} S_{32}^{3}+\left(2 B_{2}-A_{2}\right) S_{33}^{4}
$$

$$
+D_{2} S_{41}^{2}-E_{2} S_{41}^{4}+B_{3} S_{42}^{3}+C_{2} S_{43}^{4}=0
$$

$$
-2 C_{3} S_{11}^{3}+2 B_{3} S_{11}^{4}-2 E_{2} S_{12}^{3}+2 D_{3} S_{12}^{4}-E_{2} S_{31}^{2}
$$

$$
+\left(C_{1}-F_{1}\right) S_{31}^{3}+C_{2} S_{32}^{3}+\left(2 A_{2}-B_{2}\right) S_{32}^{4}+B_{3} S_{33}^{4}
$$

$$
+D_{3} S_{41}^{2}+\left(B_{1}-F_{1}\right) S_{41}^{4}-\left(A_{2}+B_{2}\right) S_{42}^{3}+C_{3} S_{43}^{4}=0
$$

$$
-D_{3} S_{21}^{3}-E_{2} S_{21}^{4}+B_{3} S_{22}^{3}+C_{3} S_{22}^{4}+D_{2} S_{31}^{3}
$$

$$
+\left(E_{1}-A_{1}\right) S_{31}^{4}-\left(A_{2}+B_{2}\right) S_{32}^{3}-C_{2} S_{32}^{4}
$$

$$
+\left(A_{1}-D_{1}\right) S_{41}^{3}-D_{2} S_{41}^{4}+\left(B_{2}-2 A_{2}\right) S_{42}^{4}=0
$$

$$
D_{3} S_{21}^{2}+\left(B_{1}-F_{1}\right) S_{21}^{4}-\left(A_{2}+B_{2}\right) S_{22}^{3}
$$

$$
+C_{3} S_{23}^{4}-D_{2} S_{31}^{2}+E_{2} S_{31}^{4}-B_{3} S_{32}^{3}-C_{2} S_{33}^{4}
$$

$$
+\left(D_{1}-B_{1}\right) S_{41}^{2}-D_{3} S_{41}^{4}-B_{3} S_{42}^{4}+\left(2 B_{2}-A_{2}\right) S_{43}^{4}=0
$$

$$
E_{2} S_{21}^{2}+\left(F_{1}-C_{1}\right) S_{21}^{3}-C_{2} S_{22}^{3}+\left(B_{2}-2 A_{2}\right) S_{22}^{4}
$$

$$
-B_{3} S_{23}^{4}+\left(C_{1}-E_{1}\right) S_{31}^{2}-E_{2} S_{31}^{3}-C_{3} S_{32}^{3}
$$

$$
+\left(2 B_{2}-A_{2}\right) S_{33}^{4}+D_{2} S_{41}^{2}+D_{3} S_{41}^{3}-C_{3} S_{42}^{4}+C_{2} S_{43}^{4}=0
$$

$$
-B_{3} S_{21}^{2}+\left(2 A_{2}-B_{2}\right) S_{21}^{3}-D_{2} S_{22}^{3}+\left(D_{1}-F_{1}\right) S_{22}^{4}
$$

$$
+E_{2} S_{23}^{4}+\left(2 B_{2}-A_{2}\right) S_{31}^{2}-D_{3} S_{32}^{3}+E_{2} S_{32}^{4}
$$

$$
+\left(D_{1}-E_{1}\right) S_{33}^{4}-2 D_{3} S_{42}^{4}+2 D_{2} S_{43}^{4}=0
$$

$$
-C_{3} S_{21}^{2}-C_{2} S_{21}^{3}+\left(A_{2}+B_{2}\right) S_{21}^{4}+\left(F_{1}-E_{1}\right) S_{22}^{3}
$$

$$
+D_{2} S_{22}^{4}-D_{3} S_{23}^{4}+2 C_{2} S_{31}^{2}-2 E_{2} S_{32}^{3}+2 D_{2} S_{33}^{4}
$$

$$
+\left(A_{2}-2 B_{2}\right) S_{41}^{2}+D_{3} S_{42}^{3}-E_{2} S_{42}^{4}+\left(E_{1}-D_{1}\right) S_{43}^{4}=0
$$

$$
\begin{align*}
&-2 C_{3} S_{21}^{3}+2 B_{3} S_{21}^{4}-2 E_{2} S_{22}^{3}+2 D_{3} S_{22}^{4}+C_{3} S_{31}^{2}+C_{2} S_{31}^{3}  \tag{E24}\\
&-\left(A_{2}+B_{2}\right) S_{31}^{4}+\left(E_{1}-F_{1}\right) S_{32}^{3}-D_{2} S_{32}^{4}+D_{3} S_{33}^{4}-B_{3} S_{41}^{2} \\
&+\left(2 A_{2}-B_{2}\right) S_{41}^{3}-D_{2} S_{42}^{3}+\left(D_{1}-F_{1}\right) S_{42}^{4}+E_{2} S_{43}^{4}=0
\end{align*}
$$

We see immediately that the following linear relations hold: $(E 5)+(E 19)=$ $(E 10),(E 6)+(E 20)=(E 16),(E 12)+(E 21)=(E 17),(E 3)+(E 13)=(E 8)$.

Hence we obtain
Proposition 4.1. The rank of the system (12) is not greater than twenty.
We shall now give the following purely algebraic definition:
Definition 4.2. Let $\mathcal{R}$ denote the space of all 4-dimensional curvature tensors on ( $\mathbf{V},<,>$ ) and let $A(\mathbf{V}, \mathbf{V})$ denote the space of all skew-symmetric endomorphismus of $(\mathbf{V},<,>)$. The $\sigma$-rank of a curvature tensor $R \in \mathcal{R}$ is defined as the difference $24-k$ where $k$ is the dimension of the subspace of all elements $S \in \mathbf{V}^{*} \otimes A(\mathbf{V}, \mathbf{V})$ satisfying the identity (7).

Hence we see that, for any orthonormal basis $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ of $\mathbf{V}$, and any fixed $R \in \mathcal{R}$, the corresponding $\sigma$-rank can be defined as the rank of the corresponding system (12) of linear algebraic equations for the 24 unknowns $S_{i j}^{k}$. Especially, we can use the Chern bases for the calculation of the $\sigma$-rank.

Now, let $\widetilde{\mathcal{R}}$ denote the orbit space of $\mathcal{R}$ (w.r. to the action of the orthogonal group $O(\mathbf{V})$ ), provided with the factor topology. Every curvature tensor $R \in \mathcal{R}$ can be represented as a point in $\mathbf{R}^{14}$ via some Chern basis. Such representation is not unique but a generic curvature tensor has only a finite number of representatives (because we have only a finite number of Chern bases). Obviously, the corresponding orbits $[R] \in \widetilde{\mathcal{R}}$ can be also represented as points in $\mathbf{R}^{14}$ and maximal orbits have only finite number of representatives. It is also obvious that all curvature tensors belonging to the same orbit have the same $\sigma$-rank and thus the $\sigma$-rank can be considered as a function on $\widetilde{\mathcal{R}}$. Now we have
Theorem 4.3. The $\sigma$-rank is equal to 20 on a dense open subset of $\widetilde{\mathcal{R}}$.
Proof. It is sufficient to give an example of a curvature tensor whose $\sigma$-rank is equal to twenty. Then using the equations (E1) - (E24) written w.r. to a Chern basis we see that the condition for the $\sigma$-rank to be less than twenty would mean a system of algebraic equations for the components (3), which is not satisfied identically. This gives a subset of zero measure in $\mathbf{R}^{14}$ and our Theorem will follow.

Let us suppose that a curvature tensor $R$ is represented w.r. to a Chern basis and

$$
\begin{equation*}
C_{2}=D_{2}=E_{2}=0, \quad B_{3}=C_{3}=D_{3}=0 \tag{13}
\end{equation*}
$$

We shall omitt the equations (E13),(E19),E20),(E21) which are linearly dependent on the others. After re-aranging the order of our unknowns $S_{i j}^{k}$ in a convenient way and after a permutation of our equations we find that the resulting coefficient
matrix decomposes into four blocks which are $(6,5)$-matrices. The corresponding subsystems of equations with separated variables are given by the following tables:

| Eq. | $S_{11}^{3}$ | $S_{12}^{4}$ | $S_{21}^{4}$ | $S_{22}^{3}$ | $S_{41}^{2}$ | $S_{43}^{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (E1) | $D_{1}-A_{1}$ | $2 A_{2}-B_{2}$ | $-\left(A_{2}+B_{2}\right)$ | $B_{1}-A_{1}$ | 0 | 0 |
| (E6) | $2 A_{2}-B_{2}$ | $D_{1}-F_{1}$ | $F_{1}-B_{1}$ | $A_{2}+B_{2}$ | 0 | 0 |
| (E15) | $F_{1}-C_{1}$ | $B_{2}-2 A_{2}$ | 0 | 0 | $2 B_{2}-A_{2}$ | $C_{1}-B_{1}$ |
| (E16) | $B_{2}-2 A_{2}$ | $F_{1}-D_{1}$ | 0 | 0 | $B_{1}-D_{1}$ | $A_{2}-2 B_{2}$ |
| (E23) | 0 | 0 | $A_{2}+B_{2}$ | $F_{1}-E_{1}$ | $A_{2}-2 B_{2}$ | $E_{1}-D_{1}$ |


| Eq. | $S_{11}^{2}$ | $S_{13}^{4}$ | $S_{31}^{4}$ | $S_{32}^{3}$ | $S_{41}^{3}$ | $S_{42}^{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (E2) | $D_{1}-B_{1}$ | $2 B_{2}-A_{2}$ | $-\left(A_{2}+B_{2}\right)$ | $B_{1}-A_{1}$ | 0 | 0 |
| (E5) | $A_{2}-2 B_{2}$ | $E_{1}-D_{1}$ | $A_{1}-E_{1}$ | $A_{2}+B_{2}$ | 0 | 0 |
| (E9) | $E_{1}-C_{1}$ | $A_{2}-2 B_{2}$ | 0 | 0 | $2 A_{2}-B_{2}$ | $C_{1}-A_{1}$ |
| (E10) | $A_{2}-2 B_{2}$ | $E_{1}-D_{1}$ | 0 | 0 | $A_{1}-D_{1}$ | $B_{2}-2 A_{2}$ |
| (E24) | 0 | 0 | $-\left(A_{2}+B_{2}\right)$ | $E_{1}-F_{1}$ | $2 A_{2}-B_{2}$ | $D_{1}-F_{1}$ |


| Eq. | $S_{11}^{4}$ | $S_{12}^{3}$ | $S_{21}^{3}$ | $S_{22}^{4}$ | $S_{31}^{2}$ | $S_{33}^{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (E7) | $A_{1}-E_{1}$ | $A_{2}+B_{2}$ | $B_{2}-2 A_{2}$ | $A_{1}-C_{1}$ | 0 | 0 |
| (E12) | $A_{2}+B_{2}$ | $F_{1}-E_{1}$ | $C_{1}-F_{1}$ | $2 A_{2}-B_{2}$ | 0 | 0 |
| (E14) | $B_{1}-F_{1}$ | $-\left(A_{2}+B_{2}\right)$ | 0 | 0 | $A_{2}-2 B_{2}$ | $B_{1}-C_{1}$ |
| (E17) | $A_{2}+B_{2}$ | $F_{1}-E_{1}$ | 0 | 0 | $C_{1}-E_{1}$ | $2 B_{2}-A_{2}$ |
| (E22) | 0 | 0 | $2 A_{2}-B_{2}$ | $D_{1}-F_{1}$ | $2 B_{2}-A_{2}$ | $D_{1}-E_{1}$ |


| Eq. | $S_{21}^{2}$ | $S_{23}^{4}$ | $S_{31}^{3}$ | $S_{32}^{4}$ | $S_{41}^{4}$ | $S_{42}^{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (E3) | $A_{2}-2 B_{2}$ | $B_{1}-C_{1}$ | $2 A_{2}-B_{2}$ | $C_{1}-A_{1}$ | 0 | 0 |
| (E4) | $B_{1}-D_{1}$ | $A_{2}-2 B_{2}$ | $A_{1}-D_{1}$ | $B_{2}-2 A_{2}$ | 0 | 0 |
| (E8) | $A_{2}-2 B_{2}$ | $B_{1}-C_{1}$ | 0 | 0 | $-\left(A_{2}+B_{2}\right)$ | $B_{1}-A_{1}$ |
| (E11) | $C_{1}-E_{1}$ | $2 B_{2}-A_{2}$ | 0 | 0 | $A_{1}-E_{1}$ | $A_{2}+B_{2}$ |
| (E18) | 0 | 0 | $C_{1}-F_{1}$ | $2 A_{2}-B_{2}$ | $B_{1}-F_{1}$ | $-\left(A_{2}+B_{2}\right)$ |

The rank of each of these matrices is five, in general, and hence the $\sigma$-rank is equal to twenty for a general $R$ satisfying (13). This concludes the proof of Theorem 4.3.

The previous tables will be used in the next section for the proof of our main Theorem. Here we only make the following remark:

It is natural to call a curvature tensor $R \in \mathcal{R} \sigma$-generic if its $\sigma$-rank is equal to twenty. One can ask, what is the relation between the notion "generic" and " $\sigma$-generic". The following examples show that there is no direct relation.
Example 1. Suppose, in addition to (13), that $A_{2}=B_{2}=0, D_{1}=E_{1}=F_{1}=0$ and the numbers $\left|A_{1}\right|,\left|B_{1}\right|,\left|C_{1}\right|$ are all distinct. Then the corresponding tensor $R$ is generic (see Section 2,d)) but the $\sigma$-rank is $\leq 17$, as we see from our tables.
Example 2. Suppose, in addition to (13), that $B_{1}=D_{1}, C_{1}=E_{1}, A_{2}=2 B_{2}$, $A_{1}, D_{1}, E_{1}, F_{1}$ are arbitrary and $B_{2} \neq 0$. Then we see that, in general, the $\sigma$-rank is maximal. But due to f ) of Section 2, the corresponding curvature tensor is not generic.

It is not known to the authors if the $\sigma$-genericity may have some geometrical meaning.

## 5. The main existence theorem

Theorem 5.1. Let $R$ be a curvature tensor on ( $\mathbf{V},<,>$ ), $\operatorname{dim} \mathbf{V}=4$. Suppose that, w.r. to some orthonormal basis $\left\{e_{1}, \ldots, e_{4}\right\}$, all the components $A_{1}, B_{1}, C_{1}$, $D_{1}, E_{1}, F_{1}$ (written shortly as $A, B, \ldots, F$ ) are distinct and all the other curvature components are zero. Further, suppose that the following two sets of inequalities hold for $A, B, \ldots, F$ :
(A) $\left|\begin{array}{ccc}D-A & B-A & 0 \\ F-C & 0 & C-B \\ 0 & F-E & E-D\end{array}\right| \neq 0,\left|\begin{array}{ccc}D-B & B-A & 0 \\ E-C & 0 & C-A \\ 0 & E-F & D-F\end{array}\right| \neq 0$,

$$
\left|\begin{array}{ccc}
A-E & A-C & 0 \\
B-F & 0 & B-C \\
0 & D-F & D-E
\end{array}\right| \neq 0,\left|\begin{array}{ccc}
B-D & A-D & 0 \\
C-E & 0 & A-E \\
0 & C-F & B-F
\end{array}\right| \neq 0
$$

(B) Under the notations

$$
\begin{aligned}
& \alpha_{1}=\frac{B-C}{A-C}, \beta_{1}=\frac{B-C}{A-B}, \alpha_{2}=\frac{E-D}{E-A}, \beta_{2}=\frac{E-D}{D-A}, \\
& \alpha_{3}=\frac{D-F}{B-F}, \beta_{3}=\frac{D-F}{B-D}, \alpha_{4}=\frac{F-E}{F-C}, \beta_{4}=\frac{F-E}{E-C},
\end{aligned}
$$

the rank of the matrix

$$
\left(\begin{array}{ccccc}
0 & 0 & 2 \alpha_{3} & 2 \alpha_{4} & A \\
0 & 2 \alpha_{2} & 0 & -2 \beta_{4} & B \\
0 & -2 \beta_{2} & -2 \beta_{3} & 0 & C \\
2 \alpha_{1} & 0 & 0 & 2 \alpha_{4} \beta_{4} & D \\
-2 \beta_{1} & 0 & 2 \alpha_{3} \beta_{3} & 0 & E \\
2 \alpha_{1} \beta_{1} & 2 \alpha_{2} \beta_{2} & 0 & 0 & F
\end{array}\right)
$$

is equal to five.
Then the curvature tensor $R$ does not belong to any curvature homogeneous space.
Remark. a) The inequalities ( $A$ ) are all independent. E.g., if we take $A=B=$ $C=u, D=E=F=v, u \neq v$, then the first three determinants are zero and the last one is non zero.
b) The coefficients $\alpha_{i}, \beta_{i}$ from ( $B$ ) are not independent but they satisfy the equalities

$$
\begin{equation*}
\alpha_{i} \beta_{i}=\beta_{i}-\alpha_{i} \quad(i=1, \ldots, 4) \tag{14}
\end{equation*}
$$

Proof of the Theorem. The four subsystems of linear equations with separated variables coming from (E1) - (E24) are now given by the tables

| $S_{11}^{3}$ | $S_{12}^{4}$ | $S_{21}^{4}$ | $S_{22}^{3}$ | $S_{41}^{2}$ | $S_{43}^{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{D}-\mathrm{A}$ | 0 | 0 | $\mathrm{~B}-\mathrm{A}$ | 0 | 0 |
| 0 | $\mathrm{D}-\mathrm{F}$ | $\mathrm{F}-\mathrm{B}$ | 0 | 0 | 0 |
| $\mathrm{~F}-\mathrm{C}$ | 0 | 0 | 0 | 0 | $\mathrm{C}-\mathrm{B}$ |
| 0 | $\mathrm{~F}-\mathrm{D}$ | 0 | 0 | $\mathrm{~B}-\mathrm{D}$ | 0 |
| 0 | 0 | 0 | $\mathrm{~F}-\mathrm{E}$ | 0 | $\mathrm{E}-\mathrm{D}$ |


| $S_{11}^{2}$ | $S_{13}^{4}$ | $S_{31}^{4}$ | $S_{32}^{3}$ | $S_{41}^{3}$ | $S_{42}^{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{D}-\mathrm{B}$ | 0 | 0 | $\mathrm{~B}-\mathrm{A}$ | 0 | 0 |
| 0 | $\mathrm{E}-\mathrm{D}$ | $\mathrm{A}-\mathrm{E}$ | 0 | 0 | 0 |
| $\mathrm{E}-\mathrm{C}$ | 0 | 0 | 0 | 0 | $\mathrm{C}-\mathrm{A}$ |
| 0 | $\mathrm{E}-\mathrm{D}$ | 0 | 0 | $\mathrm{~A}-\mathrm{D}$ | 0 |
| 0 | 0 | 0 | $\mathrm{E}-\mathrm{F}$ | 0 | $\mathrm{D}-\mathrm{F}$ |


| $S_{11}^{4}$ | $S_{12}^{3}$ | $S_{21}^{3}$ | $S_{22}^{4}$ | $S_{31}^{2}$ | $S_{33}^{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{~A}-\mathrm{E}$ | 0 | 0 | $\mathrm{~A}-\mathrm{C}$ | 0 | 0 |
| 0 | $\mathrm{~F}-\mathrm{E}$ | $\mathrm{C}-\mathrm{F}$ | 0 | 0 | 0 |
| $\mathrm{~B}-\mathrm{F}$ | 0 | 0 | 0 | 0 | $\mathrm{~B}-\mathrm{C}$ |
| 0 | $\mathrm{~F}-\mathrm{E}$ | 0 | 0 | $\mathrm{C}-\mathrm{E}$ | 0 |
| 0 | 0 | 0 | $\mathrm{D}-\mathrm{F}$ | 0 | $\mathrm{D}-\mathrm{E}$ |


| $S_{21}^{2}$ | $S_{23}^{4}$ | $S_{31}^{3}$ | $S_{32}^{4}$ | $S_{41}^{4}$ | $S_{42}^{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $\mathrm{~B}-\mathrm{C}$ | 0 | $\mathrm{C}-\mathrm{A}$ | 0 | 0 |
| $\mathrm{~B}-\mathrm{D}$ | 0 | $\mathrm{~A}-\mathrm{D}$ | 0 | 0 | 0 |
| 0 | $\mathrm{~B}-\mathrm{C}$ | 0 | 0 | 0 | $\mathrm{~B}-\mathrm{A}$ |
| $\mathrm{C}-\mathrm{E}$ | 0 | 0 | 0 | $\mathrm{~A}-\mathrm{E}$ | 0 |
| 0 | 0 | $\mathrm{C}-\mathrm{F}$ | 0 | $\mathrm{~B}-\mathrm{F}$ | 0 |

We put

$$
\begin{equation*}
U_{1}=S_{23}^{4}, U_{2}=S_{13}^{4}, U_{3}=S_{12}^{4}, U_{4}=S_{12}^{3} \tag{15}
\end{equation*}
$$

and we shall try to express the other unknown functions $S_{i j}^{k}$ through $U_{1}, U_{2}, U_{3}, U_{4}$ from the previous systems of equations. Now the inequalities $(A)$ from our Theorem guarantee that the Cramer's rule can be used in all cases. We obtain easily

$$
\begin{equation*}
S_{i i}^{j}=0, S_{j i}^{j}=0 \text { for all } i, j,(1 \leq i<j \leq 4) \tag{16}
\end{equation*}
$$

and

$$
\begin{array}{llll}
S_{32}^{4}=\alpha_{1} U_{1}, & S_{42}^{3}=\beta_{1} U_{1}, & S_{31}^{4}=\alpha_{2} U_{2}, & S_{41}^{3}=\beta_{2} U_{2}  \tag{17}\\
S_{21}^{4}=\alpha_{3} U_{3}, & S_{41}^{2}=\beta_{3} U_{3}, & S_{21}^{3}=\alpha_{4} U_{4}, & S_{31}^{2}=\beta_{4} U_{4}
\end{array}
$$

Next, we shall use the known identities

$$
\begin{equation*}
\left[\nabla_{E_{i}}, \nabla_{E_{j}}\right] E_{k}-\nabla_{\left[E_{i}, E_{j}\right]} E_{k}=R\left(E_{i}, E_{j}\right) E_{k} \tag{18}
\end{equation*}
$$

Due to (8),(9) we have

$$
\begin{equation*}
\left[E_{i}, E_{j}\right]=\nabla_{E_{i}} E_{j}-\nabla_{E_{j}} E_{i}=\sum_{l}\left(S_{i j}^{l}-S_{j i}^{l}\right) E_{l} \tag{19}
\end{equation*}
$$

and (18) can be rewritten in the form

$$
\begin{equation*}
E_{i}\left(S_{j k}^{l}\right)-E_{j}\left(S_{i k}^{l}\right)+S_{j k}^{u} S_{i u}^{l}-S_{i k}^{u} S_{j u}^{l}-S_{i j}^{u} s_{u k}^{l}+S_{j i}^{u} S_{u k}^{l}=-R_{i j k l} \tag{20}
\end{equation*}
$$

(36 equations, in which $u$ is a summation index.)
Now, let us consider the six equations of (20) for which $1 \leq i<j \leq 4,(k, l)=$ $(i, j)$. According to (16), all these equations are purely algebraic. Substituting from (15)-(17) we get easily

$$
\begin{align*}
2 \alpha_{3}\left(U_{3}\right)^{2}+2 \alpha_{4}\left(U_{4}\right)^{2} & =A, \\
2 \alpha_{2}\left(U_{2}\right)^{2}-2 \beta_{4}\left(U_{4}\right)^{2} & =B, \\
-2 \beta_{2}\left(U_{2}\right)^{2}-2 \beta_{3}\left(U_{3}\right)^{2} & =C, \\
2 \alpha_{1}\left(U_{1}\right)^{2}+2 \alpha_{4} \beta_{4}\left(U_{4}\right)^{2} & =D,  \tag{21}\\
-2 \beta_{1}\left(U_{1}\right)^{2}+2 \alpha_{3} \beta_{3}\left(U_{3}\right)^{2} & =E, \\
2 \alpha_{1} \beta_{1}\left(U_{1}\right)^{2}+2 \alpha_{2} \beta_{2}\left(U_{4}\right)^{2} & =F .
\end{align*}
$$

This is a system of six linear algebraic equations for four unknown functions $\left(U_{1}\right)^{2}$, $\left(U_{2}\right)^{2},\left(U_{3}\right)^{2},\left(U_{4}\right)^{2}$. The matrix of the corresponding homogeneous system is of rank $\leq 4$. Hence, if the condition $(B)$ of Theorem 5.1 is satisfied, then the system (21) has no solution according to the Frobenius Theorem. But this is a contradiction to Proposition 3.3, and a curvature homogeneous space with such a type of curvature tensor cannot exist. This concludes the proof of Theorem 5.1. $\square$
Remark 1. The condition $(B)$ is a bit ackward to check. But we can obtain very simple sufficient conditions which already imply contradictions. E.g., if we require $\operatorname{sgn} \alpha_{3}=\operatorname{sgn} \alpha_{4}=-\operatorname{sgn} A$, then the first equation (21) is contradictory, and we get similar conditions for the other equations.
Remark 2. For the curvature tensor of the form $R=a R_{S^{4}}+R_{C P^{2}}$ from the Introduction one can check easily that, with respect to a convenient orthonormal basis, $A_{1}=F_{1}=4 \lambda+c, B_{1}=C_{1}=D_{1}=E_{1}=\lambda+c, A_{2}=2 \lambda, B_{2}=\lambda, \lambda c \neq 0$, and the other components are zero. Hence the $\sigma$-rank of $R$ is equal to eight, and our method cannot be used in any way.

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[^0]:    1991 Mathematics Subject Classification: Primary 53C25, Secondary 53C30.
    Key words and phrases: Riemannian manifolds, curvature tensor, curvature homogeneous spaces.

    Received August 16, 1993
    This research was partly supported by the grant GA ČR 201/93/0469.

