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SINGULAR SOLUTIONS FOR THE DIFFERENTIAL EQUATION WITH p -LAPLACIAN

MIROSLAV BARTUŠEK

ABSTRACT. In the paper a sufficient condition for all solutions of the differential equation with p -Laplacian to be proper. Examples of super-half-linear and sub-half-linear equations $(|y'|^{p-1}y')' + r(t)|y|^\lambda \operatorname{sgn} y = 0$, $r > 0$ are given for which singular solutions exist (for any $p > 0$, $\lambda > 0$, $p \neq \lambda$).

Consider the differential equation with p -Laplacian

$$(1) \quad (a(t)|y'|^{p-1}y')' + r(t)f(y) = 0$$

where $p > 0$, $a \in C^0(R_+)$, $r \in C^0(R_+)$, $f \in C^0(R)$, $R_+ = [0, \infty)$, $R = (-\infty, \infty)$ and

$$(2) \quad a > 0, r \geq 0 \quad \text{on } R_+, f(x)x \geq 0 \quad \text{on } R.$$

A solution y of (1) is called proper if it is defined on R_+ and $\sup_{t \in [\tau, \infty)} |y(t)| > 0$ for every $\tau \in (0, \infty)$. It is called singular of the first kind if it is defined on R_+ , there exists $\tau \in (0, \infty)$ such that $y \equiv 0$ on $[\tau, \infty)$ and $\sup_{T \leq t < \tau} |y(t)| > 0$ for every $T \in [0, \tau)$. It is called singular of the second kind if it is defined on $[0, \tau)$, $\tau < \infty$ and $\sup_{0 \leq t < \tau} |y'(t)| = \infty$. A singular solution y is called oscillatory if there exists a sequence of its zeros $\{t_k\}_1^\infty$, $t_k \in [0, \tau)$ tending to τ .

Eq. (1) and its special case

$$(3) \quad (a(t)|y'|^{p-1}y')' + r(t)|y|^\lambda \operatorname{sgn} y = 0$$

where $\lambda > 0$ is studied by many authors now, see e.g. [5, 6, 8] and the references therein.

One important problem is the existence of proper and singular solutions, respectively. It is known that all solutions of (3) are defined on R_+ if $\lambda \leq p$ and there exists no singular solution of the first kind if $\lambda \geq p$ (see Theorem 1 below); hence in case of half-linear equations, $\lambda = p$, all solutions are proper. But the set of Eqs. (3) with solutions to be proper is larger, Mirzov [8] proved that all solutions of (3) are proper if the functions a and $r > 0$ are locally absolute continuous on R_+ . In the present paper we generalize these results to (1). Other results for

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the nonexistence of singular solutions of the second order differential equations (1) with $a \equiv 1$ and $p = 1$ see e.g. in [2], [4] and [9].

Our second goal is to generalize results of [3] and [7] concerning to the second order equation ($p \equiv 1, a \equiv 1$). We prove that for $\lambda \neq p, a \equiv 1$ there exist equations of the form (3) with singular solutions.

The following theorem is a special case of Theorems 1.1 and 1.2 in [8]; the equivalent expression of results is also given in [5].

Theorem 1. *Let $M \in (0, \infty)$ and $M_1 \in (0, \infty)$.*

- (i) *If $|f(x)| \leq M_1|x|^p$ for $|x| \leq M$, then there exists no singular solution of the 1-st kind of (1).*
- (ii) *If $|f(x)| \leq M_1|x|^p$ for $|x| \geq M$, then there exists no singular solution of the 2-nd kind of (1).*

Theorem 2. *Let the function $a^{\frac{1}{p}}r$ be locally absolute continuous on R_+ and $\frac{1}{r} \in L_{\text{loc}}(R_+)$. Then every nontrivial solution y of (1) is proper. Moreover, if $a^{\frac{1}{p}}(t)r(t) = r_0(t) - r_1(t)$, $t \in R_+$ and*

$$(4) \quad \rho(t) = a^{\frac{p+1}{p}}(t)|y'(t)|^{p+1} + \frac{p+1}{p} a^{\frac{1}{p}}(t)r(t) \int_0^{y(t)} f(s) ds$$

where r_0 and r_1 are nonnegative, nondecreasing and continuous functions, then for $0 \leq s < t < \infty$

$$(5) \quad \rho(s) \exp \left\{ - \int_s^t \frac{r'_1(\sigma)}{a^{\frac{1}{p}}(\sigma)r(\sigma)} d\sigma \right\} \leq \rho(t) \leq \rho(s) \exp \left\{ \int_s^t \frac{r'_0(\sigma)}{a^{\frac{1}{p}}(\sigma)r(\sigma)} d\sigma \right\}.$$

Proof. As $a^{\frac{1}{p}}r$ has locally bounded variation, the continuous nondecreasing functions r_0 and r_1 exist such that $a^{\frac{1}{p}}r = r_0 - r_1$ and they can be chosen to be nonnegative on R_+ . Moreover, $r_0 \in L_{\text{loc}}(R_+)$, $r_1 \in L_{\text{loc}}(R_+)$. Let y be a solution of (1) defined on $[s, t]$. Then

$$\rho'(\tau) = \frac{p+1}{p} [a^{\frac{1}{p}}(t)r(t)]'_{|t=\tau} \int_0^{y(\tau)} f(\sigma) d\sigma, \quad \tau \in [s, t] \quad \text{a.e.}$$

Let $\varepsilon > 0$ be arbitrary. Then

$$\begin{aligned} \frac{\rho'(\tau)}{\rho(\tau) + \varepsilon} &= \frac{p+1}{p} \frac{a^{\frac{1}{p}}(\tau)r(\tau)}{\rho(\tau) + \varepsilon} \int_0^{y(\tau)} f(\sigma) d\sigma \frac{r'_0(\tau) - r'_1(\tau)}{a^{\frac{1}{p}}(\tau)r(\tau)}, \\ &\quad - \frac{r'_1(\tau)}{a^{\frac{1}{p}}(\tau)r(\tau)} \leq \frac{\rho'(\tau)}{\rho(\tau) + \varepsilon} \leq \frac{r'_0(\tau)}{a^{\frac{1}{p}}(\tau)r(\tau)}, \quad \text{a.e. on } [s, t] \end{aligned}$$

and the integration and (4) yield

$$\exp \left\{ - \int_s^t \frac{r'_1(\sigma) d\sigma}{a^{\frac{1}{p}}(\sigma)r(\sigma)} \right\} \leq \frac{\rho(t) + \varepsilon}{\rho(s) + \varepsilon} \leq \exp \left\{ \int_s^t \frac{r'_0(\sigma)}{a^{\frac{1}{p}}(\sigma)r(\sigma)} d\sigma \right\}.$$

As $\varepsilon > 0$ is arbitrary, (5) holds and due to $r^{-1} \in L_{\text{loc}}(R_+)$, y is proper. \square

Remark 1. The assumption $\frac{1}{r} \in L_{\text{loc}}(R_+)$ holds e.g. if $r > 0$ on R_+ .

Theorem 3. Let the assumption of Theorem 2 be valid with $r > 0$ on R_+ and let

$$\rho_1(t) = \frac{a(t)}{r(t)} |y'(t)|^{p+1} + \frac{p+1}{p} \int_0^{y(t)} f(s) ds.$$

Then for $0 \leq s < t < \infty$ we have

$$\rho_1(s) \exp \left\{ - \int_s^t \frac{r'_0(\sigma) d\sigma}{a^{\frac{1}{p}}(\sigma)r(\sigma)} \right\} \leq \rho_1(t) \leq \rho_1(s) \exp \left\{ \int_s^t \frac{r'_1(\sigma)}{a^{\frac{1}{p}}(\sigma)r(\sigma)} d\sigma \right\}.$$

Proof. It is similar to one of Theorem 2 as

$$\rho'_1(t) = \left[\frac{(a(t)|y'(t)|^p)^{\frac{p+1}{p}}}{a^{\frac{1}{p}}(t)r(t)} + \frac{p+1}{p} \int_0^{y(t)} f(s) ds \right]' = - \frac{[a^{\frac{1}{p}}(t)r(t)]' a(t)|y'(t)|^{p+1}}{a^{\frac{1}{p}}(t)r(t) r(t)}.$$

□

Remark 2. For $p = 1$, $a \equiv 1$ and $r > 0$ on R_+ Theorems 1 and 2 are proved in [9], Th. 17.1 and Cor. 17.2; for Eq. (3), if a and $r > 0$ are locally absolutely continuous they are proved in [8], Th. 9.4.

In [1] there is an example of Eq. (3) with $a \equiv 1$, $0 < \lambda < 1$ and $p = 1$ for which there exists a solution y with infinitely many accumulation points of zeros. The following corollary gives a sufficient condition under which every solution of (1) has no accumulation points of zeros in R_+ .

Corollary 1. If the assumptions of Th. 1 are fulfilled, there every nontrivial solution of (1) has only finite number of zeros on a finite interval and it has no double zeros.

Proof. Let $\tau \in R_+$ be an accumulation point of zeros or a double zero of a solution y of (1). As y is proper, $y(\tau) = y'(\tau) = 0$ and (1) has a solution \bar{y} such that $\bar{y} = y$ for $t \leq \tau$ and $\bar{y} \equiv 0$ on (τ, ∞) . Hence \bar{y} is singular of the first kind that contradicts Th. 1. □

The following theorem shows that singular solutions exist. It enlarges the same results for the second order differential equation, obtained in [3] and [7], to (3).

Lemma 1. For an arbitrary integer k there exists $q_k \in C[0, 1]$ such that

$$(6) \quad q_k(0) = q_k(1) = 0,$$

$$(7) \quad \lim_{k \rightarrow \infty} q_k(t) = 0 \quad \text{uniformly on } [0, 1]$$

and the equation

$$(8) \quad (|u'|^{p-1}u')' + (C + q_k(t)) |u|^\lambda \operatorname{sgn} u = 0$$

has a solution u_k fulfilling

$$(9) \quad u_k(0) = 1, \quad u_k(1) = \left(\frac{k+1}{k}\right)^{\frac{2(p+1)}{\lambda-p}}, \quad u'_k(0) = u'_k(1) = 0$$

where C is a suitable positive constant. Moreover, $C + q_k(t) > 0$ on $[0, 1]$.

Proof. Consider a solution w of the problem

$$\left(|\dot{w}|^{p-1}\dot{w}\right)' + |w|^\lambda \operatorname{sgn} w = 0, \quad w(0) = 1, \quad w'(0) = 0, \quad \frac{d}{dx} = \dot{}.$$

Then $|\dot{w}(x)|^{p+1} + \frac{p+1}{p(\lambda+1)}|w(x)|^{\lambda+1} \equiv \frac{p+1}{p(\lambda+1)}$ on the definition interval and it is clear that w is a periodic function with period $T > 0$ with the local maximum at $x = T$. Transformation $x = tT$ yields the existence of a solution Z of the problem

$$(10) \quad \left(|Z'|^{p-1}Z'\right)' + C|Z|^\lambda \operatorname{sgn} Z = 0, \quad Z(0) = Z(1) = 1, \quad Z'(0) = Z'(1) = 0$$

where $C = T^{p+1} > 0$. Note that $Z' > 0$ in a left neighbourhood of $t = 1$.

Let $t_0 \in (0, 1)$ be such that

$$(11) \quad Z(t) > 0 \quad \text{and} \quad Z'(t) > 0 \quad \text{for} \quad t_0 \leq t < 1$$

and put

$$(12) \quad u_k(t) = \begin{cases} Z(t) & \text{for } t \in [0, t_0], \\ \left(\frac{k+1}{k}\right)^{\frac{2(p+1)}{\lambda-p}} - 1 + Z(t) \\ \quad + \int_{t_0}^1 Z'(s) [\alpha_k(s-t_0)^3 + \beta_k(s-T_0)^2] ds & \text{for } t \in (t_0, 1] \end{cases}$$

where α_k and β_k fulfil the system

$$(13) \quad \alpha_k \int_{t_0}^1 Z'(s)(s-t_0)^3 ds + \beta_k \int_{t_0}^1 Z'(s)(s-t_0)^2 ds = 1 - \left(\frac{k+1}{k}\right)^{\frac{2(p+1)}{\lambda-p}},$$

$$(14) \quad \alpha_k(1-t_0)^3 + \beta_k(1-t_0)^2 = 1 - \left(\frac{k+1}{k}\right)^{\frac{2(p+1)\lambda}{(\lambda-p)p}}.$$

Note that the determinant of the system is negative, as due to $Z' > 0$ we have

$$(1-t_0)^2 \int_{t_0}^1 Z'(s)(s-t_0)^3 ds - (1-t_0)^3 \int_{t_0}^1 Z'(s)(s-t_0)^2 ds < 0$$

and it is clear that

$$(15) \quad \lim_{k \rightarrow \infty} \alpha_k = 0, \quad \lim_{k \rightarrow \infty} \beta_k = 0.$$

As

$$(16) \quad u'_k(t) = Z'(t)[1 - \alpha_k(t-t_0)^3 - \beta_k(t-t_0)^2], \quad t \in (t_0, 1],$$

(13) yields $u_k \in C^1[0, 1]$ and according to (15) there exists k_0 such that

$$(17) \quad u_k(t) > 0, \quad u'_k(t) \geq 0 \quad \text{on} \quad [t_0, 1] \quad \text{for} \quad k \geq k_0.$$

Further, from this

$$\begin{aligned} (|u'_k(t)|^{p-1}u'_k(t))' &= (Z'(t)^p[1 - \alpha_k(t - t_0)^3 - \beta_k(t - t_0)^2]^p)' \\ &= -CZ^\lambda(t)(1 - \alpha_k(t - t_0)^3 - \beta_k(t - t_0)^2)^p \\ &\quad - pZ'(t)^p(1 - \alpha_k(t - t_0)^3 - \beta_k(t - t_0)^2)^{p-1} \\ &\quad \times [3\alpha_k(t - t_0)^2 + 2\beta_k(t - t_0)]. \end{aligned}$$

Hence, (11) and (12) yield

$$(18) \quad |u'_k(t)|^{p-1}u'_k(t) \in C^1[0, 1], \quad (|u'_k(t)|^{p-1}u'_k(t))' < 0$$

on $[t_0, 1]$ for large k , say, $k \geq k_1 \geq k_0$.

Define q_k by

$$(19) \quad q_k(t) = \begin{cases} 0 & \text{for } t \in [0, t_0] \\ C + [u_k(t)]^{-\lambda}(|u'_k(t)|^{p-1}u'_k(t))' & \text{for } t \in (t_0, 1]. \end{cases}$$

Then $q \in C[0, 1]$ and (14) yields (6) be valid; it is clear that u_k is a solution of (8) on $[0, 1]$ and according to (10), (12), (16), the relation (9) holds. As according to (12) and (15) $\lim_{k \rightarrow \infty} \frac{u_k(t)}{Z(t)} = 1$ uniformly on $[t_0, 1]$, then (15) and (19) yield (7). Note that according to (17), (18) and (19) $C + q_k(t) > 0$ on $[0, 1]$ for $k \geq k_1$. \square

Theorem 4. *Let $a \equiv 1$ and $p > \lambda$ ($p < \lambda$). Then there exists a positive continuous function r such that Eq. (3) has a singular solution of the first (second) kind.*

Proof. Consider the sequence $\{t_k\}_{k=1}^\infty$ such that $t_1 = 0$, $t_k = \sum_{i=1}^{k-1} \frac{1}{i^2}$, $k = 2, 3, \dots$. Then $\lim_{k \rightarrow \infty} t_k = \frac{\pi^2}{6}$. Let r and y be functions defined by

$$(20) \quad r(t) = (C + q_k(k^2(t - t_k))) , \quad y(t) = k^{\frac{2(p+1)}{\lambda-p}} u_k(k^2(t - t_k))$$

for $t \in [t_k, t_{k+1})$, $k = 1, 2, \dots$

where q_k and u_k are given by Lemma 1.

Let $k \in \{1, 2, \dots\}$ be fixed. The transformation

$$(21) \quad t = t_k + \frac{x}{k^2}, \quad x \in [0, 1], \quad y(t) = k^{\frac{2(p+1)}{\lambda-p}} u_k(x)$$

shows that y is a solution of (3) on $[t_k, t_{k+1})$ and

$$(22) \quad y_+(t_k) = k^{\frac{2(p+1)}{\lambda-p}}, \quad y_-(t_{k+1}) = (k + 1)^{\frac{2(p+1)}{\lambda-p}}, \quad y'_+(t_k) = y'_-(t_{k+1}) = 0,$$

$r_+(t_k) = r_-(t_{k+1}) = C$; here $h_+(\bar{t})$ ($h_-(\bar{t})$) denote the right-hand side (left-hand side) limit of a function h . Hence function r is continuous on $[0, \frac{\pi^2}{6})$ and (7) yields $\lim_{t \rightarrow \frac{\pi^2}{6}} r(t) = C$. Similarly the function y , defined by (20) fulfils $y \in C^1[0, \frac{\pi^2}{6})$ and it is a solution of (3) on $[0, \frac{\pi^2}{6})$. Moreover, according to (12), (16), (21) and (22)

$$\lim_{t \rightarrow \frac{\pi^2}{6}} y(t) = \lim_{t \rightarrow \frac{\pi^2}{6}} y'(t) = 0 \quad \text{if } \lambda < p$$

and

$$\limsup_{t \rightarrow \frac{\pi^2}{6}} |y(t)| = \infty \quad \text{if } \lambda > p.$$

If we put $r(t) = C$ for $t \geq \frac{\pi^2}{6}$ then y is the singular solution of the second kind if $\lambda > p$ and

$$y(t) = \begin{cases} k^{\frac{2(p+1)}{\lambda-p}} u_k(k^2(t-t_k)), & t_k \leq t < t_{k+1}, \quad k = 1, 2, \dots \\ 0, & t \geq \frac{\pi^2}{6} \end{cases}$$

is the singular solution of the first kind if $\lambda < p$. It is clear that $r > 0$ on R_+ . \square

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