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CHARACTERIZATION OF  $E\mathcal{F}$ -SUBCOMPACTIFICATION

ABDOLMAJID FATTAHI AND H. R. EBRAHIMI VISHKI

ABSTRACT. For extending the notion of  $E$ -algebra, as defined in [2], we present an example of an  $m$ -admissible algebra which is not an  $E$ -algebra. Then we define  $E$ -subcompactification and  $E\mathcal{F}$ -subcompactification to study the universal  $E$ -subcompactification and the universal  $E\mathcal{F}$ -subcompactification from the function algebras point of view.

## 1. INTRODUCTION AND PRELIMINARIES

A semigroup  $S$  is called *right reductive* if for each  $a, b \in S$ , the equality  $at=bt$  for every  $t \in S$ , implies that  $a=b$ . For example, all right cancellative semigroups and semigroups with a right identity, are right reductive.

For notation and terminology our ground reference is the extensive book of Berglund et al.[1]. From now on  $S$  will be a semitopological semigroup. By a *semigroup compactification* of  $S$  we mean a pair  $(\psi, X)$ , where  $X$  is a compact Hausdorff right topological semigroup, and  $\psi : S \rightarrow X$  is a continuous homomorphism with dense image such that, for each  $s \in S$ , the mapping  $x \rightarrow \psi(s)x : X \rightarrow X$  is continuous. The  $C^*$ -algebra of all bounded complex-valued continuous functions on  $S$ , will be denoted by  $\mathcal{C}(S)$ . For  $\mathcal{C}(S)$  the left and right translations,  $L_s$  and  $R_t$ , are defined for each  $s, t \in S$  by  $(L_s f)(t) = f(st) = (R_t f)(s)$ ,  $f \in \mathcal{C}(S)$ . A subset  $\mathcal{F}$  of  $\mathcal{C}(S)$  is said to be left translation invariant, if for all  $s \in S$ ,  $L_s \mathcal{F} \subseteq \mathcal{F}$ . A left translation invariant unital  $C^*$ -subalgebra  $\mathcal{F}$  of  $\mathcal{C}(S)$  is called  *$m$ -admissible* if the function  $s \rightarrow T_\mu f(s) = \mu(L_s f)$  is in  $\mathcal{F}$  for all  $f \in \mathcal{F}$  and  $\mu \in S^\mathcal{F}$  (=the spectrum of  $\mathcal{F}$ ). Then the product of  $\mu, \nu \in S^\mathcal{F}$  can be defined by  $\mu\nu = \mu \circ T_\nu$  and the Gelfand topology on  $S^\mathcal{F}$  makes  $(\epsilon, S^\mathcal{F})$  a semigroup compactification (called the  $\mathcal{F}$ -compactification) of  $S$ , where  $\epsilon : S \rightarrow S^\mathcal{F}$  is the evaluation mapping.

Some  $m$ -admissible subalgebras of  $\mathcal{C}(S)$  that we will need in the sequel are:  $\mathcal{LMC} :=$  left multiplicatively continuous functions,  $\mathcal{D} :=$  distal functions,  $\mathcal{MD} :=$  minimal distal functions, and  $\mathcal{SD} :=$  strongly distal functions. We also write  $\mathcal{GP}$  for  $\mathcal{MD} \cap \mathcal{SD}$ ; and we define  $\mathcal{LZ} := \{f \in \mathcal{C}(S); f(st) = f(s) \text{ for all } s, t \in S\}$

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and  $\mathcal{RZ} := \{f \in \mathcal{C}(S); f(st) = f(t) \text{ for all } s, t \in S\}$ .

For a discussion of the universal property of the corresponding compactifications of these function algebras see [1] and also [4].

Let  $(\psi, X)$  be a compactification of  $S$ , then the mapping  $\sigma : S \times X \rightarrow X$ , defined by  $\sigma(s, x) = \psi(s)x$ , is separately continuous and so  $(S, X, \sigma)$  is a flow. If  $\Sigma_X$  denotes the enveloping semigroup of the flow  $(S, X, \sigma)$  (i.e., the pointwise closure of semigroup  $\{\sigma(s, \cdot) : s \in S\}$  in  $X^X$ ) and the mapping  $\sigma_X : S \rightarrow \Sigma_X$  defined by  $\sigma_X(s) = \sigma(s, \cdot)$  for all  $s \in S$ , then  $(\sigma_X, \Sigma_X)$  is a compactification of  $S$  (see [1; 1.6.5]).

One can easily verify that  $\Sigma_X = \{\lambda_x : x \in X\}$ , where  $\lambda_x(y) = xy$  for each  $y \in X$ . If we define the mapping  $\theta : X \rightarrow \Sigma_X$  by  $\theta(x) = \lambda_x$ , then  $\theta$  is a continuous homomorphism with the property that  $\theta \circ \psi = \sigma_X$ . So  $(\sigma_X, \Sigma_X)$  is a factor of  $(\psi, X)$ , that is  $(\psi, X) \geq (\sigma_X, \Sigma_X)$ . By definition,  $\theta$  is one-to-one, if and only if  $X$  is right reductive. So we get the next proposition, which is an extension of the Lawson's result [5; 2.4(ii)].

**Proposition 1.1.** *Let  $(\psi, X)$  be a compactification of  $S$ . Then  $(\sigma_X, \Sigma_X) \cong (\psi, X)$ , if and only if  $X$  is right reductive.*

A compactification  $(\psi, X)$  is called *reductive*, if  $X$  is right reductive. For example, the  $\mathcal{MD}$ ,  $\mathcal{GP}$  and  $\mathcal{LZ}$ -compactifications, are reductive.

In [2] an  $m$ -admissible subalgebra  $\mathcal{F}$  of  $\mathcal{C}(S)$  is defined as an  $E$ -algebra if there is a compactification  $(\psi, X)$  such that  $(\sigma_X, \Sigma_X) \cong (\epsilon, S^{\mathcal{F}})$ . In this setting  $(\psi, X)$  is called an  $E\mathcal{F}$ -compactification of  $S$ . Clearly every reductive compactification is an  $E$ -compactification but the converse is not, in general true; for example see [2; 2.2].

Now we present an example of  $m$ -admissible subalgebra of  $\mathcal{C}(S)$  which is not an  $E$ -algebra. For this purpose we need the following lemma.

**Lemma 1.2.** *Let  $S$  be a factorizable semigroup, i.e.  $S = S^2$  (for instance, let  $S$  be a regular semigroup, see [3]) and  $(\psi, X)$  be a compactification of  $S$  such that  $xyz = yz$  for every  $x, y, z \in X$ . Then  $X$  is a right zero semigroup.*

**Proof.** We show that  $yz = z$  for each  $y, z \in X$ . First suppose that  $z \in \psi(S)$ . So  $z = \psi(s)$  for some  $s \in S$ . Hence  $yz = y\psi(s) = y\psi(\underline{s_1s_2}) = y\psi(s_1)\psi(s_2) = \psi(s_1)\psi(s_2) = \psi(s) = z$ . Now let  $y \in \psi(S)$  and  $z \in X = \overline{\psi(S)}$ . So  $y = \psi(t)$  for some  $t \in S$  and there exist a sequence  $\{\psi(t_n)\}$  in  $\psi(S)$  such that  $\psi(t_n) \rightarrow z$ . Since  $\psi(S) \subset \Lambda(X)$ , we have  $\psi(t_n) = \underline{y\psi(t_n)} \rightarrow yz$ . Therefore  $yz = z$ .

Now suppose that  $y \in X = \overline{\psi(S)}$  and  $z \in X$ . Then there exists a sequence  $\{\psi(s_n)\}$  in  $\psi(S)$  such that  $\psi(s_n) \rightarrow y$ . Since  $X$  is right topological,  $\psi(s_n)z \rightarrow yz$ . But  $\psi(s_n)z = z$  for all  $n$ , and so  $yz = z$  for every  $y, z \in X$ , as claimed.  $\square$

**Example 1.3.** If  $S$  is a factorizable semigroup, then  $\mathcal{RZ}$  is not an  $E$ -algebra. Indeed, let  $(\psi, X)$  be a compactification of  $S$  such that  $(\sigma_X, \Sigma_X) \cong (\epsilon, S^{\mathcal{RZ}})$ , then  $\Sigma_X$  must be a right zero semigroup. It is easy to see that  $\Sigma_X$  is a right zero semigroup if and only if  $xyz = yz$  for every  $x, y, z \in X$ . Now by Lemma 1.2  $X$  is a right zero semigroup and so  $\Sigma_X$  is a trivial semigroup.

2.  $E$ -SUBCOMPACTIFICATION

In this section we extend the notion of  $E\mathcal{F}$ -compactification (see [2]), to  $E\mathcal{F}$ -subcompactification.

**Definition 2.1.** Let  $(\psi, X)$  be a compactification of  $S$ . We say that a compactification  $(\phi, Y)$  is an  $E$ -subcompactification of  $(\psi, X)$  if  $(\sigma_Y, \Sigma_Y)$  is a factor of  $(\psi, X)$ , (in symbol,  $(\sigma_Y, \Sigma_Y) \leq (\psi, X)$ ).

Trivially, every compactification of  $S$  is an  $E$ -subcompactification of itself. Now we are going to construct the universal  $E$ -subcompactification of  $S$ .

**Lemma 2.2.** *Let  $(\phi, Y)$  be the subdirect product of the family  $\{(\phi_i, Y_i) : i \in I\}$  of compactifications of  $S$ . Then  $(\sigma_Y, \Sigma_Y)$  is isomorphic to the subdirect product of the family  $\{(\sigma_{Y_i}, \Sigma_{Y_i}) : i \in I\}$  (i.e.,  $\vee(\sigma_{Y_i}, \Sigma_{Y_i}) \cong (\sigma_Y, \Sigma_Y)$ ).*

**Proof.** By [1; 3.2.5], for each  $i \in I$ , there exists a homomorphism  $p_i$  of  $(\phi, Y)$  onto  $(\phi_i, Y_i)$ . So, by [1; 1.6.7], for each  $i \in I$ , there exists a unique continuous homomorphism  $\pi_i$  of  $(\sigma_Y, \Sigma_Y)$  onto  $(\sigma_{Y_i}, \Sigma_{Y_i})$  such that

$$\pi_i(\zeta)(p_i(y)) = p_i(\zeta(y)) \quad y \in Y, \zeta \in \Sigma_Y.$$

Suppose that  $\zeta_1, \zeta_2 \in \Sigma_Y$ . If  $\pi_i(\zeta_1) = \pi_i(\zeta_2)$  for all  $i \in I$ , then

$$p_i(\zeta_1(y)) = (\pi_i(\zeta_1))(p_i(y)) = (\pi_i(\zeta_2))(p_i(y)) = p_i(\zeta_2(y)),$$

for all  $y \in Y$  and  $i \in I$ . Thus  $\zeta_1 = \zeta_2$ . Therefore the family  $\{\pi_i : i \in I\}$  separates the points of  $\Sigma_Y$ . Now the conclusion follows from [1; 3.2.5].  $\square$

**Theorem 2.3.** *Every compactification  $(\psi, X)$  of  $S$  has the universal  $E$ -subcompactification.*

**Proof.** Let  $(\psi, X)$  be a compactification of  $S$ . Suppose  $\{(\phi_i, Y_i) : i \in I\}$  is a family of  $E$ -subcompactifications of  $(\psi, X)$ , and  $(\phi, Y)$  is the subdirect product of this family. We show that  $(\phi, Y)$  is an  $E$ -subcompactification of  $(\psi, X)$ , and so it is the universal  $E$ -subcompactification of  $(\psi, X)$ . To see this, for each  $i \in I$ , we have  $(\sigma_{Y_i}, \Sigma_{Y_i}) \leq (\psi, X)$ . So, by the subdirect product property and the previous lemma we have,  $(\sigma_Y, \Sigma_Y) \cong \vee(\sigma_{Y_i}, \Sigma_{Y_i}) \leq (\psi, X)$ . This means that  $(\phi, Y)$  is an  $E$ -subcompactification of  $(\psi, X)$ .  $\square$

**Definition 2.4.** Let  $\mathcal{F}$  be an  $m$ -admissible subalgebra of  $\mathcal{C}(S)$ . The compactification  $(\psi, X)$  of  $S$  is called an  $E\mathcal{F}$ -subcompactification of  $S$  if  $(\sigma_X, \Sigma_X) \leq (\epsilon, S^{\mathcal{F}})$ .

Now we are going to prove the next theorem which is an extension of [2; 2.6].

**Theorem 2.5.** *Every  $m$ -admissible subalgebra  $\mathcal{F}$  of  $\mathcal{C}(S)$  has the universal  $E\mathcal{F}$ -subcompactification.*

**Proof.** Set

$$G_{\mathcal{F}} := \{f \in \mathcal{L}MC : T_{\nu}f \in \mathcal{F} \text{ for all } \nu \in S^{\mathcal{L}MC}\}.$$

It is easy to verify that  $G_{\mathcal{F}}$  is an  $m$ -admissible subalgebra of  $\mathcal{C}(S)$  containing  $\mathcal{F}$ . By definition of  $G_{\mathcal{F}}$  we can define the mapping  $\theta : S^{\mathcal{F}} \rightarrow \Sigma_{S^{G_{\mathcal{F}}}}$  by  $\theta(\mu) = \lambda_{\tilde{\mu}}$ , where  $\tilde{\mu}$  is an extension of  $\mu$  to  $S^{G_{\mathcal{F}}}$ . Clearly  $\theta$  is continuous and  $\theta \circ \epsilon = \sigma_{S^{G_{\mathcal{F}}}}$ .

Thus  $(\epsilon, S^{\mathcal{F}}) \geq (\sigma_{S^{G_{\mathcal{F}}}}, \Sigma_{S^{G_{\mathcal{F}}}})$ . So  $G_{\mathcal{F}}$  is an  $E\mathcal{F}$ -subcompactification of  $S$ . Finally, if  $(\psi, X)$  is an  $E\mathcal{F}$ -subcompactification of  $S$  and  $f \in \psi^*(\mathcal{C}(X))$  (where  $\psi^*$  is the adjoint of  $\psi$ ), then by [2; 2.5.],  $T_{\mu}f \in \sigma_X^*(\mathcal{C}(\Sigma_X)) \subset \mathcal{F}$  for all  $\mu \in S^{\mathcal{L}MC}$ . Therefore  $\psi^*(\mathcal{C}(X)) \subset G_{\mathcal{F}}$  and  $(\psi, X) \leq (\epsilon, S^{G_{\mathcal{F}}})$ .  $\square$

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