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ON THE OSCILLATION OF A CLASS OF LINEAR HOMOGENEOUS THIRD ORDER DIFFERENTIAL EQUATIONS

N. Parhi and P. Das

Abstract.

$$y''' \quad a\ t\ y'' \quad b\ t\ y' \quad c\ t\ y \quad ,$$

$a \in C^2 \ \sigma, \infty\ , R \quad b \in C^1 \ \sigma, \infty\ , R \quad c \in C \ \sigma, \infty\ , R \qquad \sigma \in R$
$a\ t\ \leq \quad b\ t\ \leq \qquad c\ t\ \leq$

1. Introduction

The object of this work is to answer a question raised in [1, p. 683] for the differential equation

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0\,, \tag{1.1}$$

where $a \in C^2([\sigma, \infty), R)$, $b \in C^1([\sigma, \infty), R)$, $c \in C([\sigma, \infty), R)$ and $\sigma \in R$ such that $a(t) \leq 0$, $b(t) \leq 0$ and $c(t) \leq 0$ with $b(t) \not\equiv 0$ and $c(t) \not\equiv 0$ on any sub-interval of $[\sigma, \infty)$. A solution $y(t)$ of (1.1) on $[\sigma, \infty)$ is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if it has an oscillatory solution; otherwise, it is said to be nonoscillatory.

If $a(t)$, $b(t)$ and $c(t)$ are constants, then (1.1) reduces to

$$y''' + ay'' + by' + cy = 0 \tag{1.2}$$

By the rule of signs, the characteristic equation

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{1.3}$$

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has one and only one positive real root γ, say, and either two complex conjugate roots $\alpha + i\beta$ and $\alpha - i\beta$, where

$$2\alpha = -(\alpha^2 + \beta^2 - b)/\gamma < 0 \,,$$

or two negative real roots, counting multiplicities. In the first case every solution of (1.2) is of the form

$$c_1 e^{\gamma t} + e^{\alpha t}(c_2 \cos \beta t + c_3 \sin \beta t) \,.$$

Thus a real nontrivial solution of (1.2) has arbitrarily large zeros if and only if $c_1 = 0$. If $c_1 \neq 0$, then the solution and all of its derivatives have the same sign from a certain point on. In the second case, all solutions of (1.2) are nonoscillatory and there exist solutions whose derivatives alternate in sign. In fact, (1.3) admits two imaginary roots if

$$\frac{ab}{3} - \frac{2a^3}{27} - c - \frac{2}{3\sqrt{3}}\left(\frac{a^2}{3} - b\right)^{3/2} > 0 \,.$$

Otherwise, (1.3) has two negative roots.

In an attempt to obtain results similar to the above observations for (1.1), Ahmad and Lazer [1] have obtained the following theorems.

Theorem 1. *Suppose that $a(t) \leq 0$, $b(t) \leq 0$ and $c(t) < 0$. The following statements are equivalent:*

A) *There exists an oscillatory solution of (1.1)*
B) *If w is a nonoscillatory solution of (1.1), then there exists a $t_0 \geq \sigma$ such that $w(t)w'(t)w''(t) \neq 0$ for $t \geq t_0$ and $\operatorname{sgn} w(t) = \operatorname{sgn} w'(t) = \operatorname{sgn} w''(t)$, $t \geq t_0$.*

Theorem 2. *If $a(t) \leq 0$, $b(t) \leq 0$, $c(t) < 0$ and (1.1) admits an oscillatory solution, then there exist two linearly independent oscillatory solutions u and v of (1.1) such that any nontrivial linear combination of u and v is also oscillatory and the zeros of u and v separate.*

Theorem 3. *If $p(t) < 0$, $q(t) \leq 0$, $2p(t) - q'(t) \leq 0$ for $t \in [\sigma, \infty)$ and*

$$y''' + q(t)y' + p(t)y = 0 \tag{1.4}$$

has an oscillator solution, then there exist two linearly independent oscillatory solutions u and v of (1.4) whose zeros separate and such that a solution of (1.4) is oscillatory if and only if it is a nontrivial linear combination of u and v.

In [3, Theorem 1], Jones has obtained Theorem 3 above without the condition $2p(t) - q'(t) \leq 0$. However, in the proof of his Theorem 1, the identity $y_1' y_2 - y_1 y_2' = N$ is not clear. This identity plays the crucial role in the proof of the theorem.

In section 2 of this work we have obtained a result for (1.1) which provides an affirmative answer to the question raised by Ahmad and Lazer in [1]. However, we have additional restrictions *viz*, $a(t)$ is twice and $b(t)$ is once continuously differentiable. This is because we are taking the help of adjoints. The adjoint of (1.1) is given by

$$x''' - (a(t)x)'' + (b(t)x)' - c(t)x = 0\,. \tag{1.5}$$

Equation (1.1) may be written as

$$(r(t)y'')' + q(t)y' + p(t)y = 0\,, \tag{1.6}$$

where $r(t) = \exp\left(\int_\sigma^t a(s)\,ds\right)$, $q(t) = r(t)b(t)$ and $p(t) = r(t)c(t)$. The adjoint of (1.6) is written as

$$(r(t)z')'' + (q(t)z)' - p(t)z = 0\,. \tag{1.7}$$

We may note that the transformation $x = r(t)z$ transforms (1.5) into (1.7) and vice versa. In Section 3 we obtain sufficient conditions for oscillation of (1.1).

Equation (1.1) is said to be of Class I or C_I if any of its solutions $y(t)$ for which $y(t_0) = y'(t_0) = 0$, $y''(t_0) > 0$ $(\sigma < t_0 < \infty)$ satisfies $y(t) > 0$ for $t \in [\sigma, t_0)$. Equation (1.1) is said to be of Class II or C_{II} if any of its solutions $y(t)$ for which $y(t_0) = y'(t_0) = 0$, $y''(t_0) > 0$ $(\sigma \le t_0 < \infty)$ satisfies $y(t) > 0$ for $t > t_0$.

Remark 4. It is easy to see that (1.5) is of C_I or C_{II} if and only if (1.7) is of C_I or C_{II}. Further, (1.5) is oscillatory if and only if (1.7) is oscillatory.

2. Main results

Theorem 5. *If (1.1) has an oscillatory solution, then there exist two linearly independent oscillatory solutions $y_1(t)$ and $y_2(t)$ of (1.1) whose zeros separate and such that a solution of (1.1) is oscillatory if and only if it is a nontrivial linear combination of $y_1(t)$ and $y_2(t)$.*

Theorem 5 may be written as follows:

Theorem 6. *If (1.1) has an oscillatory solution, then the set of all oscillatory solutions of (1.1) form a two dimensional subspace of the solution space of (1.1).*

In the following we obtain some results which are interesting in themselves and which will be needed for the proof of Theorem 5.

Lemma 7. *Equation (1.6) is of Class II.*

Proof. Let $y(t)$ be a solution of (1.6) with $y(t_0) = 0 = y'(t_0)$ and $y''(t_0) > 0$, $t_0 > \sigma$. From the continuity of y'' it follows that $y(t) > 0$ in a neighbourhood of t_0. We claim that $y(t) > 0$ for $t > t_0$. If not, there exists a $t_1 > t_0$ such that $y''(t_1) = 0$ and $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ for $t \in (t_0, t_1)$. Now integrating (1.6) from t_0 to t_1 we obtain

$$0 > -r(t_0)y''(t_0) = -\int_{t_0}^{t_1} [q(t)y'(t) + p(t)y(t)]\,dt > 0\,,$$

a contradiction. Hence our claim holds and this completes the proof of the lemma.□

Lemma 8. *Equation (1.7) is of Class I.*

Proof. Equation (1.6) is of C_{II} implies that Equation (1.1) is of C_{II}. This in turn implies that, by Lemma 2.9 due to Hanan [2], Equation (1.5) is of C_I and hence Equation (1.7) is of C_I. $\square$

Lemma 9. *Equation (1.6) is oscillatory if and only if Equation (1.7) is oscillatory.*

This follows from Theorem 4.1 due to Hanan [2].

Proposition 10. *Equation (1.7) admits a nonoscillatory solution $N(t)$ satisfying $N(t) > 0$, $N'(t) < 0$ and $(rN')'(t) + q(t)N(t) > 0$ for $t \in [\sigma, \infty)$.*

The proof is similar to that of Theorem 2 due to Jones [3] and hence is omitted.

Lemma 11. *With $N(t)$ as in Proposition 10, the following statements hold:*

(i) $\lim\limits_{t\to\infty} r(t)N'(t) = 0$
(ii) $\lim\limits_{t\to\infty} tr(t)N'(t) = 0$
(iii) $\lim\limits_{t\to\infty} t^2[(rN')'(t) + q(t)N(t)] = 0\,.$

The proof is simple and along the lines of Jones [3] and hence is omitted.

Theorem 12. *If (1.1) has an oscillatory solution, then there exist two linearly independent oscillatory solutions $u_1(t)$ and $u_2(t)$ of*

$$\frac{y'}{N(t)}{}' + \frac{(rN')'(t) + q(t)N(t)}{r(t)N^2(t)}\,y = 0 \tag{2.1}$$

which satisfy (1.1).

Proof. Since Eq. (1.1) has an oscillatory solution, then Eq. (1.6) has an oscillatory solution. From Lemma 9 it follows that (1.7) is oscillatory. It is clear from a result due to Hanan [2, Theorem 3.4] that a solution of (1.7) which has at least one zero is oscillatory.

Let $z_1(t)$ and $z_2(t)$ be two linearly independent solutions of (1.7) with

$$z_1(\sigma) = 0 = z_1'(\sigma)\,, \quad z_1''(\sigma) = 1$$
$$z_2(\sigma) = 0\,, \quad z_2'(\sigma) = 1\,, \quad (rz_2')'(\sigma) = 0\,.$$

So $z_1(t)$ and $z_2(t)$ are oscillatory. It is easy to verify that

$$w_1(t) = N(t)z_1'(t) - N'(t)z_1(t) = N^2(t)\,\frac{z_1}{N}{}'(t)$$

and

$$w_2(t) = N(t)z_2'(t) - N'(t)z_2(t) = N^2(t)\,\frac{z_2}{N}{}'(t)$$

are oscillatory solutions of

$$(2.2) \qquad \frac{(r(t)x)'}{N(t)}{}' + \frac{(rN')'(t) + q(t)N(t)}{N^2(t)} \; x = 0\,.$$

Consequently, $u_1(t) = r(t)w_1(t)$ and $u_2(t) = r(t)w_2(t)$ are oscillatory solutions of (2.1). It may be shown easily that $u_1(t)$ and $u_2(t)$ satisfy (1.6) and hence (1.1).

To complete the proof of the theorem, it is to be shown that $u_1(t)$ and $u_2(t)$ are linearly independent. If possible, let $u_1(t)$ and $u_2(t)$ be linearly dependent. So there exist c_1 and c_2, not both zero, such that $c_1u_1(t) + c_2u_2(t) = 0$ for $t \in [\sigma, \infty)$, that is, $c_1w_1(t) + c_2w_2(t) = 0$ for $t \in [\sigma, \infty)$. Since $w_1(t)$ and $w_2(t)$ are nontrivial solutions of (2.2), then $c_1 = 0$ implies that $c_2 = 0$ and $c_2 = 0$ implies that $c_1 = 0$. Hence $c_1 \neq 0$ and $c_2 \neq 0$. Now $w_1(t) + \lambda w_2(t) = 0$ for $t \in [\sigma, \infty)$, where $\lambda = c_2/c_1$, implies that

$$\frac{z_1'(t) + \lambda z_2'(t)}{z_1(t) + \lambda z_2(t)} = \frac{N'(t)}{N(t)}\,.$$

Thus $N(t) = c(z_1(t) + \lambda z_2(t))$, $c \neq 0$. Consequently, $z_1(t) + \lambda z_2(t)$ is nonoscillatory. Hence there exists a $t_1 > \sigma$ such that $z_1(t) + \lambda z_2(t)$ has one sign for $t \geq t_1$. Let t_2 and $t_3 (t_1 < t_2 < t_3)$ be successive zeros of z_1. From a result in [5] it follows that the function $(z_1(t) + \lambda z_2(t)) + \mu z_1(t)$ has a double zero in (t_2, t_3), where μ is a constant, that is, $(1 + \mu)z_1(t) + \lambda z_2(t)$ has a double zero in (t_2, t_3). Clearly, $(1 + \mu)z_1(t) + \lambda z_2(t)$ is a solution of (1.7) with a zero at $t = \sigma$. This contradicts the fact that (1.7) is of C_I. Thus $u_1(t)$ and $u_2(t)$ are linearly independent.

This completes the proof of the theorem. □

Remark 13. Any solution of (2.1) is a solution of (1.1). It is possible to choose two linearly independent solutions $y_1(t)$ and $y_2(t)$ of (2.1) such that $w(t) > 0$, where $w(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$. Since (2.1) is oscillatory, $y_1(t)$ and $y_2(t)$ are oscillatory. Moreover, $y_1(t)$ and $y_2(t)$ are solutions of (1.1).

Proposition 14. *N and W are linearly dependent. In fact, $W(t) = \lambda N(t)$, where $\lambda > 0$ is a constant.*

Proof. As $y_1(t)$ and $y_2(t)$ are linearly independent solutions of (2.1), then $W(t) \neq 0$ for $t \geq \sigma$. Clearly, $y_1(t)$ and $y_2(t)$ are solutions of

$$\begin{matrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ r(t)y'' & r(t)y_1'' & r(t)y_2'' \end{matrix} = 0\,,$$

$t \in [\sigma, \infty)$, that is, of

$$(2.3) \qquad y'' - \frac{W'(t)}{W(t)} \; y' + \frac{(rW')'(t) + q(t)W(t)}{r(t)W(t)} \; y = 0\,.$$

Equation (2.1) may be written as

$$y'' - \frac{N'(t)}{N(t)} \; y' + \frac{(rN')'(t) + q(t)N(t)}{r(t)N(t)} \; y = 0\,.$$

Clearly, equations (2.1) and (2.3) have the same solution space. If $u(t)$ is a solution of (2.1), then it is a solution of (2.3) and hence $u(t)$ is a solution of the first order equation

$$a_1(t)y' + b_1(t)y = 0\,,$$

where

$$a_1(t) = \frac{N'(t)}{N(t)} - \frac{W'(t)}{W(t)}$$

and

$$b_1(t) = \frac{(rW')'(t) + q(t)W(t)}{r(t)W(t)} - \frac{(rN')'(t) + q(t)N(t)}{r(t)N(t)}\,.$$

Hence, in particular,

$$\begin{aligned} a_1(t)y_1'(t) + b_1(t)y_1(t) &= 0 \\ a_1(t)y_2'(t) + b_1(t)y_2(t) &= 0\,. \end{aligned}$$

Since $W(t) \neq 0$ for $t \geq \sigma$, then $a_1(t) = 0$ and $b_1(t) = 0$ for $t \geq \sigma$. But $a_1(t) = 0$, $t \geq \sigma$ implies that $W(t) = \lambda N(t)$, where $\lambda \neq 0$ is a constant. Further, $W(t) > 0$ and $N(t) > 0$ implies that $\lambda > 0$.

Hence the proposition is proved. □

Remark 15. In view of Proposition 14, Proposition 10, Lemma 11 and Theorem 12 will hold when $N(t)$ is replaced by $W(t)$.

Theorem 16. *For any solution $y(t)$ of*

$$\left(\frac{y'}{W(t)}\right)' + \left(\frac{(rW')'(t) + q(t)W(t)}{r(t)W^2(t)}\right)y = 0\,, \tag{2.4}$$

the function $G(y(t))$ is a decreasing function of t, where

$$G(y(t)) = r(t)W(t)(y'(t))^2 + ((rW')'(t) + q(t)W(t))y^2(t)\,.$$

The proof is similar to that of Theorem 4 due to Jones [3] and hence is omitted.

The proof of Theorem 5 of this paper proceeds along the lines of the proof of Theorem 1 in [3]. However, for completeness and clarity the proof is given here.

Proof of Theorem 5. From Remark 13 it follows that there exist two linearly independent oscillatory solutions $y_1(t)$ and $y_2(t)$ of (1.1) whose zeros separate. To complete the proof of the theorem it is enough to show that any oscillatory solution of (1.1) can be expressed as a linear combination of $y_1(t)$ and $y_2(t)$.

Let $y_3(t)$ be a solution of (1.1) with $y_3(\sigma) = 0 = y_3'(\sigma)$, $y_3''(\sigma) > 0$. From Lemma 7 it follows that $y_3(t) > 0$ for $t > \sigma$. Consequently, $y_3'(t) > 0$ and $y_3''(t) > 0$ for $t > \sigma$. Clearly $\{y_1, y_2, y_3\}$ is linearly independent. Hence

$$\begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ r(t)y_1''(t) & r(t)y_2''(t) & r(t)y_3''(t) \end{vmatrix} = k\,,$$

where $k \neq 0$ is a constant. Thus

$$\begin{vmatrix} y_1(t) & y_2(t) & u(t) \\ y_1'(t) & y_2'(t) & u'(t) \\ r(t)y_1''(t) & r(t)y_2''(t) & r(t)u''(t) \end{vmatrix} = 1\,,$$

where $u(t) = y_3(t)/k$. Expanding the determinant we get

$$r(t)W(t)u''(t) - r(t)W'(t)u'(t) + ((rW')'(t) + q(t)W(t))u(t) = 1\,. \tag{2.5}$$

Clearly, $k < 0$ implies that $u(t) < 0$, $u'(t) < 0$ and $u''(t) < 0$. This in turn leads to contradiction in (2.5) where the left hand side becomes negative. Thus $k > 0$.

Let $z(t)$ be an oscillatory solution of (1.1). We claim that $z(t)$ can be expressed as a linear combination of $y_1(t)$ and $y_2(t)$. If not, there exist c_1, c_2 and $c_3 \neq 0$, such that $z(t) = c_1y_1(t) + c_2y_2(t) + c_3u(t)$. We may note that c_1 and c_2 cannot be zero simultaneously. Writing

$$z_1(t) = z(t)/c_3 \quad \text{and} \quad y(t) = -(c_1y_1(t) + c_2y_2(t))/c_3\,,$$

we get

$$z_1(t) = u(t) - y(t)\,.$$

Clearly, $y(t)$ is an oscillatory solution (nontrivial) of (2.4) and (1.1). Thus $z_1(t)$ is a solution of (2.5). Consequently,

$$\begin{aligned} r(t)W(t)(u(t) - y(t))'' &- r(t)W'(t)(u(t) - y(t))' \\ &+ ((rW')'(t) + q(t)W(t))(u(t) - y(t)) = 1\,. \end{aligned} \tag{2.6}$$

Since $z(t)$ is oscillatory, $u(t) - y(t)$ is oscillatory. From Theorem 16 it follows that

$$((rW')'(t) + q(t)W(t))y^2(t)$$

is bounded. As

$$\begin{aligned} &[((rW')'(t) + q(t)W(t))y(t)]^2 \\ &\qquad = ((rW')'(t) + q(t)W(t))y^2(t) \cdot t^2((rW')'(t) + q(t)W(t)) \cdot t^{-2}\,, \end{aligned}$$

from Lemma 11 (iii) we obtain

$$\lim_{t\to\infty}[((rW')'(t) + q(t)W(t))y(t)] = 0\,.$$

Hence there exists a $T > \sigma$ such that

$$|((rW')'(t) + q(t)W(t))y(t)| < 1/4$$

for $t \geq T$. From (2.5) we get, for $t \geq \sigma$,

$$0[((rW')'(t) + q(t)W(t))u(t)] < 1\,.$$

Let $t_0 > T$ be a maximum of $u - y$. So $u(t_0) - y(t_0) \geq 0$ and $u'(t_0) - y'(t_0) = 0$. Now multiplying (2.6) through by $u'(t) - y'(t)$ and integrating the resulting identity from t_0 to t, we obtain

$$\begin{aligned}
&\frac{1}{2}r(t)W(t)(u'(t) - y'(t))^2 \\
&\quad - \frac{1}{2}\int_{t_0}^{t}(rW)'(s)(u'(s) - y'(s))^2\,ds \\
&\quad - \int_{t_0}^{t} r(s)W'(s)(u'(s) - y'(s))^2\,ds \\
&\quad + \frac{1}{2}((rW')'(t) + q(t)W(t))(u(t) - y(t))^2 \\
&\quad - \frac{1}{2}((rW')'(t_0) + q(t_0)W(t_0))(u(t_0) - y(t_0))^2 \\
&\quad - \frac{1}{2}\int_{t_0}^{t} p(s)W(s)(u(s) - y(s))^2\,ds \\
&= (u-y)(t) - (u-y)(t_0)\,,
\end{aligned}$$

since $W(t)$ is a solution of (1.7). As $(rW)'(t) = r'(t)W(t) + r(t)W'(t) < 0$, we have

$$(u-y)(t_0)[1 - \frac{1}{2}((rW')'(t_0) + q(t_0)W(t_0))(u-y)(t_0)] < (u-y)(t)\,.$$

But

$$\begin{aligned}
&((rW')'(t_0) + q(t_0)W(t_0))(u-y)(t_0) \\
&\qquad = ((rW')'(t_0) + q(t_0)W(t_0))u(t_0) \\
&\qquad\quad - ((rW')'(t_0) + q(t_0)W(t_0))y(t_0) \\
&\qquad < 1 + \frac{1}{4} = \frac{5}{4}\,.
\end{aligned}$$

So, for $t > t_0$,

$$(u-y)(t) > \frac{3}{8}(u-y)(t_0) \geq 0\,,$$

which contradicts the fact that $u - y$ is oscillatory. Hence our claim holds.

This completes the proof of the theorem. □

3. Sufficient conditions for oscillation

In this section we have obtained sufficient conditions for oscillation of (1.1).

Theorem 17. *Suppose that $a(t) \le 0$, $a'(t) \ge 0$, $b(t) - 2a'(t) \le 0$ and $c(t) - b'(t) + a''(t) < 0$. If*

$$
\begin{aligned}
{}^{\infty} \quad & -\frac{2}{27}a^3(t) + \frac{1}{3}a(t)b(t) - c(t) - \frac{2}{3}a(t)a'(t) + b'(t) - a''(t) \\
& - \frac{2}{3\sqrt{3}} \quad \frac{1}{3}a^2(t) - b(t) + 2a'(t) \quad ^{3/2} \qquad dt = \infty\,,
\end{aligned}
\tag{3.1}
$$

then (1.1) is oscillatory.

Proof. Since (1.1) is of C_{II}, then from Theorem 4.7 due to Hanan [2] it follows that (1.1) is oscillatory if and only if (1.5) is oscillatory, that is, if

$$
\begin{gathered}
x''' - a(t)x'' + (b(t) - 2a''(t))x' \\
-(c(t) - b'(t) + a''(t))x = 0
\end{gathered}
$$

is oscillatory. From Theorem 8 [6] it follows that (1.5) is oscillatory and hence (1.1) is oscillatory.

Hence the theorem is proved. □

Remark 18. The above theorem generalizes Theorem 2.6 due to Lazer [4].

Remark 19. We may note that no sign restriction has been imposed on $b(t)$ and $c(t)$ in the above theorem.

Theorem 20. *Suppose that $a(t) \le 0$, $b(t) \le 0$, $c(t) < 0$, $a'(t) \ge 0$, $a''(t) \le 0$ and $b'(t) \ge 0$. If (3.1) holds, then (1.1) is oscillatory.*

This follows from Theorem 17.

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