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ON A MULTIPOINT BOUNDARY VALUE PROBLEM FOR LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULARITIES

G. D. Tskhovrebadze

Abstract. A criterion for the unique solvability of and sufficient conditions for the correctness of the modified Vallèe-Poussin problem are established for the linear ordinary differential equations with singularities.

Introduction

This paper is devoted to the investigation of a certain modification of the Vallèe-Poussin's boundary value problem, and it seems natural to explain in first place which modification is meant and which factors have led to it.

Let us consider the linear ordinary differential equation

$$u^{(n)}=\sum_{k=1}^{l} p_k(t)u^{(k-1)}+q(t)\,, \tag{1}$$

where $n \geq 2$ is a natural number, $p_1,\dots,p_l,q$ are continuous functions on the segment $[a,b]$. Let $m \in \{2,\dots,n\}$, $n_i \in \{1,\dots,n-1\}$ $(i=1,\dots,m)$, $\sum\limits_{i=1}^{m} n_i = n$, $-\infty < a = t_1 < \cdots < t_m = b < +\infty$.

As is well-known, the classical Vallèe-Poussin's boundary value problem is formulated as follows: Find a solution of the differential equation (1) satisfying the conditions

$$u^{(k-1)}(t_i)=0 \quad (k=1,\dots,n_i;\ i=1,\dots,m)\,. \tag{2_1}$$

The solution, naturally, is sought for in the class of n-times continuously differentiable functions on the segment $[a,b]$.

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There are a lot of works devoted to the investigation of the Vallèe-Poussin's boundary value problem in this classical formulation (see, for example, [1] and References from [3]).

This problem has also been studied with sufficient thoroughness in the case when the coefficients of the equation (1) have singularities at the points $t_1, \dots, t_m$ (see, for example, [2,3,5]). However, in all works devoted to the study of the Vallèe-Poussin's problem it is assumed that

$$(*)\quad \int_a^b (t-a)^{n-n_1-1}(b-t)^{n-n_m-1}\prod_{i=1}^m |t-t_i|^{n_{ik}}|p_k(t)|\,dt < +\infty \quad (k=1,\dots,l),$$

where

$$n_{ik} = \begin{cases} n_i - k + 1 & \text{for} \quad k \le n_i \\ 0 & \text{for} \quad k > n_i \end{cases},$$

and

$$\int_a^b (t-a)^{n-n_1-1}(b-t)^{n-n_m-1}|q(t)|\,dt < +\infty .$$

This assumption is not casual. The matter is that if functions p_k $(k = 1, \dots, l)$ have singularities of order $n - n_1 + n_{1k}$ and $n - n_m + n_{mk}$ at the points a and b, respectively (in particular, the function p_1 has singularities of order n at the points a and b), then Problem (1), (2_1) is not, generally speaking, uniquely solvable even in the simplest case. For example, given boundary conditions (2_1), the equation

$$u^{(n)} = \frac{(-1)^{n-1}\delta}{(t-a)^n} u$$

has an infinite number of solutions for $n_1 = 1$ and sufficiently small $\delta > 0$.

Therefore, to provide the solution uniqueness, we have to introduce an additional and, of course, natural condition such as, for example,

$$(2_2)\qquad \sup\left\{(t-a)^{l-1-\lambda_1}(b-t)^{l-1-\lambda_2}|u^{(l-1)}(t)| : a < t < b\right\} < +\infty ,$$

where $\lambda_1 \in]n_1 - 1,\, n_1[$, $\lambda_2 \in]n_m - 1,\, n_m[$.

This condition is natural because if the condition (*) is fulfilled, than (2_1), yields (2_2), i.e. Problem (1), (2_1), (2_2) coincides with the Vallèe-Poussin's problem. However, if the condition (*) is not fulfilled, than, as follows from the above example, this is not so.

Problem (1), (2_1), (2_2) is the generalization of the Vallèe-Poussin's boundary value problem and has not yet been studied with sufficient completeness. Here an attempt is made to fill up somehow this gap. In particular, the conditions are established, guaranteeing Problem (1), (2_1), (2_2) to be Fredholmian and its solution to be stable with respect to integrally small perturbations of the coefficients of equation (1). It is assumed that the functions $p_k : I_m \to \mathbb{R}$ $(k = 1, \dots, l)$, $q :]a, b[\to \mathbb{R}$ be locally integrable on I_m and $]a, b[$, respectively, where $I_m = [a, b] \setminus \{t_1, \dots, t_m\}$, and the condition (*) is not fulfilled. Note that the solution of Problem (1), (2_1), (2_2) is sought for in the class of functions $u :]a,b[\to \mathbb{R}$ absolutely continuous together with $u^{(k)}$ $(k=1,\dots,n-1)$ inside $]a,b[$.[1)]

The following notation will be used throughout this paper:

$$\sigma_{k,\lambda_1,\lambda_2}(t) = (t-a)^{\lambda_1-k+1}(b-t)^{\lambda_2-k+1}\prod_{i=2}^{m-1}|t-t_i|^{n_{ik}}\,;\ ^{1)}$$

$$\sigma_{k,n}(t) = (t-a)^{n-k}(b-t)^{n-k}\prod_{i=2}^{m-1}|t-t_i|^{n_{ik}}\,;$$

$$\nu_{kl}(\lambda) = |(k-1-\lambda)\dots(l-2-\lambda)| \quad (k=1,\dots,l-1), \nu_{ll}(\lambda)=1\,;$$

$L([a,b];\mathbb{R})$ is a set of Lebesgue-integrable functions $g:[a,b]\to\mathbb{R}$; $L_{\mathrm{loc}}(]a,b[;\mathbb{R})$ is a set of functions $g :]a,b[\to \mathbb{R}$ which are Lebesgue-integrable inside $]a,b[$; $L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ is a set of measurable functions $g :]a,b[\to\mathbb{R}$ such that

$$|g(\cdot)|_{n-1-\lambda_1,\,n-1-\lambda_2} = \\ = \sup\Big\{(t-a)^{n-1-\lambda_1}(b-t)^{n-1-\lambda_2}\Big|\int_{\frac{a+b}{2}}^{t} g(\tau)\,d\tau\Big| \,:\, a<t<b\Big\} < +\infty\,.$$

1. Lemmas on A Priori Estimates

In this section we consider Problem

(1) $$u^{(n)} = \sum_{k=1}^{l} p_k(t)u^{(k-1)} + q(t)\,,$$

(2_1) $$u^{(k-1)}(t_i) = 0 \quad (k=1,\dots,n_i;\ i=1,\dots,m)\,,$$

(2_2) $$\sup\big\{(t-a)^{l-1-\lambda_1}(b-t)^{l-1-\lambda_2}|u^{(l-1)}(t)| : a<t<b\big\} < +\infty\,,$$

where $\lambda_1 \in]n_1-1, n_1[$, $\lambda_2 \in]n_m-1,\ n_m[$, $q\in L_{n-1-\lambda_1,\ n-1-\lambda_2}(]a,b[;\mathbb{R})$ and $p_k :]a,b[\to\mathbb{R}$ $(k=1,\dots,l)$ are measurable functions satisfying inequalities

(3) $$p_{1k}(t) \le p_k(t) \le p_{2k}(t) \quad \text{for} \quad a<t<b \quad (k=1,\dots,l)\,.$$

On imposing certain restrictions on the vector function $(p_{11},\dots,p_{1l};p_{21},\dots,p_{2l})$, we obtain an a priori estimate of the solution of Problem (1), (2_1), (2_2) which is unique for the considered set of coefficients.

Before formulating the main lemma, some definitions will be given.

[1)] i.e., on each segment contained in $]a,b[$.

[2)] in the case $m=2$ $\prod_{i=2}^{m-1}|t-t_i|^{n_{ik}}$ denotes unity.

Definition 1. Let $n_0 \in \{1,\dots,n-1\}$ and $\lambda \in]n_0-1, n_0[$. The vector function $(h_1,\dots,h_l)$ with measurable components $h_k :]a,b[\to \mathbb{R}$ $(k=1,\dots,l)$ will be said to belong to the set $S^+(a,b;n,n_0;\lambda)$ $\Big(S^-(a,b;n,n_0;\lambda)\Big)$ if there exists $\alpha \in]a,b[$ such that we have the inequality

$$\limsup_{t\to a} \frac{(t-a)^{l-1-\lambda}}{(n-l)!} \sum_{k=1}^{l} \frac{1}{\nu_{kl}(\lambda)} \int_t^\alpha (\tau-t)^{n-l}(\tau-a)^{\lambda-k+1}|h_k(\tau)|\,d\tau < 1$$

$$\Big(\limsup_{t\to b} \frac{(b-t)^{l-1-\lambda}}{(n-l)!} \sum_{k=1}^{l} \frac{1}{\nu_{kl}(\lambda)} \int_\alpha^t (t-\tau)^{n-l}(b-\tau)^{\lambda-k+1}|h_k(\tau)|\,d\tau < 1\Big)$$

in the case $l \in \{n_0+1,\dots,n\}$ and the inequality

$$\limsup_{t\to a} \frac{(t-a)^{l-1-\lambda}}{(n-n_0-1)!(n_0-l)!}\times$$
$$\times \sum_{k=1}^{l} \frac{1}{\nu_{kl}(\lambda)} \int_a^t (t-s)^{n_0-l} \int_s^\alpha (\tau-s)^{n-n_0-1}(\tau-a)^{\lambda-k+1}|h_k(\tau)|\,d\tau\,ds < 1$$

$$\Big(\limsup_{t\to b} \frac{(b-t)^{l-1-\lambda}}{(n-n_0-1)!(n_o-l)!}\times$$
$$\times \sum_{k=1}^{l} \frac{1}{\nu_{kl}(\lambda)} \int_t^b (s-t)^{n_0-l} \int_\alpha^s (s-\tau)^{n-n_0-1}(b-\tau)^{\lambda-k+1}|h_k(\tau)|\,d\tau\,ds < 1\Big)$$

in the case $l \in \{1,\dots,n_0\}$.

Definition 2. Let

$$\sigma_{k,\lambda_1,\lambda_2}(\cdot)p_{jk}(\cdot) \in L_{\text{loc}}(]a,b[;\mathbb{R}) \quad (j=1,2;\ k=1,\dots,l),$$
$$p_{1k}(t) \le p_{2k}(t) \quad \text{for } a<t<b \quad (k=1,\dots,l),$$
$$(p_1^*,\dots,p_l^*) \in S^+(a,b;n,n_1;\lambda_1) \cap S^-(a,b;n,n_m;\lambda_2), \tag{4}$$

where $p_k^*(t) = \max\{|p_{1k}(t)|,|p_{2k}(t)|\}$ $\quad (k=1,\dots,l)$ and moreover, under the boundary conditions (2_1), (2_2) the differential equation

$$u^{(n)} = \sum_{k=1}^{l} p_k(t)u^{(k-1)} \tag{1_0}$$

does not have a nontrivial solution no matter what measurable functions $p_k :]a,b[\to \mathbb{R}$ $(k=1,\dots,l)$, satisfying inequalities (3), are. Then the vector function $(p_{11},\dots,p_{1l};\ p_{21},\dots,p_{2l})$ will be said to belong to the class $V(t_1,\dots,t_m;n_1,\dots,n_m;\ \lambda_1,\lambda_2)$.

Lemma 1. *Let*

$$(p_{11},\dots,p_{1l};\ p_{21},\dots,p_{2l})\in V(t_1,\dots,t_m;\ n_1,\dots,n_m;\ \lambda_1,\lambda_2)\,. \tag{5}$$

Then there exists a positive number ρ_0 such that for any $q\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ and measurable functions $p_k:]a,b[\to\mathbb{R}$ $(k=1,\dots,l)$, satisfying inequalities (3), an arbitrary solution u of Problem (1), (2_1), (2_2) admits the estimate

$$|u^{(k-1)}(t)|\le \rho_0\sigma_{k,\lambda_1,\lambda_2}(t)|q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} \quad \textit{for } a<t<b \quad (k=1,\dots,l)\,. \tag{6}$$

Proof. By condition (5) there are numbers $\alpha\in]a,t_2[$, $\beta\in]t_{m-1},b[$ and $\eta\in]0,1[$ such that

$$\alpha-a<1,\quad b-\beta<1 \tag{7}$$

and the functions $p^*_k(t)=\max\{|p_{1k}(t)|,|p_{2k}(t)|\}$ $(k=1,\dots,l)$ satisfy the inequalities

$$\frac{(t-a)^{l-1-\lambda_1}}{(n-n_1-1)!(n_1-l)!}\times \sum_{k=1}^{l}\frac{1}{\nu_{kl}(\lambda_1)}\int_a^t (t-s)^{n_1-l}\int_s^\alpha (\tau-s)^{n-n_1-1}(\tau-a)^{\lambda_1-k+1}p^*_k(\tau)\,d\tau\,ds\le\eta \quad \text{for } a<t\le\alpha,\quad l\in\{1,\dots,n_1\}, \tag{8_l}$$

$$\frac{(t-a)^{l-1-\lambda_1}}{(n-l)!}\sum_{k=1}^{l}\frac{1}{\nu_{kl}(\lambda_1)}\int_t^\alpha (\tau-t)^{n-l}(\tau-a)^{\lambda_1-k+1}p^*_k(\tau)\,d\tau\le\eta \quad \text{for } a<t\le\alpha,\quad l\in\{n_1+1,\dots,n\}\,, \tag{8_l}$$

$$\frac{(b-t)^{l-1-\lambda_2}}{(n-n_m-1)!(n_m-l)!}\times \sum_{k=1}^{l}\frac{1}{\nu_{kl}(\lambda_2)}\int_t^b (s-t)^{n_m-l}\int_\beta^s (s-\tau)^{n-n_m-1}(b-\tau)^{\lambda_2-k+1}p^*_k(\tau)\,d\tau\,ds\le\eta \quad \text{for } \beta\le t<b,\quad l\in\{1,\dots,n_m\}\,, \tag{9_l}$$

$$\frac{(b-t)^{l-1-\lambda_2}}{(n-l)!}\sum_{k=1}^{l}\frac{1}{\nu_{kl}(\lambda_2)}\int_\beta^t (t-\tau)^{n-l}(b-\tau)^{\lambda_2-k+1}p^*_k(\tau)\,d\tau\le\eta \quad \text{for } \beta\le t<b,\quad l\in\{n_m+1,\dots,n\}\,. \tag{9_l}$$

Let us assume that the Lemma is not true. Then for each j there exist $q_j \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$, measurable functions $\tilde{p}_{kj} :]a,b[\to \mathbb{R} (k=1,\dots,l)$ satisfying inequalities

$$p_{1k}(t) \le \tilde{p}_{kj}(t) \le p_{2k}(t) \quad \text{for } a<t<b \quad (k=1,\dots,l)\,, \tag{10}$$

and solution u_j of the differential equation

$$u^{(n)} = \sum_{k=1}^{l} \tilde{p}_{kj}(t) u^{(k-1)} + q_j(t) \tag{11}$$

satisfying the boundary conditions (2_1), (2_2) such that

$$\gamma_{1j} + \gamma_{0j} + \gamma_{2j} > j|q_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\,, \tag{12}$$

where

$$\gamma_{1j} = \sup\Big\{ \sum_{k=1}^{l} \frac{|u_j^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a<t\le\alpha \Big\}\,, \tag{13_1}$$

$$\gamma_{0j} = \operatorname{vraimax}\Big\{ \sum_{k=1}^{l} \frac{|u_j^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : \alpha\le t\le\beta \Big\}\,, \tag{13_2}$$

$$\gamma_{2j} = \sup\Big\{ \sum_{k=1}^{l} \frac{|u_j^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : \beta\le t<b \Big\}\,. \tag{13_3}$$

It is assumed for each j that

$$\gamma_j = \max\Big\{ \sum_{k=1}^{n} |u_j^{(k-1)}(t)| : \alpha\le t\le\beta \Big\}\,, \tag{14}$$

$$\gamma_j^* = \begin{cases} 1 & \text{for } \gamma_j = 0 \\ \gamma_j & \text{for } \gamma_j > 0 \end{cases}\,, \tag{15}$$

$$v_j(t) = \frac{u_j(t)}{\gamma_j^*}\,. \tag{16}$$

By (2_2), (11) and (16) we have

$$r_j = \sup\Big\{ (t-a)^{l-1-\lambda_1} |v_j^{(l-1)}(t)| : a<t\le\alpha \Big\} < +\infty \tag{17}$$

and

$$v_j^{(n)}(t) = \sum_{k=1}^{l} \tilde{p}_{kj}(t) v_j^{(k-1)}(t) + \frac{g_j(t)}{\gamma_j^*}\,. \tag{18}$$

If $l \in \{1,\dots,n_1\}$, then by virtue of (2_1) and (17) we have

$$|v_j^{(k-1)}(t)| \le \frac{r_j}{\nu_{kl}(\lambda_1)}(t-a)^{\lambda_1-k+1} \quad \text{for} \quad a<t\le\alpha \quad (k=1,\dots,l). \tag{19}$$

Taking (7), (10) and (19) into account, from (18) we find

$$|v_j^{(n_1)}(t)| \le \frac{\gamma_j}{\gamma_j^*}+$$
$$+\frac{r_j}{(n-n_1-1)!}\sum_{k=1}^{l}\frac{1}{\nu_{kl}(\lambda_1)}\int_t^\alpha(\tau-t)^{n-n_1-1}(\tau-a)^{\lambda_1-k+1}p_k^*(\tau)\,d\tau+$$
$$+(t-a)^{\lambda_1-n_1}\frac{c_1}{\gamma_j^*(n_1-\lambda_1)}\ |q_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\,,$$

where $c_1=2[\frac{b-a}{(b-\alpha)^2}]^{n-1-\lambda_2}$. Taking (2_1) and the last inequality into account, we easily obtain

$$|v_j^{(l-1)}(t)| \le \frac{\gamma_j}{\gamma_j^*}(t-a)^{n_1-l+1}+\frac{r_j}{(n-n_1-1)!(n_1-l)!}\times$$
$$\times\sum_{k=1}^{l}\frac{1}{\nu_{kl}(\lambda_1)}\int_a^t(t-s)^{n_1-l}\int_s^\alpha(\tau-s)^{n-n_1-1}(\tau-a)^{\lambda_1-k+1}p_k^*(\tau)\,d\tau\,ds+$$
$$+(t-a)^{\lambda_1-l+1}\frac{c_1}{\gamma_j^*(n_1-\lambda_1)(\lambda_1-n_1+1)}\ |q_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}$$
$$\text{for} \quad a<t\le\alpha\,.$$

Hence by virtue of (7), (8_l) we have

$$r_j \le \frac{\gamma_j}{\gamma_j^*}+r_j\eta+\frac{c_1}{\gamma_j^*(n_1-\lambda_1)(\lambda_1-n_1+1)}\ |q_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\,.$$

Therefore

$$r_j \le \frac{c_2}{\gamma_j^*}\left(\gamma_j+|q_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\right)\,,$$

where

$$c_2=(1-\eta)^{-1}\left(1+\frac{c_1}{(n_1-\lambda_1)(\lambda_1-n_1+1)}\right)\,.$$

Using this estimate, from (19) we obtain

$$|v_j^{(k-1)}(t)| \le \frac{c_3}{\gamma_j^*}\left(\gamma_j+|q_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\right)\ \sigma_{k,\lambda_1,\lambda_2}(t) \tag{20}$$
$$\text{for} \quad a<t\le\alpha \quad (k=1,\dots,l)\,,$$

where

$$c_3 = \sum_{k=1}^{l} \frac{c_2}{\nu_{kl}(\lambda_1)} \sup\Big\{ \frac{(t-a)^{\lambda_1-k+1}}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t \le \alpha \Big\}.$$

If $l \in \{n_1+1, \dots, n\}$, then by virtue of (7) and (17) we have

$$|v_j^{(k-1)}(t)| \le \Big(\frac{\gamma_j}{\gamma_j^*} + \frac{r_j}{\nu_{kl}(\lambda_1)} \Big) (t-a)^{\lambda_1-k+1} \quad \text{for} \quad a < t \le \alpha \quad (k = n_1+1, \dots, l). \tag{21}$$

Hence on account of (2_1) we find

$$|v_j^{(k-1)}(t)| \le \Big(\frac{\gamma_j}{\gamma_j^*(\lambda_1 - n_1 + 1)} + \frac{r_j}{\nu_{kl}(\lambda_1)} \Big) (t-a)^{\lambda_1-k+1} \quad \text{for} \quad a < t \le \alpha \quad (k = 1, \dots, n_1). \tag{22}$$

Using inequalities (7), (10), (21) and (22), from (18) we obtain

$$|v_j^{(l-1)}(t)| \le \Big(1 + \frac{1}{\lambda_1 - n_1 + 1} \sum_{k=1}^{l} \int_t^\alpha (\tau - t)^{n-l} (\tau - a)^{\lambda_1-k+1} p_k^*(\tau)\, d\tau \Big) \frac{\gamma_j}{\gamma_j^*} +$$
$$+ \frac{r_j}{(n-l)!} \sum_{k=1}^{l} \frac{1}{\nu_{kl}(\lambda_1)} \int_t^\alpha (\tau - t)^{n-l} (\tau - a)^{\lambda_1-k+1} p_k^*(\tau)\, d\tau +$$
$$+ \frac{c_1}{\gamma_j^*} |q_j(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2} (t-a)^{\lambda_1-l+1} \quad \text{for } a < t \le \alpha.$$

Hence by (8_l) and (17) we have

$$(t-a)^{l-1-\lambda_1} |v_j^{(l-1)}(t)| \le$$
$$\le \frac{\tilde c_1}{\gamma_j^*} \Big(\gamma_j + |q_j(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2} \Big) + \eta r_j \quad \text{for} \quad a < t \le \alpha$$

and

$$r_j \le \frac{\tilde c_1}{\gamma^*} \Big(\gamma_j + |q_j(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2} \Big) + \eta r_j \,,$$

where

$$\tilde c_1 = 1 + c_1 + \frac{(n-l)!}{\lambda_1 - n_1 + 1} \sum_{k=1}^{l} \nu_{kl}(\lambda_1).$$

Therefore

$$r_j \le \frac{\tilde c_2}{\gamma_j^*} \Big(\gamma_j + |q_j(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2} \Big) \,,$$

where

$$\tilde{c}_2 = \frac{\tilde{c}_1}{1-\eta}\,.$$

By virtue of this inequality (21) and (22) again yield estimates (20), where

$$c_3 = \sum_{k=1}^{l}\Big(\frac{1}{\lambda_1 - n_1 + 1} + \frac{\tilde{c}_2}{\nu_{kl}(\lambda_1)}\Big)\sup\Big\{\frac{(t-a)^{\lambda_1-k+1}}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t \le \alpha\Big\}.$$

It will be shown quite similarly that

$$|v_j^{(k-1)}(t)| \le \frac{c_4}{\gamma_j^*}\Big(\gamma_j + |q_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\Big)\,\sigma_{k,\lambda_1,\lambda_2}(t) \quad \text{for} \quad \beta \le t < b \quad (k=1,\dots,l)\,, \tag{23}$$

where c_4 is a positive constant independent of j and k.

By (2_1), (14) and (16)

$$|v_j^{(k-1)}(t)| \le \frac{\gamma_j}{\gamma_j^*}\text{for} \quad \alpha \le t \le \beta \quad (k=1,\dots,n)$$

and

$$|v_j^{(k-1)}(t)| \le \frac{\gamma_j}{\gamma_j^*}\sigma_k(t) \quad \text{for} \quad \alpha \le t \le \beta \quad (k=1,\dots,l)$$

where

$$\sigma_k(t) = \begin{cases} |t-t_2|^{n_{2k}} & \text{for } \alpha \le t \le \frac{t_2+t_3}{2}, \\ |t-t_i|^{n_{ik}} & \text{for } \frac{t_{i-1}+t_i}{2} \le t \le \frac{t_i+t_{i+1}}{2} \ (i=3,\dots,m-2), \\ |t-t_{m-1}|^{n_{m-1\,k}} & \text{for } \frac{t_{m-2}+t_{m-1}}{2} \le t \le \beta. \end{cases}$$ [1)]

Therefore

$$|v_j^{(k-1)}(t)| \le \frac{c_5}{\gamma_j^*}\gamma_j\,\sigma_{k,\lambda_1,\lambda_2}(t) \quad \text{for} \quad \alpha \le t \le \beta \quad (k=1,\dots,l)\,, \tag{24}$$

where

$$c_5 = \text{vraimax}\Big\{\sum_{k=1}^{l}\frac{\sigma_k(t)}{\sigma_{k,\lambda_1,\lambda_2}(t)} : \alpha \le t \le \beta\Big\}.$$

Using (20), (23) and (24), from (13_1)-(13_3) we find

$$\gamma_{1j} \le c_0\Big(\gamma_j + |q_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\Big)\,,$$
$$\gamma_{2j} \le c_0\Big(\gamma_j + |q_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\Big)\,,$$
$$\gamma_{0j} \le c_0\gamma_j\,,$$

[1)] In the case $m=2$ it is assumed that $\sigma_k(t) \equiv 1$ $(k=1,\dots,l)$.

where

$$c_0 = l \max\Big\{c_3, c_4, c_5\Big\}.$$

Inequality (12) therefore yields

$$\gamma_j > \frac{j-2c_0}{3c_0} |q_j(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2}.$$

Therefore

$$\gamma_j^* = \gamma_j > 0, \quad v_j(t) = \frac{u_j(t)}{\gamma_j} \quad \text{for} \ \ j \geq [2c_0]+1, \tag{25}$$

$$\max\Big\{\sum_{k=1}^{n} |v_j^{(k-1)}(t)| \colon \alpha \leq t \leq \beta\Big\} = 1 \quad \text{for} \ \ j \geq [2c_0]+1, \tag{26}$$

$$v_j^{(n)}(t) = \sum_{k=1}^{l} \tilde{p}_{kj}(t) v_j^{(k-1)}(t) + \frac{q_j(t)}{\gamma_j} \quad \text{for } j \geq [2c_0]+1 \tag{27}$$

and

$$\frac{|q_j(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2}}{\gamma_j} < \frac{3c_0}{j-2c_0} \quad \text{for } j \geq [2c_0]+1. \tag{28}$$

By virtue of (25) and (28) we obtain from (20), (23) and (24)

$$|v_j^{(k-1)}(t)| \leq c^* \sigma_{k,\lambda_1,\lambda_2}(t) \text{ for } a < t < b \quad (k = 1, \dots, l; j > [2c_0]+2), \tag{29}$$

where

$$c^* = c_0(1+3c_0).$$

It will be shown now that the sequences $\Big(v_j^{(k-1)}\Big)_{j=1}^{+\infty}$ $(k = 1, \dots, n)$ are uniformly bounded and equicontinuous inside $]a, b[$. For this we shall need in the first place the estimate of the integral

$$\int_s^t \frac{q_j(\tau)}{\gamma_j} \, d\tau.$$

Let a_0 and b_0 be arbitrary points from the intervals $]a, \alpha[$ and $]\beta, b[$, respectively. It is easy to see that

$$\left| \int_s^t \frac{q_j(\tau)}{\gamma_j} \, d\tau \right| \leq \frac{2}{\gamma_j (a_0-a)^{n-1-\lambda_1}(b-b_0)^{n-1-\lambda_2}} |q_j(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2}$$
$$\text{for} \quad a_0 \leq s, \quad t \leq b_0 \quad (j \geq [2c_0]+1).$$

From which by virtue of (28) we have

$$\left|\int_s^t \frac{q_j(\tau)}{\gamma_j}\,d\tau\right| \le \frac{\rho(a_0,b_0)}{j} \text{ for } a_0 \le s, \quad t \le b_0 \quad (j \ge 2[2c_0]+2)\,, \tag{30}$$

where

$$\rho(a_0,b_0) = \frac{12c_0}{(a_0-a)^{n-1-\lambda_1}(b-b_0)^{n-1-\lambda_2}}\,|q_j(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\,.$$

We introduce the functions

$$f^*(t) = c^* \sum_{k=1}^{l} p_k^*(t)\sigma_{k,\lambda_1,\lambda_2}(t)\,,$$

$$f_1^*(t) = \frac{c_{01}}{(t-a)^{n-1-\lambda_1}} + \int_t^\alpha f^*(\tau)\,d\tau\,,\ f_2^*(t) = \frac{c_{02}}{(b-t)^{n-1-\lambda_2}} + \int_\beta^t f^*(\tau)\,d\tau\,,$$

where

$$c_{01} = \frac{12c_0}{(b-\alpha)^{n-1-\lambda_2}}\,, \quad c_{02} = \frac{12c_0}{(\beta-a)^{n-1-\lambda_1}}\,.$$

Using (4) and (5) we have

$$f^* \in L_{\mathrm{loc}}(]a,b[;\mathbb{R}),$$

$$\int_a^\alpha ds \int_s^\alpha (\tau-s)^{n-n_1-2} f_1^*(\tau)\,d\tau < +\infty \quad \text{for} \quad l \in \{1,\dots,n_1\}$$

and

$$\int_\beta^b ds \int_\beta^s (s-\tau)^{n-n_m-2} f_2^*(\tau)\,d\tau < +\infty \quad \text{for } l \in \{1,\dots,n_m\}\,.$$

Therefore, taking into account conditions (10), (26), (29), (30), also the equalities

$$v_j^{(k-1)}(a) = 0 \quad (k=1,\dots,n_1)\,, \quad v_j^{(k-1)}(b) = 0 \quad (k=1,\dots,n_m)\,,$$

from (27) we obtain

$$|v_j^{(n-1)}(t)| \le 1 + \rho(a_0,b_0) + \left|\int_\alpha^t f^*(\tau)\,d\tau\right| \quad \text{for} \quad a_0 \le t \le b_0 \quad (j \ge 2[2c_0]+2)\,, \tag{31}$$

$$|v_j^{(n-1)}(t) - v_j^{(n-1)}(s)| \le \left|\int_s^t f^*(\tau)\,d\tau\right| + \frac{\rho(a_0,b_0)}{j} \quad \text{for } a_0 \le s,\ t \le b_0 \quad (j \ge 2[2c_0]+2)\,, \tag{32}$$

$$|v_j^{(k-1)}(t)| \le (t-a)^{n_1-k+1} + \int_a^t (t-s)^{n_1-k} \int_s^\alpha (\tau-s)^{n-n_1-2} f_1^*(\tau)\,d\tau\,ds$$
$$\text{for } a < t \le \alpha\,, \quad l \in \{1,\dots,n_1\} \quad (k=1,\dots,n_1)\,, \tag{29$_1$}$$

$$|v_j^{(k-1)}(t)| \le (b-t)^{n_m-k+1} + \int_t^b (s-t)^{n_m-k} \int_\beta^s (s-\tau)^{n-n_m-2} f_2^*(\tau)\,d\tau\,ds$$
$$\text{for } \beta \le t < b\,, \quad l \in \{1,\dots,n_m\} \quad (k=1,\dots,n_m)\,. \tag{29$_2$}$$

From (26), (31) and (32) it follows that on the interval $[a_0, b_0]$ sequences $\Big(v_j^{(k-1)}\Big)_{j=1}^{+\infty}$ $(k=1,\dots,n)$ are uniformly bounded and equicontinuous. Therefore, since a_0 and b_0 are arbitrary, by virtue of the Arzela-Askoli lemma it can be assumed without loss of generality that they converge uniformly inside $]a,b[$.

Let

$$u(t) = \lim_{j\to+\infty} v_j(t) \quad \text{for } a < t < b\,.$$

Then

$$u^{(k-1)}(t) = \lim_{j\to+\infty} v_j^{(k-1)}(t) \quad \text{for } a < t < b \quad (k=1,\dots,n)\,. \tag{33}$$

On the other hand, on account of (26), (29), (29_1) and (29_2) we have

$$\max\Big\{ \sum_{k=1}^n |u^{(k-1)}(t)| \colon \alpha \le t \le \beta \Big\} = 1\,, \tag{34}$$

$$|u^{(k-1)}(t)| \le c^* \sigma_{k,\lambda_1,\lambda_2}(t) \quad \text{for } a < t < b \quad (k=1,\dots,l)\,, \tag{35}$$

$$|u^{(k-1)}(t)| \le (t-a)^{n_1-k+1} + \int_a^t (t-s)^{n_1-k} \int_s^\alpha (\tau-s)^{n-n_1-2} f_1^*(\tau)\,d\tau\,ds$$
$$\text{for } a < t \le \alpha\,, \quad l \in \{1,\dots,n_1\} \quad (k=1,\dots,n_1), \tag{35$_1$}$$

$$|u^{(k-1)}(t)| \le (b-t)^{n_m-k+1} + \int_t^b (s-t)^{n_m-k} \int_\beta^s (s-\tau)^{n-n_m-2} f_2^*(\tau)\,d\tau\,ds$$
$$\text{for } \beta \le t < b\,, \quad l \in \{1,\dots,n_m\} \quad (k=1,\dots,n_m). \tag{35$_2$}$$

Our aim is to prove that u is a solution of equation (1_0), where $p_k :]a,b[\to \mathbb{R}$ $(k=1,\dots,l)$ are the measurable functions satisfying inequalities (3).

It is assumed for each $i \in \{1,\dots,m-1\}$ that

$$s_i = \frac{t_i + t_{i+1}}{2}$$

and

$$P_{ikj}(t) = \int_{s_i}^{t} \tilde{p}_{kj}(\tau)\,d\tau \quad \text{for } t_i < t < t_{i+1} \quad (k=1,\dots,l).$$

From (4) and (10) it follows that sequence $\Big(P_{ikj}\Big)_{j=1}^{+\infty}$ $(k=1,\dots,l)$ are uniformly bounded and equicontinuous inside $]t_i,t_{i+1}[$. Therefore, by the Arzela-Askoli lemma it can be assumed without loss of generality that these sequences converge uniformly inside $]t_i,t_{i+1}[$.

Let

$$P_{ik}(t) = \lim_{j\to+\infty} P_{ikj}(t).$$

Passing to the limit in the inequality

$$\int_s^t p_{1k}(\tau)\,d\tau \le P_{ikj}(t) - P_{ikj}(s) \le \int_s^t p_{2k}(\tau)\,d\tau \quad \text{for } t_i < s < t < t_{i+1}$$

when $j \to +\infty$, we obtain

$$\int_s^t p_{1k}(\tau)\,d\tau \le P_{ik}(t) - P_{ik}(s) \le \int_s^t p_{2k}(\tau)\,d\tau \quad \text{for} \quad t_i < s < t < t_{i+1},$$

from which it is clear that P_{ik} are absolutely continuous inside $]t_i,t_{i+1}[$ and

$$p_{1k}(t) \le P'_{ik}(t) \le p_{2k}(t) \quad \text{for } t_i < t < t_{i+1} \quad (k=1,\dots,l).$$

Therefore the functions

$$p_k(t) = P'_{ik}(t) \quad \text{for } t_i < t < t_{i+1},\ i=1,\dots,m-1 \quad (k=1,\dots,l) \tag{36}$$

satisfy inequalities (3).

Due to (30) and (36) it is clear that

$$\lim_{j\to+\infty} \int_{s_i}^{t} \tilde{p}_{kj}(\tau)\,d\tau = \int_{s_i}^{t} p_k(\tau)\,d\tau \quad (k=1,\dots,l), \quad \lim_{j\to+\infty} \int_{s_i}^{t} \frac{q_j(\tau)}{\gamma_j} = 0$$

uniformly inside $]t_i,t_{i+1}[$. By virtue of Theorem 1.2 from Ref. [4], from these conditions and equalities (33) it follows that u is a solution of equation (1_0) on each interval $]t_i,t_{i+1}[$ $(i=1,\dots,m-1)$. Since, besides, $u \in \tilde{C}^{m-1}_{\mathrm{loc}}(]a,b[;\mathbb{R})$ and estimates (35), (35_1) and (35_2) are fulfilled, it is obvious that u is a solution of

Problem (1_0), (2_1), (2_2). Therefore by condition (5) $u(t) \equiv 0$. But this contradicts the equality (34). The contradiction obtained proves the lemma. □

It is assumed for each natural number i that

$$\delta_{ji} = \begin{cases} \left[t_j - \frac{t_j - t_{j-1}}{3i}, t_j + \frac{t_{j+1} - t_j}{3i}\right] & \text{for } j \in \{2, \dots, m-1\} \\ \left[a, a + \frac{t_2 - a}{3i}\right] & \text{for } j = 1 \\ \left[b - \frac{b - t_{m-1}}{3i}, b\right] & \text{for } j = m \end{cases},$$

$$\delta_i = \bigcup_{j=1}^{m} \delta_{ji}\,. \tag{37}$$

Lemma 2. *Let condition (5) be fulfilled. Then there exists a natural number i_0 such that for $i \ge i_0$*

$$(p_{11i}, \dots, p_{1li}; p_{21i}, \dots, p_{2li}) \in V(t_1, \dots, t_m; n_1, \dots, n_m; \lambda_1, \lambda_2)\,,$$

where

$$\begin{aligned} &p_{1ki}(t) = p_{1k}(t)\,, && p_{2ki}(t) = p_{2k}(t) && \textit{for } t \in [a,b]\backslash \delta_i, \\ &p_{1ki}(t) = -|p_{1k}(t)|\,, && p_{2ki}(t) = |p_{2k}(t)| && \textit{for } t \in \delta_i. \end{aligned} \tag{38}$$

Proof. In view of (38)

$$\max\Big\{|p_{1ki}(t)|, |p_{2ki}(t)|\Big\} = p_k^*(t) \quad (k = 1, \dots, l;\ i = 1, 2, \dots)\,, \tag{39}$$

where

$$p_k^*(t) = \max\Big\{|p_{1k}(t)|, |p_{2k}(t)|\Big\}.$$

Therefore by virtue of (5) to prove the lemma it remains for us to verify the existence of a natural number i_0 such that for any $i \ge i_0$ and measurable functions $p_k :]a, b[\to \mathbb{R}$ $(k = 1, \dots, l)$ satisfying inequalities

$$p_{1ki}(t) \le p_k(t) \le p_{2ki}(t) \quad \text{for } a < t < b \quad (k = 1, \dots, l)$$

Problem (1_0), (2_1), (2_2) is solved only trivially.

Let us assume the opposite. Then for each natural j there exists a natural number i_j and measurable functions $\tilde{p}_{kj} :]a, b[\to \mathbb{R}$ $(k = 1, \dots, l)$ such that

$$p_{1ki_j}(t) \le \tilde{p}_{kj}(t) \le p_{2ki_j}(t) \quad \text{for } a < t < b \quad (k = 1, \dots, l) \tag{40}$$

and the equation

$$u^{(n)} = \sum_{k=1}^{l} \tilde{p}_{kj}(t) u^{(k-1)} \tag{41}$$

has a nontrivial solution v_j satisfying the condition

$$\operatorname{vraimax}\Big\{\sum_{k=1}^{l}\frac{|v_j^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)}:\ a<t<b\Big\}=1\,. \tag{42}$$

We choose $\alpha\in]a,t_2[$, $\beta\in]t_{m-1},b[$ and $\eta\in]0,1[$ such that inequalities (7), (8_l), (9_l) be fulfilled, and assume

$$\gamma_{1j}=\sup\Big\{\sum_{k=1}^{l}\frac{|v_j^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)}:\ a<t\le\alpha\Big\},$$
$$\gamma_{0j}=\operatorname{vraimax}\Big\{\sum_{k=1}^{l}\frac{|v_j^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)}:\ \alpha\le t\le\beta\Big\},$$
$$\gamma_{2j}=\sup\Big\{\sum_{k=1}^{l}\frac{|v_j^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)}:\ \beta\le t<b\Big\}.$$

Then by (42) we have

$$\gamma_{1j}+\gamma_{0j}+\gamma_{2j}\ge 1\quad (j=1,2,\dots)\,. \tag{43}$$

Let

$$\begin{aligned}\gamma_j&=\max\Big\{\sum_{k=1}^{n}|v_j^{(k-1)}(t)|:\ \alpha\le t\le\beta\Big\},\\ \gamma_j^*&=\begin{cases}1 & \text{for } \gamma_j=0\\ \gamma_j & \text{for } \gamma_j>0\end{cases},\\ u_j(t)&=\frac{v_j(t)}{\gamma_j^*}\,.\end{aligned} \tag{44}$$

Repeating the reasoning used in proving Lemma 1, we shall prove that there exists a natural number j_0 and a positive constant c_0 such that

$$\gamma_{0j}\le c_0\gamma_j\,,\quad \gamma_{1j}\le c_0\gamma_j\,,\quad \gamma_{2j}\le c_0\gamma_j\,,\quad \text{for } j\ge j_0\,.$$

On account of these inequalities it follows from (41), (43) and (44) that

$$\max\Big\{\sum_{k=1}^{n}|u_j^{(k-1)}(t)|:\ \alpha\le t\le\beta\Big\}=1\quad (j=j_0,j_0+1,\dots)\,, \tag{45}$$

$$\begin{gathered}|u_j^{(k-1)}(t)|\le 3c_0\sigma_{k,\lambda_1,\lambda_2}(t)\\ \text{for } a<t<b\quad (k=1,\dots,l;\ j=j_0,j_0+1,\dots)\,,\end{gathered} \tag{46}$$

$$|u_j^{(k-1)}(t)| \le f^*(t) \quad \text{for } a < t < b \quad (j = j_0, j_0+1, \dots), \tag{47}$$

$$\begin{gathered} |u_j^{(k-1)}(t)| \le (t-a)^{n_1-k+1} + \\ + \int_a^t (t-s)^{n_1-k} \int_s^\alpha (\tau - s)^{n-n_1-1} f^*(\tau)\, d\tau\, ds \\ \text{for } a < t \le \alpha, \quad l \in \{1, \dots, n_1\} \quad (k = 1, \dots, n_1), \end{gathered} \tag{46_1}$$

$$\begin{gathered} |u_j^{(k-1)}(t)| \le (b-t)^{n_m-k+1} + \\ + \int_t^b (s-t)^{n_m-k} \int_\beta^s (s-\tau)^{n-n_m-1} f^*(\tau)\, d\tau\, ds \\ \text{for } \beta \le t < b, \quad l \in \{1, \dots, n_m\} \ (k = 1, \dots, n_m), \end{gathered} \tag{46_2}$$

where

$$f^*(t) = 3c_0 \sum_{k=1}^{l} p_k^*(t) \sigma_{k,\lambda_1,\lambda_2}(t) .$$

From (46) and (47) it follows that sequences $\left(u_j^{(k-1)}\right)_{j=1}^{+\infty}$ $(k = 1, \dots, n)$ are uniformly bounded and equicontinuous inside $]a, b[$. It can be assumed without loss of generality that these sequences converge uniformly inside $]a, b[$.

Let

$$u(t) = \lim_{j \to +\infty} u_j(t) .$$

Then

$$u^{(k-1)}(t) = \lim_{j \to +\infty} u_j^{(k-1)}(t) \quad (k = 1, \dots, n) . \tag{48}$$

Therefore by (45) equality (34) will be fulfilled.

In view of (38) and (40) it can be assumed without loss of generality that for any $i \in \{1, \dots, m-1\}$

$$\lim_{j \to +\infty} \int_{s_i}^t \tilde{p}_{kj}(\tau)\, d\tau = \int_{s_i}^t p_k(\tau) d\tau \tag{49}$$

uniformly inside $]t_i, t_{i+1}[$, where

$$s_i = \frac{t_i + t_{i+1}}{2} ,$$

and $p_k :]a, b[\to \mathbb{R}$ $(k = 1, \dots, l)$ are measurable functions satisfying inequalities (3).

By Theorem 1.2 from Ref. [4] it follows from conditions (46), (46_1), (46_2), (48) and (49) that u is a solution of Problem (1_0), (2_1), (2_2). Therefore in view of (5) $u(t) \equiv 0$, which contradicts (34). The lemma is proved. □

Lemma 3. *Let condition (5) be fulfilled. Then there exist a positive number ρ_0 and a natural number i_0 such that for any $i \geq i_0$, $q \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ and measurable functions $p_k :]a,b[\to \mathbb{R}$ $(k = 1,\dots,l)$, satisfying the conditions*

$$p_{1k}(t) \leq p_k(t) \leq p_{2k}(t) \quad \text{for } t \in [a,b]\backslash\delta_i\,, \qquad p_k(t) = 0 \quad \text{for } t \in \delta_i\,, \quad (k = 1,\dots,l), \tag{50}$$

an arbitrary solution of Problem (1), (2_1), (2_2) admits estimate (6).

Proof. By virtue of Lemma 2 there exists a natural number i_0 such that

$$(p_{11i_0},\dots,p_{1li_0};p_{21i_0},\dots,p_{2li_0}) \in V(t_1,\dots,t_m;n_1,\dots,n_m;\lambda_1,\lambda_2)\,,$$

where p_{1ki_0} and p_{2ki_0} are functions given by (38).

On the other hand, by virtue of (50) we have inequalities

$$p_{1ki}(t) \leq p_k(t) \leq p_{2ki}(t) \quad \text{for } a < t < b \quad (k = 1,\dots,l)\,,$$

for each $i \geq i_0$. If we now use Lemma 1, then the validity of Lemma 3 becomes obvious. □

2. Unique Solvability of Problem (1), (2_1), (2_2)

In this section we are going to establish the conditions for Problem (1), (2_1), (2_2) to be Fredholmian when it is assumed that

$$p_k(\cdot)\sigma_{k,\lambda_1,\lambda_2}(\cdot) \in L_{\text{loc}}(]a,b[;\mathbb{R}) \quad (k = 1,\dots,l)\,, \tag{51}$$

$$q \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})\,. \tag{52}$$

In particular, we have

Theorem 1. *Let*

$$(p_1,\dots,p_l) \in S^+(a,b;n,n_1;\lambda_1) \cap S^-(a,b;n,n_m;\lambda_2)\,. \tag{53}$$

Then for Problem (1), (2_1), (2_2) to be uniquely solvable, it is necessary and sufficient that the corresponding homogeneous Problem (1_0), (2_1), (2_2) have the trivial solution only.

Proof. Since the necessity is obvious, we shall prove the sufficiency. Let

$$p_{1k}(t) \equiv p_{2k}(t) \equiv p_k(t) \quad (k = 1,\dots,l)\,.$$

From condition (53) and the fact that the homogeneous Problem (1_0), (2_1), (2_2) has no nontrivial solution it follows that

$$(p_{11},\dots,p_{1l};p_{21},\dots,p_{2l}) \in V(t_1,\dots,t_m;n_1,\dots,n_m;\lambda_1,\lambda_2)\,.$$

By virtue of Lemma 3 there exists a natural number i_0 such that for any $i \geq i_0$ the equation

$$u^{(n)} = \sum_{k=1}^{l} p_{ki}^*(t) u^{(k-1)}, \tag{54}$$

where

$$p_{ki}^*(t) = \begin{cases} p_k(t) & \text{for } t \in [a,b] \backslash \delta_i \ ^{1)} \\ 0 & \text{for } t \in \delta_i \end{cases}, \tag{55}$$

has no nontrivial solution, satisfying the boundary conditions (2_1), in the class $\tilde{C}^{n-1}([a,b];\mathbb{R})$. Therefore by the well-known theorem on the unique solvability of the general boundary value problem[2] for each $i \geq i_0$ the equation

$$u^{(n)} = \sum_{k=1}^{l} p_{ki}^*(t) u^{(k-1)} + q_i(t), \tag{56}$$

where

$$q_i(t) = \begin{cases} q(t) & \text{for } t \in \left[a + \frac{b-a}{3i}, b - \frac{b-a}{3i}\right] \\ 0 & \text{for } t \in [a,b] \backslash \left[a + \frac{b-a}{3i}, b - \frac{b-a}{3i}\right] \end{cases}, \tag{57}$$

has the unique solution $u_i \in \tilde{C}^{n-1}([a,b];\mathbb{R})$, satisfying the boundary conditions (2_1). As for conditions (2_2) they automatically follow from (2_1), since $u_i \in \tilde{C}^{n-1}([a,b];\mathbb{R})$.

By Lemma 3 there exists a positive number ρ_0 such that

$$|u_i^{(k-1)}(t)| \leq \rho_0 \sigma_{k,\lambda_1,\lambda_2}(t) |\, q_i(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2}$$
$$\text{for } a < t < b \quad (k = 1, \dots, l; i \geq i_0).$$

However, in view of (57)

$$|q_i(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2} \leq \left(2^{n-1-\lambda_1} + 2^{n-1-\lambda_2}\right) |q(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2}.$$

Therefore

$$|u_i^{(k-1)}(t)| \leq \rho_1 \sigma_{k,\lambda_1,\lambda_2}(t) |q(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2}$$
$$\text{for } a < t < b \quad (k = 1, \dots, l; i \geq i_0), \tag{58}$$

---

1) δ_i is a set determined by (37)

2) see, e.g., Ref. [4], Theorem 1.1

where
$$\rho_1 = \rho_0 \left(2^{n-1-\lambda_1} + 2^{n-1-\lambda_2} \right) .$$
Besides, for some $\alpha \in]a, t_2[$, $\beta \in]t_{m-1}, b[$ such that
$$\alpha - a < 1, \quad b - \beta < 1$$
we have
$$\Big| \int_t^\alpha (\tau - t)^{n-n_1-1} q_i(\tau)\, d\tau \Big| \le c_1 (t-a)^{\lambda_1 - n_1} |q(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2}$$
$$\text{for } a < t \le \alpha \,,$$
$$\Big| \int_\beta^t (t - \tau)^{n-n_m-1} q_i(\tau)\, d\tau \Big| \le c_2 (b-t)^{\lambda_2 - n_m} |q(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2}$$
$$\text{for } \beta \le t < b \,,$$
where
$$c_1 = \frac{2^{n-\lambda_1} + 2^{n-\lambda_2}}{n_1 - \lambda_1} (b-\alpha)^{\lambda_2 - n + 1}, \quad c_2 = \frac{2^{n-\lambda_1} + 2^{n-\lambda_2}}{n_m - \lambda_2} (\beta - a)^{\lambda_1 - n + 1} ,$$
and also by virtue of (53) we have
$$\int_a^\alpha ds \int_s^\alpha (\tau - s)^{n-n_1-1} f^*(\tau) d\tau < +\infty \quad \text{for } l \in \{1, \dots, n_1\} ,$$
$$\int_\beta^b ds \int_\beta^s (s - \tau)^{n-n_m-1} f^*(\tau) d\tau < +\infty \quad \text{for } l \in \{1, \dots, n_m\} ,$$
where
$$f^*(t) = \rho_1 |q(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2} \sum_{k=1}^{l} |p_k(t)| \sigma_{k,\lambda_1,\lambda_2}(t) ,$$
and $f^*(t) \in L_{\mathrm{loc}}(]a, b[; \mathbb{R})$. Therefore, taking into account (55), (57) (58) and the equalities
$$u_i^{(k-1)}(a) = 0 \quad (k = 1, \dots, n_1), \quad u_i^{(k-1)}(b) = 0 \quad (k = 1, \dots, n_m),$$
from (56) we find

(58$_1$)
$$|u_i^{(k-1)}(t)| \le (t-a)^{n_1-k+1} \sum_{j=n_1+1}^{n} |u_i^{(j-1)}(\alpha)| +$$
$$+ \int_a^t (t-s)^{n_1-k} \int_s^\alpha (\tau - s)^{n-n_1-1} f^*(\tau)\, d\tau\, ds +$$
$$+ \frac{c_1}{\lambda_1 - n_1 + 1} |q(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2} (t-a)^{\lambda_1 - k + 1}$$
$$\text{for } a < t \le \alpha , \quad l \in \{1, \dots, n_1\} \quad (k = 1, \dots, n_1) ,$$

$$
(58_2)\qquad \begin{aligned}
&|u_i^{(k-1)}(t)| \le (b-t)^{n_m-k+1} \sum_{j=n_m+1}^{n} |u_i^{(j-1)}(\beta)|+\\
&+\int_t^b (s-t)^{n_m-k} \int_\beta^s (s-\tau)^{n-n_m-1} f^*(\tau)\,d\tau\,ds+\\
&+\frac{c_2}{\lambda_2-n_m+1}|q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}(b-t)^{\lambda_2-k+1}\\
&\text{for } \beta\le t<b\,,\quad l\in\{1,\dots,n_m\}\ (k=1,\dots,n_m),
\end{aligned}
$$

$$
(59)\qquad |u_i^{(n)}(t)| = \Big|\sum_{k=1}^{l} p_{ki}^*(t)u_i^{(k-1)}(t)+q_i(t)\Big| \le f^*(t)+|q(t)|\,.
$$

Proceeding from (52), (58) and (59), by the Arzela-Askoli lemma it can be assumed without loss of generality that sequences $\Big(u_i^{(k-1)}\Big)_{i=1}^{+\infty}$ $(k=1,\dots,n)$ converge uniformly inside $]a,b[$.

Let

$$
u(t)=\lim_{i\to+\infty} u_i(t) \quad \text{for } a<t<b\,.
$$

Then

$$
(60)\qquad u^{(k-1)}(t)=\lim_{i\to+\infty} u_i^{(k-1)}(t) \qquad \text{for } a<t<b \quad (k=1,\dots,n)\,.
$$

Therefore by virtue of (58), (58_1) and (58_2)

$$
(61)\qquad \begin{aligned}
&|u^{(k-1)}(t)| \le \rho_1\sigma_{k,\lambda_1,\lambda_2}(t)|q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\\
&\text{for}\quad a<t<b \quad (k=1,\dots,l)\,,
\end{aligned}
$$

$$
(61_1)\qquad \begin{aligned}
&|u^{(k-1)}(t)| \le (t-a)^{n_1-k+1} \sum_{j=n_1+1}^{n} |u^{(j-1)}(\alpha)|+\\
&+\int_a^t (t-s)^{n_1-k} \int_s^\alpha (\tau-s)^{n-n_1-1} f^*(\tau)\,d\tau\,ds+\\
&+\frac{c_1}{\lambda_1-n_1+1}|q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}(t-a)^{\lambda_1-k+1}\\
&\text{for } a<t\le\alpha\,,\ l\in\{1,\dots,n_1\}\ (k=1,\dots,n_1)\,,
\end{aligned}
$$

$$
(61_2)\qquad \begin{aligned}
&|u^{(k-1)}(t)| \le (b-t)^{n_m-k+1} \sum_{j=n_m+1}^{n} |u^{(j-1)}(\beta)|\\
&+\int_t^b (s-t)^{n_m-k} \int_\beta^s (s-\tau)^{n-n_m-1} f^*(\tau)\,d\tau\,ds\\
&+\frac{c_2}{\lambda_2-n_m+1}|q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}(b-t)^{\lambda_2-k+1}\\
&\text{for } \beta\le t<b,\ l\in\{1,\dots,n_m\}\ (k=1,\dots,n_m)\,.
\end{aligned}
$$

On the other hand, for any $s,\, t\in]a,b[$ (56) yield

$$u_i^{(n-1)}(t)-u_i^{(n-1)}(s)=\int_s^t \Big(\sum_{k=1}^l p_{ki}^*(\tau)u_i^{(k-1)}(\tau)+q_i(\tau)\Big)\,d\tau\,.$$

Taking (55), (57)-(59) also (60) into account and using the Lebesgue theorem on the limit passage under the integral sign, from the latter equality we obtain

$$u^{(n-1)}(t)-u^{(n-1)}(s)=\int_s^t \Big(\sum_{k=1}^l p_k(\tau)u^{(k-1)}(\tau)+q(\tau)\Big)\,d\tau\,.$$

Hence, with (61), (61_1) and (61_2) taken into account, we conclude that u is the solution of Problem (1), (2_1), (2_2).

The solution uniqueness of this problem follows from the unique solvability of the homogeneous Problem (1_0), (2_1), (2_2). The Theorem is proved. □

Corollary 1. *Let*

$$p_k(t)=\frac{g_{1k}(t)}{(t-a)^{n-k+1}}+\frac{g_{2k}(t)}{(b-t)^{n-k+1}}+p_{0k}(t)\quad (k=1,\dots,l)\,, \tag{62}$$

where

$$\sigma_{k,n}(\cdot)p_{0k}(\cdot)\in L([a,b];\mathbb{R})\quad (k=1,\dots l)\,, \tag{63}$$

and $g_{1k},\ g_{2k}:[a,b]\to\mathbb{R}\quad (k=1,\dots,l)$ *are continuous functions satisfying the inequalities*

$$\sum_{k=1}^l \frac{|g_{1k}(a)|}{\nu_{kl}(\lambda_1)\nu_{ln+1}(\lambda_1)}<1\,,\quad \sum_{k=1}^l \frac{|g_{2k}(b)|}{\nu_{kl}(\lambda_2)\nu_{ln+1}(\lambda_2)}<1\,. \tag{64}$$

Then for Problem (1), (2_1), (2_2) to be uniquely solvable it is necessary and sufficient that the corresponding homogeneous Problem (1_0), (2_1), (2_2) have the trivial solution only.

Proof. By Theorem 1 to prove the corollary it suffices to verify that functions p_k $(k=1,\dots,l)$ satisfy condition (53).

First it will be shown that

$$(p_1,\dots,p_l)\in S^+(a,b;n,n_1;\lambda_1)\,. \tag{65}$$

Since functions g_{1k} $(k=1,\dots,l)$ are continuous and the first of inequalities (64), there exists $\alpha\in]a,t_2[$ such that

$$\sum_{k=1}^l \frac{c_k}{\nu_{kl}(\lambda_1)\nu_{ln+1}(\lambda_1)}<1\,, \tag{66}$$

where

$$c_k = \max\Big\{|g_{1k}(t)| : a \le t \le \alpha\Big\} \quad (k = 1, \dots, l)\,.$$

Let $l \in \{1, \dots, n_1\}$. It will be shown that for any $k \in \{1, \dots, l\}$

$$\limsup_{t\to a}(t-a)^{l-1-\lambda_1}\times \tag{67}$$
$$\times\int_a^t (t-s)^{n_1-l}\int_s^\alpha (\tau-s)^{n-n_1-1}(\tau-a)^{\lambda_1-k+1}|\tilde p_{0k}(\tau)|\,d\tau\,ds = 0\,,$$

where

$$\tilde p_{0k}(t) = \frac{g_{2k}(t)}{(b-t)^{n-k+1}} + p_{0k}(t)\,.$$

Because of the continuity of g_{2k} and condition (63) we have

$$\int_a^\alpha (\tau-a)^{n-k}|\tilde p_{0k}(\tau)|\,d\tau < +\infty\,.$$

Therefore for an arbitrary $\varepsilon > 0$ there exists $\alpha_\varepsilon \in]a, \alpha[$ such that

$$\int_a^{\alpha_\varepsilon}(\tau-a)^{n-k}|\tilde p_{0k}(\tau)|\,d\tau < (\lambda_1 - n_1 + 1)\varepsilon\,.$$

Next, for $a < s \le \alpha_\varepsilon$

$$\int_s^\alpha (\tau-a)^{n-n_1+\lambda_1-k}|\tilde p_{0k}(\tau)|\,d\tau \le \int_s^{\alpha_\varepsilon}(\tau-a)^{n-n_1+\lambda_1-k}|\tilde p_{0k}(\tau)|\,d\tau + r_0 \le$$
$$\le (s-a)^{\lambda_1-n_1}\int_s^{\alpha_\varepsilon}(\tau-a)^{n-k}|\tilde p_{0k}(\tau)|\,d\tau + r_0 \le$$
$$\le (\lambda_1-n_1+1)\varepsilon(s-a)^{\lambda_1-n_1} + r_0\,,$$

where

$$r_0 = \int_{\alpha_\varepsilon}^\alpha (\tau-a)^{n-n_1+\lambda_1-k}|\tilde p_{0k}(\tau)|\,d\tau\,.$$

Using this estimate, we find for $a < t \le \alpha_\varepsilon$

$$(t-a)^{l-1-\lambda_1}\int_a^t (t-s)^{n_1-l}\,ds\int_s^\alpha (\tau-s)^{n-n_1-1}(\tau-a)^{\lambda_1-k+1}|\tilde p_{0k}(\tau)|\,d\tau \le$$
$$\le (t-a)^{n_1-1-\lambda_1}\int_a^t ds\int_s^\alpha (\tau-a)^{n-n_1+\lambda_1-k}|\tilde p_{0k}(\tau)|\,d\tau \le \varepsilon + r_0(t-a)^{n_1-\lambda_1}\,,$$

and therefore

$$\limsup_{t\to a}(t-a)^{l-1-\lambda_1}\int_a^t (t-s)^{n_1-l}\int_s^\alpha (\tau-s)^{n-n_1-1}(\tau-a)^{\lambda_1-k+1}|\tilde p_{0k}(\tau)|\,d\tau\,ds \le \varepsilon\,,$$

from which due to the arbitrarity of ε we obtain (67).

By (62), (66) and (67)

$$\limsup_{t\to a}\frac{(t-a)^{l-1-\lambda_1}}{(n-n_1-1)!(n_1-l)!}\times$$
$$\times\sum_{k=1}^{l}\frac{1}{\nu_{kl}(\lambda_1)}\int_a^t(t-s)^{n_1-l}\int_s^\alpha(\tau-s)^{n-n_1-1}(\tau-a)^{\lambda_1-k+1}|p_k(\tau)|\,d\tau\,ds\le$$
$$\le\limsup_{t\to a}\frac{(t-a)^{l-1-\lambda_1}}{(n-n_1-1)!(n_1-l)!}\times$$
$$\times\sum_{k=1}^{l}\frac{c_k}{\nu_{kl}(\lambda_1)}\int_a^t(t-s)^{n_1-l}\int_s^\alpha(\tau-s)^{n-n_1-1}(\tau-a)^{\lambda_1-n}\,d\tau\,ds\le$$
$$\le\sum_{k=1}^{l}\frac{c_k}{\nu_{kl}(\lambda_1)\nu_{ln+1}(\lambda_1)}<1\,.$$

The validity of (65) in the case $l\in\{1,\dots,n_1\}$ is thereby proved. The case $l\in\{n_1+1,\dots,n\}$ is treated similarly.

The inclusion

$$(p_1,\dots,p_l)\in S^-(a,b;n,n_m;\lambda_2)$$

is proved by the same technique. The corollary is proved. □

Remark 1. Condition (64) is unprovable in the sense that none of the inequalities contained in it cannot be replaced by the corresponding equality. To verify this we shall consider the boundary value problem

$$u^{(n)}=\frac{\lambda_1(\lambda_1-1)\dots(\lambda_1-n+1)}{t^n}u+t^{\lambda_1-n}\,, \tag{68}$$

$$u^{(i-1)}(0)=0\quad(i=1,\dots n_1)\,,\qquad u^{(j-1)}(1)=0\quad(j=1,\dots,n_2)\,, \tag{69_1}$$

$$\sup\Big\{t^{-\lambda_1}(1-t)^{-\lambda_2}|u(t)|:0<t<1\Big\}<+\infty\,, \tag{69_2}$$

where n_1, $n_2\in\{1,\dots,n-1\}$, $n_1+n_2=n$, $\lambda_2\in]n_2-1,n_2[$, and $\lambda_1\in]n_1-1,n_1[$ is choosen so that the equation

$$x(x-1)\dots(x-n+1)=\lambda_1(\lambda_1-1)\dots(\lambda_1-n+1)$$

has n roots $x_1,\dots,x_n$ such that

$$x_1<x_2<\dots<x_n\quad\text{and } x_{n_1+1}=\lambda_1\,. \tag{70}$$

Here

$$m=2,\quad l=1,\quad g_{11}(t)\equiv(-1)^{n-n_1}\nu_{1n+1}(\lambda_1),\quad g_{21}(t)\equiv0\,,$$
$$p_{01}(t)\equiv0,\quad q(t)=t^{\lambda_1-n}\,.$$

Instead of condition (64) which for $l=1$, $a=0$, $b=1$ takes the form

$$(64_1) \qquad \frac{|g_{11}(0)|}{\nu_{ln+1}(\lambda_1)} < 1\,, \quad \frac{|g_{21}(0)|}{\nu_{ln+1}(\lambda_2)} < 1\,,$$

the condition

$$(64'_1) \qquad \frac{|g_{11}(0)|}{\nu_{ln+1}(\lambda_1)} = 1\,, \frac{|g_{21}(0)|}{\nu_{ln+1}(\lambda_2)} < 1\,,$$

is fulfilled in the case under consideration.

First it will be shown that under the boundary conditions (69_1), (69_2) the homogeneous equation

$$(68_0) \qquad u^{(n)} = \frac{\lambda_1(\lambda_1-1)\dots(\lambda_1-n+1)}{t^n}u$$

has the trivial solution only. Indeed, the general solution of the equation (68_0) has the form

$$(71) \qquad u(t) = \sum_{k=1}^{l} c_k t^{x_k}\,,$$

where $c_1,\dots,c_n$ are arbitrary constants. It is obvious that for (69_1) and (69_2) to be fulfilled it is necessary that

$$(72) \qquad \limsup_{t\to 0} \frac{|u(t)|}{t^{\lambda_1}} < +\infty$$

and

$$(73) \qquad u^{(j-1)}(1) = 0 \quad (j=1,\dots,n_2)\,.$$

In view of (70), (71) we find from (72) and (73)

$$c_1 = \dots = c_{n_1} = 0$$

and

$$\sum_{k=n_1+1}^{l} a_{ik} c_k = 0 \quad (i=1,\dots,n_2)\,,$$

where

$$a_{1k} = 1, \quad a_{ik} = x_k(x_k-1)\dots(x_k-i+2) \quad (i=2,\dots,n_2)\,.$$

Since the determinant of the latter system differs from zero, we write

$$c_{n_1+1} = \dots = c_n = 0\,.$$

Therefore $u(t) \equiv 0$.

Finally, it remains for us to show that Problem (68), (69_1), (69_2) has no solution.

It is not difficult to verify that the function

$$u_0(t) = c_0 t^{\lambda_1} ln\, t\,,$$

where

$$c_0 = \Big[\sum_{k=0}^{n-1} \prod_{\substack{j=0\\ j\neq k}}^{n-1} (\lambda_1 - j)\Big]^{-1},$$

is the solution of (68). Thus the general solution of this equation has the form

$$u(t) = \sum_{k=1}^{l} c_k t^{x_k} + c_0 t^{\lambda_1} ln\, t\,.$$

Therefore by virtue of (70)

$$\limsup_{t\to 0} \frac{|u(t)|}{t^{\lambda_1}} = \limsup_{t\to 0} \Big|\sum_{k=1}^{l} c_k t^{x_k} + c_0 t^{\lambda_1} ln\, t\Big| = +\infty\,,$$

no matter what the constants $c_1, \dots, c_n$ are.

Thus Problem (68), (69_1), (69_2) has no solution though all the conditions of Corollary 1 were fulfilled except for condition (64_1) which was replaced by condition ($64'_1$).

3. Problem (1), (2_1), (2_2) in the case $l = 1$

In this section we consider the following boundary value problem

$$u^{(n)} = p(t)u + q(t)\,, \tag{74}$$

$$u^{(k-1)}(t_i) = 0 \quad (k = 1, \dots, n_i;\; i = 1, \dots, m)\,, \tag{75_1}$$

$$\sup\Big\{(t-a)^{-\lambda_1}(b-t)^{-\lambda_2}|u(t)| : a < t < b\Big\} < \infty\,, \tag{75_2}$$

where $p(\cdot)\sigma_{1,\lambda_1,\lambda_2}(\cdot) \in L_{\mathrm{loc}}(]a,b[;\mathbb{R})$, $\quad q \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$.

By Theorem 1 for any $r \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ the differential equation

$$u^{(n)} = r(t)$$

has the unique solution $u_0(r)(\cdot) \colon]a,b[\to \mathbb{R}$ satisfying the boundary conditions (75_1), (75_2).

Below we use the following notation

$$\rho_{\lambda_1,\lambda_2}(r) = \mathrm{vraimax}\Big\{\frac{|u_0(r)(t)|}{\sigma_{1,\lambda_1,\lambda_2}(t)} : a < t < b\Big\}\,.$$

From Ref. [6] we easily obtain the following results.

Theorem 2. *Let*

$$|p(t)| \le \frac{r(t)}{\sigma_{1,\lambda_1,\lambda_2}(t)} \quad \text{for} \quad a < t < b\,,$$

where

$$r \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})\,,$$

and

$$\rho_{\lambda_1,\lambda_2}(r) < 1\,.$$

Then Problem (74), (75_1), (75_2) has the unique solution.

Theorem 3. *Let $m = 2$ and*

$$|p(t)| \le \frac{r_0}{(t-a)^n(b-t)^n} \quad \text{for } a < t < b\,,$$

where r_0 is the number satisfying the inequality

$$r_0 < \frac{n_1!n_2!(n_1-\lambda_1)(n_2-\lambda_2)(\lambda_1-n_1+1)(\lambda_2-n_2+1)}{\gamma_1(n_1,n_2,\lambda_1,\lambda_2)+\gamma_2(n_1,n_2,\lambda_1,\lambda_2)}(b-a)^n\,,$$

where

$$\gamma_1(n_1,n_2,\lambda_1,\lambda_2) = 2n_1(n_1-\lambda_1)(\lambda_2-n_2+1)(\lambda_1-n_1+1+n_2-\lambda_2)\,,$$
$$\gamma_2(n_1,n_2,\lambda_1,\lambda_2) = 2n_2(n_2-\lambda_2)(\lambda_1-n_1+1)(\lambda_2-n_2+1+n_1-\lambda_1)\,.$$

Then Problem (74), (75_1), (75_2) has the unique solution.

For the two points boundary value problem

$$u'' = p(t)u + q(t)\,, \tag{76}$$

$$u(a) = u(b) = 0\,, \tag{77_1}$$

$$\sup\Big\{(t-a)^{-\lambda_1}(b-t)^{-\lambda_2}|u(t)| : a < t < b\Big\} < +\infty\,, \tag{77_2}$$

where $0 < \lambda_1,\ \lambda_2 < 1$, $p \in L_{\mathrm{loc}}(]a,b[;\mathbb{R})$ and $q \in L_{1-\lambda_1,1-\lambda_2}(]a,b[;\mathbb{R})$, from Theorem 2 we obtain

Corollary 2. *Let*

$$|p(t)| \le r_0\left(\frac{\lambda_1(1-\lambda_1)}{(t-a)^2} + \frac{2\lambda_1\lambda_2}{(t-a)(b-t)} + \frac{\lambda_2(1-\lambda_2)}{(b-t)^2}\right) \quad \text{for } a < t < b\,,$$

where r_0 is the number satisfying the inequality

$$r_0 < 1\,. \tag{78}$$

Then Problem (76), (77_1), (77_2) has the unique solution.

The proof of Corollary 2 see in Ref. [6].

Remark 2. Condition (78) in Corollary 2 is unprovable in the sense that the inequality contained in it cannot be replaced by the corresponding equality (see the example in Ref. [6]).

For the three-points boundary value problem

$$u''' = p(t)u + q(t)\,, \tag{79}$$

$$u(a) = u(t_0) = u(b) = 0\,, \tag{80_1}$$

$$\sup\Big\{(t-a)^{-\lambda_1}(b-t)^{-\lambda_2}|u(t)|:\, a<t<b\Big\} < +\infty\,, \tag{80_2}$$

where $a<t_0<b$, $0<\lambda_1,\ \lambda_2<1$, $t\to|t-t_0|p(t)$ is locally integrable on $]a,b[$ and $q\in L_{2-\lambda_1,2-\lambda_2}(]a,b[;\mathbb{R})$, from Theorem 2 we obtain

Corollary 3. *Let*

$$(81)\qquad |p(t)|\le r_0\Big|\frac{\lambda_1(1-\lambda_1)(2-\lambda_1)}{(t-a)^3}-\frac{3\lambda_1(1-\lambda_1)}{(t-a)^2(t-t_0)}+\frac{3\lambda_1\lambda_2(1-\lambda_1)}{(t-a)^2(b-t)}-$$
$$-\frac{6\lambda_1\lambda_2}{(t-a)(t-t_0)(b-t)}-\frac{3\lambda_1\lambda_2(1-\lambda_2)}{(t-a)(b-t)^2}-\frac{3\lambda_2(1-\lambda_2)}{(t-t_0)(b-t)^2}-\frac{\lambda_2(1-\lambda_2)(2-\lambda_2)}{(b-t)^3}\Big|$$
$$\text{for } a<t<b\,,$$

where r_0 is the number satisfying the inequality (78). Then Problem(79), (80_1), (80_2) has the unique solution.

Proof. It is assumed

$$r(t)=r_0(t-a)^{\lambda_1}(t_0-t)(b-t)^{\lambda_2}\Big[\frac{\lambda_1(1-\lambda_1)(2-\lambda_1)}{(t-a)^3}-\frac{3\lambda_1(1-\lambda_1)}{(t-a)^2(t-t_0)}+$$
$$+\frac{3\lambda_1\lambda_2(1-\lambda_1)}{(t-a)^2(b-t)}-\frac{6\lambda_1\lambda_2}{(t-a)(t-t_0)(b-t)}-\frac{3\lambda_1\lambda_2(1-\lambda_2)}{(t-a)(b-t)^2}-$$
$$-\frac{3\lambda_2(1-\lambda_2)}{(t-t_0)(b-t)^2}-\frac{\lambda_2(1-\lambda_2)(2-\lambda_2)}{(b-t)^3}\Big]\,.$$

It is obvious that $r\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ and we can write the condition (81) in the view

$$|p(t)|\le\frac{r(t)}{(t-a)^{\lambda_1}(b-t)^{\lambda_2}|t-t_0|}\,.$$

On the other side, it is not difficult to see that the function

$$u_0(r)(t)=-r_0(t-a)^{\lambda_1}(t-t_0)(b-t)^{\lambda_2}$$

is the solution of the problem

$$u'''=r(t),\quad u(a)=u(t_0)=u(b)=0\,.$$

According to condition (78)

$$\rho_{\lambda_1,\lambda_2}(r) = r_0 < 1 \,.$$

Therefore all conditions of Theorem 2 are fulfilled. Corollary is proved. □

Remark 3. Condition (78) in Corollary 3 is unprovable in the sense that the inequality contained in it cannot be replaced by the corresponding equality. Really, for each number $c \in \mathbb{R}$ the function

$$u_0(t) = c(t-a)^{\lambda_1}(t-t_0)(b-t)^{\lambda_2}$$

is the solution of the differential equation

$$u''' = p(t)u \,,$$

where

$$\begin{aligned} p(t) = {} & \frac{\lambda_1(1-\lambda_1)(2-\lambda_1)}{(t-a)^3} - \frac{3\lambda_1(1-\lambda_1)}{(t-a)^2(t-t_0)} \\ & + \frac{3\lambda_1\lambda_2(1-\lambda_1)}{(t-a)^2(b-t)} - \frac{6\lambda_1\lambda_2}{(t-a)(t-t_0)(b-t)} - \frac{3\lambda_1\lambda_2(1-\lambda_2)}{(t-a)(b-t)^2} \\ & - \frac{3\lambda_2(1-\lambda_2)}{(t-t_0)(b-t)^2} - \frac{\lambda_2(1-\lambda_2)(2-\lambda_2)}{(b-t)^3} \,, \end{aligned}$$

and satisfy the boundary conditions (80_1), (80_2).

4. The Continuous Dependence of Solutions on Equation Coefficients

Alongside with (1), for each natural number j we consider the equation

$$(1_j) \qquad u^{(n)} = \sum_{k=1}^{l} \tilde{p}_{kj}(t)u^{(k-1)} + q_j(t) \,,$$

where

$$\tilde{p}_{kj}(\cdot)\sigma_{k,\lambda_1,\lambda_2}(\cdot) \in L_{\mathrm{loc}}(]a,b[;\mathbb{R}), \ q_j \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R}) \,.$$

As before, it will be assumed here that p_k $(k=1,\dots,l)$ and q satisfy conditions (51), (52).

Theorem 4. *Let condition (53) be fulfilled and Problem (1), (2_1), (2_2) has the unique solution u. Let, besides,*

$$(82) \qquad \lim_{j\to+\infty} \int_{\frac{a+b}{2}}^{t} \tilde{p}_{kj}(\tau)\sigma_{k,\lambda_1,\lambda_2}(\tau)\,d\tau = \int_{\frac{a+b}{2}}^{t} p_k(\tau)\sigma_{k,\lambda_1,\lambda_2}(\tau)\,d\tau$$

uniformly inside $]a,b[$ $\quad (k=1,\dots,l)$,

$$(83)\quad \sup\Big\{(t-a)^{n-1-\lambda_1}\int_t^s |\tilde{p}_{kj}(\tau)-p_k(\tau)|\sigma_{k,\lambda_1,\lambda_2}(\tau)\,d\tau:\ a<t\le s\Big\}\to 0$$
$$\textit{for } s\to a,\ j\to+\infty\ (k=1,\dots,l),$$

$$(84)\quad \sup\Big\{(b-t)^{n-1-\lambda_2}\int_s^t |\tilde{p}_{kj}(\tau)-p_k(\tau)|\sigma_{k,\lambda_1,\lambda_2}(\tau)\,d\tau:\ s\le t<b\Big\}\to 0$$
$$\textit{for } s\to b,\ j\to+\infty\quad (k=1,\dots,l),$$

$$(85)\qquad \lim_{j\to+\infty}|q_j(\cdot)-q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}=0$$

and

$$(86)\qquad |\tilde{p}_{kj}(t)|\sigma_{k,\lambda_1,\lambda_2}(t)\le\tilde{p}(t)\quad \textit{for } a<t<b\ (k=1,\dots,l;\ j=1,2,\dots),$$

where

$$\tilde{p}\in L_{\mathrm{loc}}(]a,b[;\mathbb{R}).$$

Then, starting from some j_0 Problem (1_j), (2_1), (2_2) also has the unique solution u_j and

$$(87)\qquad \operatorname{vraimax}\Big\{\sum_{k=1}^{l}\frac{|u_j^{(k-1)}(t)-u^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)}:\ a<t<b\Big\}\longrightarrow 0\quad \textit{for } j\to+\infty.$$

Proof. From(53), (83) and (84) it readily follows that there exists a natural number j_1 such that

$$(88)\qquad (\tilde{p}_{1j},\dots,\tilde{p}_{lj})\in S^+(a,b;n,n_1;\lambda_1)\cap S^-(a,b;n,n_m;\lambda_2)\quad \text{for } j\ge j_1.$$

Let us now prove that starting from some $j_0\ge j_1$ the homogeneous equation

$$(89)\qquad u^{(n)}=\sum_{k=1}^{l}\tilde{p}_{kj}(t)u^{(k-1)}$$

has no nontrivial solution satisfying the boundary conditions (2_1), (2_2).

Let the opposite be true. Then there exists a sequence of natural numbers $\Big(j_i\Big)_{i=1}^{+\infty}$ such that for each i the equation

$$u^{(n)}=\sum_{k=1}^{l}\tilde{p}_{kj_i}(t)u^{(k-1)}$$

has the solution u_i satisfying conditions (2_1) and (2_2) and

$$\text{vraimax}\Big\{\sum_{k=1}^{l}\frac{|u_i^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)}: a<t<b\Big\}=1 \quad (i=1,2,\dots)\,. \tag{90}$$

In view of (86) and (90)

$$|u_i^{(k-1)}(t)|\le\sigma_{k,\lambda_1,\lambda_2}(t) \quad \text{for } a<t<b \quad (k=1,\dots,l;\, i=1,2,\dots) \tag{91}$$

and

$$|u^{(n)}(t)|\le l\tilde p(t) \quad \text{for } a<t<b \quad (i=1,2,\dots)\,.$$

From these estimates it follows that sequences $\left(u_i^{(k-1)}\right)_{i=1}^{+\infty}$ $(k=1,\dots,n)$ are uniformly bounded and equicontinuous inside $]a,b[$.

It is obvious that for each i

$$u_i^{(n)}(t)=\sum_{k=1}^{l}p_k(t)u_i^{(k-1)}(t)+\tilde q_i(t)\,, \tag{92}$$

where

$$\tilde q_i(t)=\sum_{k=1}^{l}\Big[\tilde p_{kj_i}(t)-p_k(t)\Big]u_i^{(k-1)}(t)\,.$$

According to conditions (83), (84) and (91)

$$\sup\Big\{(t-a)^{n-1-\lambda_1}\int_t^s|\tilde q_i(\tau)|\,d\tau: a<t\le s\Big\}\longrightarrow 0 \quad \text{for } s\to a,\ i\to+\infty$$

and

$$\sup\Big\{(b-t)^{n-1-\lambda_2}\int_s^t|\tilde q_i(\tau)|\,d\tau: s\le t<b\Big\}\longrightarrow 0 \quad \text{for } s\to b,\ i\to+\infty\,.$$

On the other hand, by virtue of Lemma 3.1 from Ref. [4] the uniform boundedness, equicontinuity and conditions (82), (86) and (91) imply that

$$\lim_{i\to+\infty}\int_{\frac{a+b}{2}}^{t}\tilde q_i(\tau)\,d\tau=0$$

uniformly inside $]a,b[$. Therefore

$$\lim_{i\to+\infty}|\tilde q_i(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}=0\,. \tag{93}$$

By Lemma 1 there exists a positive number ρ_0 such that

$$|u_i^{(k-1)}(t)| \le \rho_0 \sigma_{k,\lambda_1,\lambda_2}(t) |\tilde{q}_i(\cdot)|_{n-1-\lambda_1, n-1-\lambda_2}$$
$$\text{for } a < t < b \quad (k = 1, \dots, l;\ i = 1, 2, \dots).$$

Hence with (93) taken into account we have

$$\operatorname{vraimax}\Big\{ \sum_{k=1}^{l} \frac{|u_i^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t < b \Big\} \longrightarrow 0 \quad \text{for } i \to +\infty,$$

which contradicts condition (90). The contradiction obtained proves that if $j_0 \ge j_1$ is a sufficiently great number, then the homogeneous Problem (89), (2_1), (2_2) has the trivial solution only for each $j \ge j_0$. Since, besides, condition (88) is fulfilled too, by virtue of Theorem 1 Problem (1_j), (2_1), (2_2) has the unique solution u_j for each $j \ge j_0$.

To complete the proof of Theorem it remains for us to show that condition (87) is fulfilled. Let us assume the opposite is true. Then there exist a positive number ε_0 and a sequence of natural numbers $\Big(j_i\Big)_{i=1}^{+\infty}$ such that

$$\gamma_i = \operatorname{vraimax}\Big\{ \sum_{k=1}^{l} \frac{|u_{j_i}^{(k-1)}(t) - u^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t < b \Big\} \ge \varepsilon_0 \tag{94}$$
$$(i = 1, 2, \dots).$$

Let

$$v_i(t) = \frac{u_{j_i}(t) - u(t)}{\gamma_i}.$$

It is obvious that each v_i satisfies the boundary conditions (2_1), (2_2),

$$\operatorname{vraimax}\Big\{ \sum_{k=1}^{l} \frac{|v_i^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t < b \Big\} = 1 \quad (i = 1, 2, \dots) \tag{95}$$

and

$$v_i^{(n)}(t) = \sum_{k=1}^{l} p_k(t) v_i^{(k-1)}(t) + q_{1i}(t) + q_{2i}(t) \quad (i = 1, 2, \dots), \tag{96}$$

where

$$q_{1i}(t) = \sum_{k=1}^{l} \Big(\tilde{p}_{kj_i}(t) - p_k(t)\Big) \bigg(v_i^{(k-1)}(t) + \frac{u^{(k-1)}(t)}{\gamma_i} \bigg)$$

and

$$q_{2i}(t) = \frac{1}{\gamma_i} \Big[q_{j_i}(t) - q(t) \Big].$$

By (85), (86), (94) and (95)

$$\lim_{i\to+\infty} |q_{2i}(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} = 0 \tag{97}$$

and

$$|q_{1i}(t)| \le p^*(t), \quad \Big|\sum_{k=1}^{l} p_k(t) v_i^{(k-1)}(t)\Big| \le p^*(t)\,, \tag{98}$$

where

$$p^*(t) = l r_0 \tilde{p}(t) + r_0 \sum_{k=1}^{l} |p_k(t)| \sigma_{k,\lambda_1,\lambda_2}(t)\,,$$

$$r_0 = 1 + \frac{1}{\varepsilon_0}\ \text{vraimax}\Big\{\sum_{k=1}^{l} \frac{|u^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t < b\Big\}\,,$$

and

$$p^* \in L_{\rm loc}(]a,b[;\mathbb{R})\,.$$

From (95) – (98) it follows that sequences $\Big(v_i^{(k-1)}\Big)_{i=1}^{+\infty}$ $(k = 1,\dots,n)$ are uniformly bounded and equicontinuous inside $]a,b[$. Therefore by Lemma 3.1 from Ref. [4] and conditions (82)–(84)

$$\lim_{i\to+\infty} |q_{1i}(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} = 0\,. \tag{99}$$

By virtue of Lemma 1 there exists a positive number ρ_0 such that

$$|v_i^{(k-1)}(t)| \le \rho_0 \sigma_{k,\lambda_1,\lambda_2}(t)\times$$
$$\times\Big[|q_{1i}(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} + |q_{2i}(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\Big]$$
$$\text{for } a < t < b \quad (k = 1,\dots,l;\ i = 1,2,\dots)\,.$$

Hence with (97) and (99) taken into account we have

$$\text{vraimax}\Big\{\sum_{k=1}^{l} \frac{|v_i^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a < t < b\Big\} \longrightarrow 0 \quad \text{for } i \to +\infty\,,$$

which contradicts (95). The contradiction obtained proves Theorem. □

We shall consider, as an example, the differential equation

$$u^{(n)} = \sum_{k=1}^{l} \frac{g_k(t)\sin jt + h_{kj}(t)}{\gamma_k(t)} u^{(k-1)} + \frac{j g_0(t)\sin j^2 t + h_{0j}(t)}{(t-a)^{n-1-\lambda_1}(b-t)^{n-1-\lambda_2}}\,, \tag{$1'_j$}$$

where $\gamma_k(t)=(t-a)^{n+1-k}(b-t)^{n+1-k}\prod_{i=2}^{m-1}|t-t_i|^{n_{ik}}$, $g_k:[a,b]\to\mathbb{R}$ $(k=0,\dots,l)$ are continuously differentiable functions satisfying the conditions

$$g_k(a)=g_k(b)=0 \quad (k=0,\dots,l),$$

and $h_{kj}:[a,b]\to[-1,1]$ $(k=0,\dots,l;\ j=1,2,\dots)$ are measurable functions such that

$$\lim_{j\to+\infty} h_{kj}(t)=0 \qquad \text{uniformly on } [a,b].$$

By Theorem 4, starting from some j, Problem $(1'_j)$, (2_1), (2_2) has the unique solution u_j and

$$\operatorname{vraimax}\Big\{\sum_{k=1}^{l}\frac{|u_j^{(k-1)}(t)|}{\sigma_{k,\lambda_1,\lambda_2}(t)} : a<t<b\Big\} \longrightarrow 0 \quad \text{for } j\to+\infty.$$

5. Green's Operator and its Properties

In this section it is assumed that $p_k:]a,b[\to\mathbb{R}$ $(k=1,\dots,l)$ are fixed functions satisfying conditions (51) and (53), and Problem (1_0), (2_1), (2_2) has the trivial solution only. Then by virtue of Theorem 1 for any $q\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ Problem (1), (2_1), (2_2) has the unique solution. The operator $\mathcal{G}L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})\to C_{\mathrm{loc}}(]a,b[;\mathbb{R})$ that puts into correspondence to each $q\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ the solution $u(t)=\mathcal{G}(q)(t)$ of Problem (1), (2_1), (2_2), will called Green's operator of Problem (1_0), (2_1), (2_2).

From Lemma 1 yields

Corollary 4. *There exists the positive number ρ_0 such that for any $q\in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R})$ the inequalities*

$$\Big|\frac{d^{k-1}\mathcal{G}(q)(t)}{dt^{k-1}}\Big|\le\rho_0\sigma_{k,\lambda_1,\lambda_2}(t)|q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} \quad \text{for } a<t<b\ (k=1,\dots,l), \tag{100}$$

$$\Big|\frac{d^{n-1}\mathcal{G}(q)(t)}{dt^{n-1}}-\frac{d^{n-1}\mathcal{G}(q)(s)}{ds^{n-1}}\Big|\le\int_s^t p^*(\tau)\,d\tau+\Big|\int_s^t q(\tau)\,d\tau\Big| \quad \text{for} \quad a<s\le t<b,$$

where

$$p^*(t)=\rho_0|q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2}\sum_{k=1}^{l}|p_k(t)|\sigma_{k,\lambda_1,\lambda_2}(t),$$

are hold.

Theorem 5. *Let*

$$q \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R}),\, q_j \in L_{n-1-\lambda_1,n-1-\lambda_2}(]a,b[;\mathbb{R}) \quad (j=1,2,\dots)\,, \tag{101}$$

$$\lim_{j\to+\infty}\int_{\frac{a+b}{2}}^{t} q_j(\tau)\,d\tau = \int_{\frac{a+b}{2}}^{t} q(\tau)\,d\tau \quad \textit{uniformly inside } \ ]a,b[\,, \tag{102}$$

$$\lim_{j\to+\infty} |q_j(\cdot)-q(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} < +\infty\,. \tag{103}$$

Then

$$\lim_{j\to+\infty}\frac{d^{k-1}\mathcal{G}(q_j)(t)}{dt^{k-1}} = \frac{d^{k-1}\mathcal{G}(q)(t)}{dt^{k-1}} \quad (k=1,\dots,n) \quad \textit{uniformly inside } \ ]a,b[\,. \tag{104}$$

Proof. Let us assume that Theorem is not true. Then there exist points $a_0 \in]a,b[$ and $b_0 \in]a_0,b[$, the positive number ε_0 and the sequence $(j_i)_{i=1}^{+\infty}$ natural numbers such that

$$\gamma_i = \max\Big\{\sum_{k=1}^{n}\Big|\frac{d^{k-1}\mathcal{G}(q_{j_i}-q)(t)}{dt^{k-1}}\Big| : a_0 \le t \le b_0\Big\} \ge \varepsilon_0 \quad (i=1,2,\dots)\,. \tag{105}$$

It is assumed for each i that

$$\tilde q_i(t) = \frac{1}{\gamma_i}\Big[q_{j_i}(t)-q(t)\Big]\,,$$
$$u_i(t) = \mathcal{G}(\tilde q_i)(t)\,.$$

Then

$$u_i^{(n)}(t) = \sum_{k=1}^{l} p_k(t)u_i^{(k-1)}(t) + \tilde q_i(t)\,, \tag{106}$$

and $u(\cdot)=u_i(\cdot)$ satisfies the boundary conditions (2_1), (2_2). On the other hand, according to conditions (102), (103) and Corollary 4

$$\sup\Big\{|\tilde q_i(\cdot)|_{n-1-\lambda_1,n-1-\lambda_2} : i=1,2,\dots\Big\} < +\infty\,, \tag{107}$$

$$\lim_{i\to+\infty}\int_{\frac{a+b}{2}}^{t}\tilde q_i(\tau)\,d\tau = 0 \qquad \text{uniformly inside }]a,b[\,, \tag{108}$$

$$\max\Big\{\sum_{k=1}^{n}|u_i^{(k-1)}(t)| : a_0 \le t \le b_0\Big\} = 1 \quad (i=1,2,\dots) \tag{109}$$

and

$$|u_i^{(k-1)}(t)| \le r_0 \sigma_{k,\lambda_1,\lambda_2}(t) \quad \text{for } a < t < b \quad (k = 1, \dots, l;\ i = 1, 2, \dots), \tag{110}$$

where r_0 is a positive number independent of i. According to conditions (106) and (110)

$$|u_i^{(n)}(t) - \tilde{q}_i(t)| \le \left| \sum_{k=1}^{l} p_k(t) u_i^{(k-1)}(t) \right| \le p^*(t) \quad \text{for } \ a < t < b \quad (i = 1, 2, \dots), \tag{111}$$

where

$$p^*(t) = r_0 \sum_{k=1}^{l} |p_k(t)| \sigma_{k,\lambda_1,\lambda_2}(t)$$

and $p^* \in L_{\text{loc}}(]a, b[; \mathbb{R})$.

On account of conditions (53), (107), (109), (111) and equalities

$$u_i^{(k-1)}(a) = 0 \quad (k = 1, \dots, n_1), \quad u_i^{(k-1)}(b) = 0 \quad (k = 1, \dots, n_m)$$

from (106) we find

$$\begin{aligned} |u_i^{(k-1)}(t)| &\le r_1 (t-a)^{\lambda_1 - k + 1} \\ &+ \int_a^t (t-s)^{n_1 - k} \int_s^{t_0} (\tau - s)^{n - n_1 - 1} p^*(\tau)\, d\tau\, ds \\ &\text{for } a < t \le t_0, \quad l \in \{1, \dots, n_1\} \quad (k = 1, \dots, n_1) \end{aligned} \tag{112_1}$$

and

$$\begin{aligned} |u_i^{(k-1)}(t)| &\le r_2 (b-t)^{\lambda_2 - k + 1} \\ &+ \int_t^b (s-t)^{n_m - k} \int_{t_0}^{s} (s - \tau)^{n - n_m - 1} p^*(\tau)\, d\tau\, ds \\ &\text{for } t_0 \le t < b, \quad l \in \{1, \dots, n_m\} \quad (k = 1, \dots, n_m), \end{aligned} \tag{112_2}$$

where $t_0 = \frac{a_0 + b_0}{2}$, r_1 and r_2 are positive constant s independent of i.

From (108) – (111) it follows that sequences $\left(u_i^{(k-1)} \right)_{i=1}^{+\infty}$ $(k = 1, \dots, n)$ are uniformly bounded and equicontinuous inside $]a, b[$. Therefore by the Arzela-Alkoli lemma it can be assumed without loss of generality that sequences $\left(u_i^{(k-1)} \right)_{i=1}^{+\infty}$ $(k = 1, \dots, n)$ converge uniformly inside $]a, b[$.

On account of conditions (108), (110), (111), (112_1) and (12_2) we conclude that

$$u(t) = \lim_{i\to+\infty} u_i(t) \tag{113}$$

is the solution of Problem (1_0), (2_1), (2_2). Therefore $u(t) \equiv 0$. But it is impossible, because on account of (109) and (113) we have

$$\max\left\{\sum_{k=1}^{n} |u^{(k-1)}(t)| \colon a_0 \le t \le b_0\right\} = 1\,.$$

The contradiction obtained proves Theorem. □

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