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# GEODESIC GRAPHS ON SPECIAL 7-DIMENSIONAL G.O. MANIFOLDS

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Abstract. In [3], the present authors and S. Nikčević constructed the 2-parameter family of invariant Riemannian metrics on the homogeneous manifolds $M = [\mathrm{SO}(5) \times \mathrm{SO}(2)]/\mathrm{U}(2)$ and $M = [\mathrm{SO}(4,1) \times \mathrm{SO}(2)]/\mathrm{U}(2)$. They proved that, for the open dense subset of this family, the corresponding Riemannian manifolds are g.o. manifolds which are not naturally reductive. Now we are going to investigate the remaining metrics (in the compact case).

## 1. G.o. spaces and geodesic graphs

Let $G \subset I(M)$ be a connected Lie group which acts transitively on a Riemannian manifold $M$ and let $o \in M$ be a fixed point. If we denote by $H$ the isotropy group at $o$, then $M$ can be identified with the homogeneous manifold $G/H$. In general, there may exist more than one such group $G \subset I(M)$. If, for example, we take a connected Lie group $G'$ such that $G \neq G' \subset I(M)$ and $G'$ also acts transitively on $M$, then there is another expression of $M$ as $G'/H'$ (where $H'$ is the new isotropy group).

For any fixed choice $M = G/H$, $G$ acts effectively on $G/H$ from the left. The Riemannian metric $g$ on $M$ can be considered as a $G$-invariant metric on $G/H$. The pair $(G/H, g)$ is then called a *Riemannian homogeneous space*. Such space is always a *reductive homogeneous space* in the following sense (cf. [5]): we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $G$ and $H$ respectively and consider the adjoint representation $\mathrm{Ad}\colon H \times \mathfrak{g} \to \mathfrak{g}$ of $H$ on $\mathfrak{g}$. There exists a direct sum decomposition (*reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space $T_oM$ via the projection $\pi\colon G \to G/H = M$. Using this natural identification and the scalar product $g_o$ on $T_oM$ we obtain a scalar product $\langle\, ,\rangle$ on $\mathfrak{m}$. This scalar product is obviously $\mathrm{Ad}(H)$-invariant.

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**Definition 1.1.** A Riemannian homogeneous space $(G/H, g)$ is said to be naturally reductive if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of $\mathfrak{g}$ satisfying the condition

$$\langle [X, Z]_{\mathfrak{m}}, Y \rangle + \langle X, [Z, Y]_{\mathfrak{m}} \rangle = 0 \quad \text{for all} \quad X, Y, Z \in \mathfrak{m}\,. \tag{1}$$

Here the subscript $\mathfrak{m}$ indicates the projection of an element of $\mathfrak{g}$ into $\mathfrak{m}$.

It is also well-known that the condition (1) is equivalent to the following more geometrical property:

$$\text{The curve } \exp(tX)(o) \text{ is a geodesic for all } X \in \mathfrak{m}\,. \tag{2}$$

**Definition 1.2.** Let $(M, g)$ be a homogeneous Riemannian manifold. Then $(M, g)$ is said to be naturally reductive if there is a transitive group $G$ of isometries for which the corresponding Riemannian homogeneous space $(G/H, g)$ is naturally reductive in the sense of Definition 1.1.

Examples are known such that $M = G/H$ is not naturally reductive for some small group $G \subset I_0(M)$ but it becomes naturally reductive if we write $M = G'/H'$ for a bigger group of isometries $G' \subset I_0(M)$. By the straightforward generalization of the property (2) we get the following definition.

**Definition 1.3.** A Riemannian homogeneous space $(G/H, g)$ is called a g.o. space if each geodesic of $(G/H, g)$ (with respect to the Riemannian connection) is an orbit of a one-parameter subgroup $\{\exp(tZ)\}, Z \in \mathfrak{g}$, of the group of isometries $G$. A homogeneous Riemannian manifold $(M, g)$ is called a Riemannian g.o. manifold if each geodesic of $(M, g)$ is an orbit of a one-parameter group of isometries.

For more information about the relation between naturally reductive spaces and g.o. spaces and also for the references to related topics see [3].
Our technique used for the characterization of Riemannian g.o. spaces and g.o. manifolds is based on the concept of "geodesic graph". The original idea (not using any explicit name) comes from J. Szenthe [9].

**Definition 1.4.** Let $(G/H, g)$ be a g.o. space. A vector $Z \in \mathfrak{g}$ is called a *geodesic vector* if the curve $\exp(tZ)(o)$ is a geodesic.

**Definition 1.5.** Let $(G/H, g)$ be a g.o. space and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ an $\mathrm{Ad}(H)$-invariant decomposition of the Lie algebra $\mathfrak{g}$. A (general) geodesic graph is an $\mathrm{Ad}(H)$-equivariant map $\eta\colon \mathfrak{m} \to \mathfrak{h}$ which is rational on an open dense subset of $\mathfrak{m}$ and such that $X + \eta(X)$ is a geodesic vector for each $X \in \mathfrak{m}$.

On every Riemannian g.o. space $(G/H, g)$ there exists at least one geodesic graph. The constuction of the *canonical geodesic graph* and *general geodesic graphs* is described in details in [3], [6], [7]. The components $\eta_i$ of a geodesic graph are always rational functions in the form $\eta_i = P_i/P$, where $P_i$ and $P$ are homogeneous polynomials (of the coordinates on $T_o(M)$) and $\deg(P_j) = \deg(P) + 1$. The *degree of a geodesic graph* is defined as the degree of the denominator $P$ in the situation when $P_i$ and $P$ are relatively prime.

**Definition 1.6.** If $(M, g)$ is a g.o. manifold, then the degree of $(M, g)$ is the minimum of degrees of all geodesic graphs (either canonical or general ) constructed for all possible g.o. spaces $(G/H, g)$ where $G \subset I_0(M)$ and $M = G/H$.

According to the results of J. Szenthe, the degree of $(M, g)$ is zero if and only if $(M, g)$ can be made a naturally reductive space $(G/H, g)$ for a suitable choice $G \subset I_0(M)$.
For the examples of geodesic graphs of various degrees ve refer to [6], [7], [1], [2]. The systematic description of temporary results was given in [3].

## 2. G.o. spaces in dimension 7

First, let us quote explicitly the main result of [3], Proposition 1.6.

**Proposition 2.1.** *On a homogeneous space* $G/H = (\mathrm{SO}(5) \times \mathrm{SO}\,(2))/\mathrm{U}(2)$ (*or* $G/H = (\mathrm{SO}(4,1) \times \mathrm{SO}\,(2))/\mathrm{U}(2)$, *respectively*) *there is a family* $\{g_{p,q}\}$ *of invariant metrics depending on two parameters* $p > 0$, $q > 0$ (*or* $p > 0$, $q < 0$, *respectively*) *with the following properties:*

(A) *If* $p$, $q$ *satisfy the system of inequalities*

$$p \neq 2\,, \quad q^2 \neq 4p^2\frac{1-p}{2+p}\,, \quad q^2 \neq 2p^3\frac{6-p}{3p^2+4}\,, \quad q^2 \neq p^2, \tag{3}$$

*then* $G$ *is the maximal connected group of isometries of* $(G/H, g_{p,q})$.

(B) *If* $p$, $q$ *satisfy the inequality* $p \neq 1$, *then* $(G/H, g_{p,q})$ *is a Riemannian g.o. space which is not naturally reductive; for* $p = 1$ *it is naturally reductive.*

(C) *If* $p, q$ *satisfy the inequalities*

$$p \neq 2\,, \quad q^2 \neq p^2(2-p)\,, \quad q^2 \neq 4\,(p-2)^2, \tag{4}$$

*then* $(G/H, g_{p,q})$ *is locally irreducible.*

(D) *The group* $\mathrm{SO}(5)$ (*or* $\mathrm{SO}(4,1)$, *respectively*) *acts as a transitive group of isometries on* $(G/H, g_{p,q})$ *but the corresponding Riemannian homogeneous space* $(\mathrm{SO}(5)/\mathrm{SU}(2), g_{p,q})$ (*or* $(\mathrm{SO}(4,1)/\mathrm{SU}(2), g_{p,q})$, *respectively*) *is never a g.o. space.*

Let us recall the construction of the examples in [3] at the Lie algebra level. Let $\mathfrak{m}$ be a 7-dimensional vector space with the (positive) scalar product $\langle\, , \rangle$. Choose an orthonormal basis $(E_1, \dots, E_4, Z_1, Z_2, Z_3)$ in $\mathfrak{m}$. We denote $\mathfrak{v} = \mathrm{span}(E_1, \dots, E_4)$, $\mathfrak{z} = \mathrm{span}(Z_1, Z_2, Z_3)$ and thus $\mathfrak{m} = \mathfrak{v} + \mathfrak{z}$. Further we denote $A_{ij}$ (for $1 \le i < j \le 4$) the elements of $\mathfrak{so}(\mathfrak{v})$, $B_{\alpha\beta}$ (for $1 \le \alpha < \beta \le 3$) the elements from $\mathfrak{so}(\mathfrak{z})$ and $C_{i\alpha}$ (for $1 \le i \le 4$ and $1 \le \alpha \le 3$) the elements from $\mathfrak{so}(\mathfrak{m})$ with the corresponding action given by the formulas

$$
\begin{aligned}
&A_{ij}(E_k) = \delta_{ik}E_j - \delta_{jk}E_i\,, \\
&B_{\alpha\beta}(Z_\gamma) = \delta_{\alpha\gamma}Z_\beta - \delta_{\beta\gamma}Z_\alpha\,, \\
&C_{i\alpha}(E_j) = \delta_{ij}Z_\alpha\,,\ C_{i\alpha}(Z_\beta) = -\delta_{\alpha\beta}E_i \\
&(\text{for}\quad i,j,k = 1,\dots,4 \quad \text{and} \quad \alpha,\beta,\gamma = 1,\dots,3)\,.
\end{aligned} \tag{5}
$$

We consider now the algebra $\mathfrak{h} = \operatorname{span}(A,B,C,D) \simeq \mathfrak{u}(2)$, where

$$
\begin{aligned}
A &= A_{34} - A_{12}\,, \\
B &= A_{13} + A_{24}\,, \\
C &= A_{14} - A_{23}\,, \\
D &= 2\,B_{12} + A_{14} + A_{23}
\end{aligned} \tag{6}
$$

and we put $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. Now we define the Lie algebra structure on $\mathfrak{g}$ by the additional relations

$$
\begin{aligned}
&[E_1,E_2] = p(Z_1 - A)\,, \quad [E_2,E_3] = qZ_3 - pC\,, \\
&[E_1,E_3] = p(Z_2 + B)\,, \quad [E_2,E_4] = -p(Z_2 - B)\,, \\
&[E_1,E_4] = qZ_3 + pC\,, \quad [E_3,E_4] = p(Z_1 + A)\,, \\
&[Z_1,Z_2] = \frac{2q}{p}Z_3\,, \quad [Z_2,Z_3] = \frac{2p}{q}Z_1\,, \quad [Z_3,Z_1] = \frac{2p}{q}Z_2\,,
\end{aligned} \tag{7}
$$

where $p$ and $q$ are the parameters satisfying $p > 0$, $q \neq 0$ and $p \neq |q|$, and by the adjoint action of the elements from $\mathfrak{z}$ on $\mathfrak{v}$ given by:

$$
\begin{aligned}
\operatorname{ad}(Z_1)|_{\mathfrak{v}} &= (A_{12} + A_{34})\,, \\
\operatorname{ad}(Z_2)|_{\mathfrak{v}} &= (A_{13} - A_{24})\,, \\
\operatorname{ad}(Z_3)|_{\mathfrak{v}} &= \frac{p}{q}(A_{14} + A_{23})\,.
\end{aligned} \tag{8}
$$

If we denote $\widehat{\mathfrak{g}} = \operatorname{span}(\mathfrak{m}, A, B, C)$, then for $q > 0$ the algebra $\widehat{\mathfrak{g}}$ is isomorphic to $\mathfrak{so}(5)$ via the map $\varphi\colon \widehat{\mathfrak{g}} \to \mathfrak{so}(5)$ given by

$$
\begin{aligned}
&\varphi(E_i) = \sqrt{2p}A_{i5} && \text{for } i = 1,\dots,4\,, \\
&\varphi(Z_1) = A_{12} + A_{34}\,, && \varphi(A) = A_{34} - A_{12}\,, \\
&\varphi(Z_2) = A_{13} - A_{24}\,, && \varphi(B) = A_{13} + A_{24}\,, \\
&\varphi(Z_3) = \frac{p}{q}(A_{14} + A_{23})\,, && \varphi(C) = A_{14} - A_{23}
\end{aligned}
$$

and for $q < 0$ the algebra $\widehat{\mathfrak{g}}$ is isomorphic to $\mathfrak{so}(4{,}1)$. Since the vector $Z_3 - \frac{p}{q}D$ is the central element in $\mathfrak{g}$, the algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(5) + \mathfrak{so}(2)$ for $q > 0$ or $\mathfrak{g} \simeq \mathfrak{so}(4,1) + \mathfrak{so}(2)$ for $q < 0$.

We can choose for the corresponding Lie groups $G = \mathrm{SO}(5) \times \mathrm{SO}(2)$, $H = \mathrm{U}(2)$ or $G = \mathrm{SO}(4,1) \times \mathrm{SO}(2)$, $H = \mathrm{U}(2)$. Because the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ is $\operatorname{Ad}(H)$-invariant, we obtain a $G$-invariant Riemannian metric $g_{p,q}$ on $M = G/H$ which comes from the inner product $\langle\,,\rangle$ on $\mathfrak{m}$. We use the symbol $\langle\,,\rangle$ also for the scalar product $(g_{p,q})_o$ on $T_oM$ and the notation (5) also for the corresponding operators on $T_oM$.

In [3], the curvature tensor and the Ricci form of the space $G/H$ was computed. Using the notation

$$
\begin{aligned}
r &= \frac{1}{4p}(p^3 - 4p^2 + 2q^2)\,, & s &= \frac{1}{p^2}(3q^2 - 4p^2)\,,\\
t &= \frac{1}{4}(p^2 - q^2)\,, & u &= \frac{1}{4}q(p-2)\,,\\
v &= \frac{3}{4}p^2 - 2p\,, & w &= \frac{3}{4}q^2 - 2p
\end{aligned}
$$

the components of the curvature operator are

$$
\begin{aligned}
R(E_1,E_2) &= vA_{12} + tA_{34} + 2uB_{23}\\
R(E_1,E_3) &= vA_{13} - tA_{24} - 2uB_{13}\\
R(E_1,E_4) &= wA_{14} - 2tA_{23} + 2rB_{12}\\
R(E_1,Z_1) &= -1/4p^2C_{11} - uC_{33} + rC_{42}\\
R(E_1,Z_2) &= -1/4p^2C_{12} + uC_{23} - rC_{41}\\
R(E_1,Z_3) &= -1/4q^2C_{13} - uC_{22} + uC_{31}\\
R(E_2,E_3) &= -2tA_{14} + wA_{23} + 2rB_{12}\\
R(E_2,E_4) &= -tA_{13} + vA_{24} + 2uB_{13}\\
R(E_2,Z_1) &= -1/4p^2C_{21} + rC_{32} + uC_{43}\\
R(E_2,Z_2) &= -uC_{13} - 1/4p^2C_{22} - rC_{31}\\
R(E_2,Z_3) &= uC_{12} - 1/4q^2C_{23} - uC_{41}\\
R(E_3,E_4) &= tA_{12} + vA_{34} + 2uB_{23}\\
R(E_3,Z_1) &= uC_{13} - rC_{22} - 1/4p^2C_{31}\\
R(E_3,Z_2) &= rC_{21} - 1/4p^2C_{32} + uC_{43}\\
R(E_3,Z_3) &= -uC_{11} - 1/4q^2C_{33} - uC_{42}\\
R(E_4,Z_1) &= -rC_{12} - uC_{23} - 1/4p^2C_{41}\\
R(E_4,Z_2) &= rC_{11} - uC_{33} - 1/4p^2C_{42}\\
R(E_4,Z_3) &= uC_{21} + uC_{32} - 1/4q^2C_{43}\\
R(Z_1,Z_3) &= -2uA_{13} + 2uA_{24} - q^2/p^2B_{13}\\
R(Z_1,Z_2) &= 2rA_{14} + 2rA_{23} + sB_{12}\\
R(Z_2,Z_3) &= 2uA_{12} + 2uA_{34} - q^2/p^2B_{23}\,.
\end{aligned}
\tag{9}
$$

The components of the Ricci form (which are also the components of the corresponding Ricci operator) are given by the diagonal matrix $\rho$, where

$$\operatorname{diag}(\rho) = (\rho_1, \rho_1, \rho_1, \rho_1, \rho_2, \rho_2, \rho_3), \tag{10}$$

and the only (multiple) eigenvalues $\rho_i$ of the Ricci operator are

$$\rho_1 = 6\,p - p^2 - \frac{1}{2}\,q^2\,, \quad \rho_2 = \frac{p^4 + 4\,p^2 - 2\,q^2}{p^2}\,, \quad \rho_3 = \frac{q^2(p^2+2)}{p^2}\,. \tag{11}$$

One can easily check that the inequalities (3) correspond to the condition of distinct Ricci eigenvalues $\rho_1$, $\rho_2$, $\rho_3$. We are now going to study the other possibilities for the eigenvalues. These are:

**Case 1:** $\rho_1 \neq \rho_2 = \rho_3$. This is satisfied if $q = \pm p$.

**Case 2:** $\rho_2 \neq \rho_1 = \rho_3$. This is satisfied if $q^2 = 2p^3 \frac{6-p}{3p^2+4}$.

**Case 3:** $\rho_1 = \rho_2 \neq \rho_3$. This splits into
Subcase 1: $q^2 = 4p^2 \frac{1-p}{2+p}$,
Subcase 2: $p = 2$.

**Case 4:** $\rho_1 = \rho_2 = \rho_3$. This splits into
Subcase 1: $p = 2$, $q = \pm 2$,
Subcase 2: $p = 2/5$, $q = \pm 2/5$.
This is the special situation of the previous cases.

We are now going to find the maximal isometry groups in these cases, and if the group is bigger than that considered in [3], we compute geodesic graphs with respect to this group. For the sake of brevity, we shall consider further only the compact case. The calculations for the non-compact case are analogous.

## 3. New computations for the compact case

**Case 1:** $\rho_1 \neq \rho_2 = \rho_3$, $p = q$.

For the eigenvalues of the Ricci operator it holds

$$\rho_1 = 6\,p - \frac{3}{2}\,p^2\,, \quad \rho_2 = \rho_3 = p^2 + 2\,. \tag{12}$$

First we find the maximal isotropy algebra $\mathfrak{h}$. We are going to find the necessary condition for the skew-symmetric operator $\mathcal{D}$ on $T_oM$ to preserve the eigenspaces of the Ricci operator and to satisfy the condition $\mathcal{D} \cdot R = 0$.

To preserve the eigenspaces of $\rho$, the operator $\mathcal{D}$ should be of the form

$$\mathcal{D} = \sum_{1 \leq i < j \leq 4} a_{ij} A_{ij} + \sum_{1 \leq k < l \leq 3} b_{kl} B_{kl}\,. \tag{13}$$

The condition $\mathcal{D} \cdot R = 0$ can be rewritten explicitly in the form

$$\big\langle (\mathcal{D} \cdot R)(X,Y,Z), W \big\rangle = 0 \quad \text{for all} \quad X, Y, Z, W \in \mathfrak{m}\,. \tag{14}$$

For the choices of the quadruplet $(X, Y, Z, W)$ as

$$(E_1, Z_3, E_4, Z_1)\,, \quad (E_1, Z_3, E_4, Z_2)\,, \quad (E_1, Z_3, Z_1, E_2)\,,$$

we obtain the necessary conditions

$$\begin{aligned} (p-2)\cdot(a_{12}+a_{34}-b_{23}) &= 0\,,\\ (p-2)\cdot(a_{13}-a_{24}+b_{13}) &= 0\,,\\ (p-2)\cdot(a_{14}+a_{23}-b_{12}) &= 0\,. \end{aligned} \tag{15}$$

Let us consider the case $p \neq 2$ ($p = q = 2$ will be studied in Case 4). The basis of the operators which satisfy these conditions can be chosen as

$$\begin{aligned} A &= A_{34} - A_{12}\,, & D_1 &= 2\,B_{12} + A_{14} + A_{23}\,,\\ B &= A_{13} + A_{24}\,, & D_2 &= 2\,B_{23} + A_{12} + A_{34}\,,\\ C &= A_{14} - A_{23}\,, & D_3 &= 2\,B_{13} - A_{13} + A_{24}\,. \end{aligned}$$

Let us denote the algebra generated by these operators as $\mathfrak{h}'$. It is clear that $\mathfrak{h}' \simeq \mathfrak{so}(3) + \mathfrak{so}(3)$. Next, let us denote as $\mathfrak{g}'$ the algebra generated by the vector space $\mathfrak{m}$ (with the Lie brackets (7) and (8)) and the algebra $\mathfrak{h}'$. Denote as $\mathfrak{k}$ the subalgebra of $\mathfrak{g}'$ generated by the elements $D_1 - Z_3, D_2 - Z_1$ and $D_3 + Z_2$. We have $\mathfrak{k} \simeq \mathfrak{so}(3)$ and the elements of $\mathfrak{k}$ commute with $\widehat{\mathfrak{g}}$. Because $\widehat{\mathfrak{g}} \simeq \mathfrak{so}(5)$ and $\mathfrak{g}' = \widehat{\mathfrak{g}} + \mathfrak{k}$, we see that $\mathfrak{g}' \simeq \mathfrak{so}(5) + \mathfrak{so}(3)$.

We choose for the corresponding Lie groups $G' = \mathrm{SO}(5) \times \mathrm{SO}(3)$, $H' = \mathrm{SO}(3) \times \mathrm{SO}(3)$. Because the decomposition $\mathfrak{g}' = \mathfrak{m} + \mathfrak{h}'$ is $\mathrm{Ad}(H')$-invariant, we obtain (from the inner product $\langle\,,\rangle$ on $\mathfrak{m}$) a $G'$-invariant Riemannian metric $g$ on $M = G'/H'$.

We are now going to compute the canonical geodesic graph $\xi\colon \mathfrak{m} \to \mathfrak{h}'$. We use the following lemma.

**Lemma 3.1** ([8]). *A vector $Z \in \mathfrak{g}$ is geodesic if and only if*

$$\langle [Z, Y]_{\mathfrak{m}}, Z_{\mathfrak{m}} \rangle = 0 \text{ for all } Y \in \mathfrak{m}. \tag{16}$$

*Here the subscript $\mathfrak{m}$ indicates the projection of an element of $\mathfrak{g}$ into $\mathfrak{m}$.*

We write each vector $X \in \mathfrak{m}$ in the form

$$X = x_1 E_1 + \cdots + x_4 E_4 + z_1 Z_1 + \cdots + z_3 Z_3\,,$$

each vector $F \in \mathfrak{h}'$ in the form

$$F = \xi_1 A + \xi_2 B + \xi_3 C + \xi_4 D_1 + \xi_5 D_2 + \xi_6 D_3$$

and consider the equation (16) in the form

$$\langle [X + F, Y]_{\mathfrak{m}}, X \rangle = 0\,, \tag{17}$$

where $Y$ runs over all $\mathfrak{m}$. We have to determine the corresponding $F$ to the given $X$. For $Y \in \mathfrak{m}$ we substitute, step by step, all 7 elements $E_1, \ldots, E_4, Z_1, \ldots, Z_3$ of the given orthonormal basis into the formula (17) and we obtain a system of 7 linear equations for the parameters $\xi_1, \ldots, \xi_6$ (satisfying the Frobenius criterion of compatibility). Now, for a *generic* vector $X$, the rank of this system is 5. We select, in a convenient way, a subsystem of 5 linearly independent equations. The matrix $\mathbf{A}$ of the coefficients of the corresponding homogeneous system and the vector $\mathbf{b}$ of the right-hand sides are given by

$$\mathbf{A} = \begin{bmatrix} -x_2 & x_3 & x_4 & x_4 & x_2 & -x_3 \\ x_1 & x_4 & -x_3 & x_3 & -x_1 & x_4 \\ x_4 & -x_1 & x_2 & -x_2 & x_4 & x_1 \\ 0 & 0 & 0 & 2\,z_2 & 0 & 2\,z_3 \\ 0 & 0 & 0 & -2\,z_1 & 2\,z_3 & 0 \end{bmatrix},$$
$$\mathbf{b} = \begin{bmatrix} (p-1)\,(x_2\,z_1 + x_3\,z_2 + x_4\,z_3) \\ -\,(p-1)\,(x_1\,z_1 + x_4\,z_2 - x_3\,z_3) \\ (p-1)\,(x_4\,z_1 - x_1\,z_2 - x_2\,z_3) \\ 0 \\ 0 \end{bmatrix}. \tag{18}$$

Now we are going to add the condition $\xi(X) \perp \mathfrak{q}_X$. Here $\mathfrak{q}_X$ is the subalgebra defined by the condition

$$\mathfrak{q}_X = \{A \in \mathfrak{h}' \mid [A, X] = 0\} \tag{19}$$

and the orthogonality is considered with respect to some invariant scalar product on $\mathfrak{h}'$. The algebra $\mathfrak{q}_X$ is generated by the vectors, whose components (with respect to the basis $\{A, B, C, D_1, D_2, D_3\}$ of $\mathfrak{h}'$) are the solutions of the homogeneous system of equations whose matrix is equal to $\mathbf{A}$ (see [6] or [3] for the details about the construction of the canonical geodesic graph). In our case $\dim\mathfrak{q}_X = 1$ and the components of the generator $Q_X$ can be obtained by the Cramer's rule. We obtain

$$\begin{aligned} Q_X = \big[&(-{x_3}^2 - {x_4}^2 + {x_2}^2 + {x_1}^2)z_1 + (2\,x_3\,x_2 + 2\,x_4\,x_1)z_2 \\ &+ (2\,x_4\,x_2 - 2\,x_1\,x_3)z_3, (-2\,x_3\,x_2 + 2\,x_4\,x_1)z_1 \\ &+ ({x_2}^2 - {x_1}^2 + {x_4}^2 - {x_3}^2)z_2 + (-2\,x_1\,x_2 - 2\,x_4\,x_3)z_3, \\ &(-2\,x_4\,x_2 - 2\,x_1\,x_3)z_1 + (-2\,x_4\,x_3 + 2\,x_1\,x_2)z_2 \\ &+ ({x_2}^2 + {x_3}^2 - {x_4}^2 - {x_1}^2)z_3, ({x_2}^2 + {x_1}^2 + {x_3}^2 + {x_4}^2)z_3\,, \\ &({x_2}^2 + {x_1}^2 + {x_3}^2 + {x_4}^2)z_1, (-{x_2}^2 - {x_1}^2 - {x_3}^2 - {x_4}^2)z_2\big]\,. \end{aligned} \tag{20}$$

Now, let us denote by $q_j$ the components of the vector $Q_X$. The condition $\xi(X) \perp \mathfrak{q}_X$ can be described by the equation

$$\sum_{j=1}^{6} q_j \cdot \xi_j = 0\,. \tag{21}$$

The system of equations described by the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ in (18) and the equation (21) give the system of 6 equation for 6 variables. For the unique solution $\xi$ we obtain by the Cramer's rule

$$\begin{aligned}
\xi_1 &= \frac{p-1}{2\,|x|^2}\Big( - \big({x_1}^2 + {x_2}^2 - {x_3}^2 - {x_4}^2\big)\, z_1 \\
&\qquad\qquad - (2\,x_1\,x_4 + 2\,x_2\,x_3)\, z_2 - (-2\,x_1\,x_3 + 2\,x_2\,x_4)\, z_3\Big)\,, \\
\xi_2 &= \frac{p-1}{2\,|x|^2}\Big( - (2\,x_1\,x_4 - 2\,x_2\,x_3)\, z_1 \\
&\qquad\qquad - \big(-{x_1}^2 + {x_2}^2 - {x_3}^2 + {x_4}^2\big)\, z_2 - (-2\,x_1\,x_2 - 2\,x_3\,x_4)\, z_3\Big)\,, \\
\xi_3 &= \frac{p-1}{2\,|x|^2}\Big( - (-2\,x_1\,x_3 - 2\,x_2\,x_4)\, z_1 \\
&\qquad\qquad - (2\,x_1\,x_2 - 2\,x_3\,x_4)\, z_2 - \big(-{x_1}^2 + {x_2}^2 + {x_3}^2 - {x_4}^2\big)\, z_3\Big)\,, \\
\xi_4 &= \frac{p-1}{2} z_3\,, \\
\xi_5 &= \frac{p-1}{2} z_1\,, \\
\xi_6 &= -\frac{p-1}{2} z_2\,.
\end{aligned} \tag{22}$$

Here we denote $|x|^2 = {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2$. The degree of the canonical geodesic graph is equal to 2. Now we are going to find the geodesic graph $\eta\colon \mathfrak{m} \to \mathfrak{h}'$ of lower degree. Let us define for every $X \in \mathfrak{m}$ the vector $\eta(X) \in \mathfrak{h}'$ by the relation

$$\eta(X) = \xi(X) + \frac{p-1}{2|x|^2} Q_X\,. \tag{23}$$

We easily check that it holds

$$\begin{aligned}
\eta_1 = \eta_2 = \eta_3 &= 0 \\
\eta_4 &= (p-1) z_3\,, \\
\eta_5 &= (p-1) z_1\,, \\
\eta_6 &= -(p-1) z_2\,.
\end{aligned} \tag{24}$$

For each $X \in \mathfrak{m}$, the vector $X + \eta(X)$ is geodesic and the map $\eta$ is $\mathrm{Ad}(H')$-equivariant. It is a *general* geodesic graph. (See [2] or [7] for detailed information about general geodesic graphs.) This geodesic graph is linear and we conclude that the manifold $M = G'/H'$ is naturally reductive.

**Case 2:** $\rho_2 \neq \rho_1 = \rho_3$.

In this case we obtain $q^2 = 2p^3 \frac{6-p}{3p^2+4}$ and for the eigenvalues of the Ricci operator it holds

$$\rho_1 = \rho_3 = -2\,\frac{p\,(p-6)\,\big(p^2+2\big)}{3\,p^2+4}\,, \quad \rho_2 = \frac{3\,p^4 + 20\,p^2 - 24\,p + 16}{3\,p^2+4}\,. \tag{25}$$

To preserve the eigenspaces of $\rho$, the skew-symmetric operator $\mathcal{D}$ on $T_oM$ should be of the form

$$\mathcal{D} = \sum_{1\le i<j\le 4} a_{ij}A_{ij} + b_{12}B_{12} + \sum_{1\le k\le 4} c_{k3}C_{k3}\,. \tag{26}$$

¿From the condition

$$\langle(\mathcal{D}\cdot R)(X,Y,Z),W\rangle = 0 \tag{27}$$

applied step by step to the quadruplets
$(E_1,E_4,E_1,E_2)$, $(E_1,E_4,E_1,E_3)$, $(E_1,E_3,Z_3,Z_2)$,
$(E_1,E_2,Z_3,E_1)$, $(E_1,E_2,Z_3,E_2)$, $(E_1,E_2,Z_3,E_3)$, $(E_1,E_2,Z_3,E_4)$,
we obtain the necessary conditions

$$\begin{aligned}
p^3(5p-2)(p-2)\cdot(a_{13}-a_{24}) &= 0\,,\\
p^3(5p-2)(p-2)\cdot(a_{12}+a_{34}) &= 0\,,\\
q(p-2)\cdot(a_{14}+a_{23}-b_{12}) &= 0\,,\\
p(p-2)(7p^2+2p+16)\cdot c_{23} &= 0\,,\\
p(p-2)(7p^2+2p+16)\cdot c_{13} &= 0\,,\\
p^2(5p-2)(p-2)\cdot c_{43} &= 0\,,\\
p^2(5p-2)(p-2)\cdot c_{33} &= 0\,.
\end{aligned} \tag{28}$$

If $p\neq 2$ and $p\neq 2/5$, we obtain

$$\begin{gathered}
a_{13}-a_{24} = a_{12}+a_{34} = a_{14}+a_{23}-b_{12} = 0\,,\\
c_{13} = c_{23} = c_{33} = c_{43} = 0
\end{gathered} \tag{29}$$

and the basis of all the operators of the form (26) which satisfy the conditions (28) is given by the formula (6). It follows that the isotropy algebra $\mathfrak{h}$ (and also the isometry group $G$) considered in [3] is maximal. The special cases $p=2$ and $p=2/5$ are considered in Case 4.

**Case 4:** $\rho_1=\rho_2=\rho_3$.

**Subcase 1:** $p=2$, $q=2$.
For the eigenvalues of the Ricci operator we have

$$\rho_1=\rho_2=\rho_3=6. \tag{30}$$

To preserve the eigenspaces of $\rho$, the skew-symmetric operator $\mathcal{D}$ on $T_oM$ should be of the form

$$\mathcal{D} = \sum_{1\le i<j\le 4} a_{ij}A_{ij} + \sum_{1\le\alpha<\beta\le 3} b_{\alpha\beta}B_{\alpha\beta} + \sum_{1\le k\le 4,1\le\gamma\le 2} c_{k\gamma}C_{k\gamma}. \tag{31}$$

It is possible to verify by the direct computation that all the elementary operators

$$A_{ij}, B_{\alpha\beta}, C_{k\gamma}\,, \qquad i,j,k=1,\dots,4,\ \ \alpha,\beta,\gamma=1,\dots,3 \tag{32}$$

satisfy the condition $\mathcal{D} \cdot R = 0$. Further, it can be verified by the direct computation that $\nabla R = 0$. Hence, the maximal isotropy algebra is generated by all the operators $A_{ij}, B_{\alpha\beta}, C_{k\gamma}$ and it is isomorphic to $\mathfrak{so}(7)$. We conclude that $M$ is locally isometric to the symmetric space SO(8)/SO(7).

It is worth mentioning here, that the algebra $\mathfrak{g} = \mathfrak{so}(5) + \mathfrak{so}(2)$ cannot be extended to $\mathfrak{g}' = \mathfrak{so}(8)$ simply by considering the vector space $\mathfrak{m}$ (with the Lie brackets (7) and (8)) and the operators $A_{ij}, B_{\alpha\beta}, C_{k\gamma}$. Even though $\mathfrak{so}(5) + \mathfrak{so}(2)$ is the subalgebra of $\mathfrak{so}(8)$, the operators $C_{k\gamma}$ do not act as derivations on $\mathfrak{m}$ (hence on $\mathfrak{g} = \mathfrak{so}(5) + \mathfrak{so}(2)$). In other words, they are not 'compatible' with the original Lie algebra structure on $\mathfrak{g}$.

**Subcase 2:** $p = 2/5$, $q = 2/5$.
For the eigenvalues of the Ricci operator we have

$$\rho_1 = \rho_2 = \rho_3 = \frac{54}{25}\,. \tag{33}$$

For the operator of the form (31) we apply the condition (27) for example to the quadruplets
$(E_1, E_2, E_2, Z_1)$, $(E_1, E_3, E_3, Z_2)$, $(E_1, E_4, E_4, Z_3)$,
$(E_1, E_2, E_1, Z_2)$, $(E_2, E_4, E_4, Z_2)$, $(E_2, E_3, E_3, Z_3)$,
$(E_3, E_4, E_4, Z_1)$, $(E_1, E_3, E_1, Z_2)$, $(E_2, E_3, E_2, Z_3)$,
$(E_3, E_4, E_3, Z_1)$, $(E_2, E_4, E_2, Z_2)$, $(E_1, E_4, E_1, Z_3)$.
We obtain the conditions

$$c_{k\gamma} = 0 \qquad k = 1, \dots, 4,\ \gamma = 1, \dots 3\,. \tag{34}$$

Further, for the quadruplets
$(E_1, Z_3, E_4, Z_1)$, $(E_1, Z_3, E_4, Z_2)$, $(E_1, Z_3, Z_1, E_2)$
we obtain the conditions (15). Here $\nabla R \neq 0$ and the manifold $M$ is not symmetric. But we are in the same situation as in Case 1 and the manifold is naturally reductive. In particular, the group considered in Case 1 is the maximal isometry group also here.

**Case 3:** $\rho_1 = \rho_2 \neq \rho_3$.

In this case we obtain $q^2 = 4p^2\frac{1-p}{2+p}$ or $p = 2$.

**Subcase 1:** $q^2 = 4p^2\frac{1-p}{2+p}$.
For the eigenvalues of the Ricci operator we have

$$\rho_1 = \rho_2 = \frac{p\left(12 + 2\,p + p^2\right)}{2 + p}\,, \qquad \rho_3 = -4\,\frac{(p-1)\left(p^2+2\right)}{2+p}\,. \tag{35}$$

To preserve the eigenspaces of $\rho$, the skew-symmetric operator $\mathcal{D}$ on $T_oM$ should be of the form

$$\mathcal{D} = \sum_{1\leq i<j\leq 4} a_{ij}A_{ij} + b_{12}B_{12} + \sum_{1\leq k\leq 4, 1\leq\gamma\leq 2} c_{k\gamma}C_{k\gamma}\,. \tag{36}$$

If we apply the condition (27) step by step to the quadruplets
$(E_1, E_4, E_1, E_2)$, $(E_1, E_4, E_1, E_3)$, $(E_1, E_2, Z_3, Z_1)$,
$(E_1, E_2, E_2, Z_1)$, $(E_1, E_2, E_2, Z_2)$, $(E_2, E_4, E_4, Z_1)$, $(E_2, E_4, E_4, Z_2)$,
$(E_1, E_3, E_1, Z_1)$, $(E_1, E_3, E_1, Z_2)$, $(E_2, E_4, E_2, Z_1)$, $(E_2, E_4, E_2, Z_2)$,
we obtain the necessary conditions

$$\begin{aligned} p^2(2-5p)\cdot(a_{13}-a_{24}) &= 0\,,\\ p^2(2-5p)\cdot(a_{12}+a_{34}) &= 0\,,\\ q(2-p)\cdot(a_{14}+a_{23}-b_{12}) &= 0\,,\\ (p^2-4)\cdot c_{11} = (p^2-4)\cdot c_{12} &= 0\,,\\ (p^2-4)\cdot c_{21} = (p^2-4)\cdot c_{22} &= 0\,,\\ (p^2-4)\cdot c_{31} = (p^2-4)\cdot c_{32} &= 0\,,\\ (p^2-4)\cdot c_{41} = (p^2-4)\cdot c_{42} &= 0\,. \end{aligned} \tag{37}$$

It follows that, if $p \neq 2$ and $p \neq \frac{2}{5}$, the maximal isotropy algebra is u(2), like in [3]. Hence the manifold $M$ is not naturally reductive.

**Subcase 2:** $p = 2$.
For the eigenvalues of the Ricci operator we have

$$\rho_1 = \rho_2 = -\frac{1}{2}\,(q-4)\,(q+4)\;, \qquad \rho_3 = \frac{3}{2}\,q^2\,. \tag{38}$$

Again, let us consider the operator of the form (36) and the equation (27). For the quadruplets
$(E_1, E_2, E_1, E_4)$, $(E_1, E_3, E_1, E_4)$, $(E_1, E_2, E_3, Z_2)$, $(E_1, E_2, E_3, Z_1)$,
$(E_1, E_2, E_4, Z_2)$, $(E_1, E_2, E_4, Z_1)$
we obtain the necessary conditions

$$\begin{aligned} &(q^2-4)\cdot(a_{13}-a_{24}) = (q^2-4)\cdot(a_{12}+a_{34}) =\\ &(q^2-4)\cdot(c_{11}-c_{42}) = (q^2-4)\cdot(c_{12}+c_{41}) =\\ &(q^2-4)\cdot(c_{21}-c_{32}) = (q^2-4)\cdot(c_{22}+c_{31}) = \quad 0\,. \end{aligned} \tag{39}$$

We are going to consider the situation when $q \neq \pm 2$. The operators which satisfy these equalities are

$$\begin{aligned} A &= A_{34} - A_{12}\,, & C_1 &= C_{11} + C_{42}\,,\\ B &= A_{13} + A_{24}\,, & C_2 &= C_{12} - C_{41}\,,\\ & & C_3 &= C_{21} + C_{32}\,,\\ A_{14}, A_{23}, B_{12}, C_4 &= C_{22} - C_{31}\,. \end{aligned}$$

It can be verified by the direct computation that these 9 operators satisfy also the conditions $\mathcal{D}\cdot R=0$, $\mathcal{D}\cdot\nabla R=0$, $\mathcal{D}\cdot\nabla^2 R=0$, $\mathcal{D}\cdot\nabla^3 R=0$ (the necessary conditions for the operator $\mathcal{D}$ to belong to the full isotropy algebra $\mathfrak{h}'$).
For the Lie bracket of these operators it holds

$$
\begin{aligned}
[A,B] &= 2(A_{14}-A_{23})\,,\\
-[A,A_{14}] &= [A,A_{23}] = B\,,\\
[B,A_{14}] &= -[B,A_{23}] = A\,,\\
[A_{14},A_{23}] &= [B_{12},A_{14}] = [B_{12},A_{23}] = [B_{12},A] = [B_{12},B] = 0\,,\\
[C_1,C_2] &= 2(B_{12}-A_{14})\,,\\
[C_3,C_4] &= 2(B_{12}-A_{23})\,,\\
[C_1,C_3] &= [C_2,C_4] = -A\,,\\
-[C_2,C_3] &= [C_1,C_4] = -B\,,\\
[A,C_1] &= -C_3\,, \quad [B,C_1] = -C_4\,,\\
[A,C_2] &= -C_4\,, \quad [B,C_2] = C_3\,,\\
[A,C_3] &= C_1\,, \quad [B,C_3] = -C_2\,,\\
[A,C_4] &= C_2\,, \quad [B,C_4] = C_1\,,\\
[A_{14},C_1] &= -C_2\,, \quad [A_{23},C_1] = 0\,, \quad [B_{12},C_1] = C_2\,,\\
[A_{14},C_2] &= C_1\,, \quad [A_{23},C_2] = 0\,, \quad [B_{12},C_2] = -C_1\,,\\
[A_{14},C_3] &= 0\,, \quad [A_{23},C_3] = -C_4\,, \quad [B_{12},C_3] = C_4\,,\\
[A_{14},C_4] &= 0\,, \quad [A_{23},C_4] = C_3\,, \quad [B_{12},C_4] = -C_3\,.
\end{aligned}
$$

It is easy to verify that $A_{14}+A_{23}+B_{12}$ is the central element. It can be also verified that the algebra $\mathfrak{h}'$ generated by the 9 operators (40) is isomorphic to $\mathfrak{u}(3)$. The isomorphism is given by the identification of the element $h\in\mathfrak{h}'$ (whose coordinates with respect to the basis $\{A,B,A_{14},A_{23},B_{12},C_1,C_2,C_3,C_4\}$ are $(a,b,a_{14},a_{23},b_{12},c_1,c_2,c_3,c_4)$) with the matrix

$$
\begin{bmatrix}
ia_{14} & a+ib & -c_1+ic_2 & 0\\
-a+ib & ia_{23} & -c_3+ic_4 & 0\\
c_1+ic_2 & c_3+ic_4 & ib_{12} & 0\\
0 & 0 & 0 & 0
\end{bmatrix}. \tag{40}
$$

Here it is natural to expect that the algebra $\mathfrak{g}'$ (corresponding to $\mathfrak{h}'$) is isomorphic to $\mathfrak{u}(4)$. However, as in Case 4, the operators $C_i$ do not act as derivations on $\mathfrak{m}$ (and hence on $\mathfrak{g}=\mathfrak{so}(5)+\mathfrak{so}(2)$) and we cannot simply extend the algebra $\mathfrak{g}=\mathfrak{so}(5)+\mathfrak{so}(2)$ into $\mathfrak{g}'=\mathfrak{u}(4)$ by adding the 5 operators $C_i, B_{12}$.
We are going to investigate the properties of the Riemannian manifold $M'=G'/H'=\mathrm{U}(4)/\mathrm{U}(3)$ with a 1-parameter family of metrics which has the same curvature tensor as the manifold $M=[\mathrm{SO}(5)\times\mathrm{SO}(2)]/\mathrm{U}(2)$.

We decompose the algebra $\mathfrak{g}' = \mathfrak{u}(4)$ as $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$, where the algebra $\mathfrak{h}' = \mathfrak{u}(3)$ is given by the matrices of the form (40) and the vector space $\mathfrak{m}'$ is identified with $T_e(G'/H')$. To preserve the adjoint action of $\mathfrak{h}'$ on $\mathfrak{m}'$ (given by the operators (40) and the formulas (5)), we identify every element $X \in \mathfrak{m}'$ (whose coordinates with respect to the basis $\{E_1, E_2, E_3, E_4, Z_1, Z_3, Z_3\}$ are $(x_1, x_2, x_3, x_4, z_1, z_2, z_3)$) with the matrix

$$\begin{bmatrix} (q-\frac{2}{q})iz_3 & 0 & 0 & x_1+ix_4 \\ 0 & (q-\frac{2}{q})iz_3 & 0 & x_2+ix_3 \\ 0 & 0 & (q-\frac{2}{q})iz_3 & z_1+iz_2 \\ -x_1+ix_4 & -x_2+ix_3 & -z_1+iz_2 & -\frac{2}{q}iz_3 \end{bmatrix}. \tag{41}$$

It is clear that the algebra $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$ is isomorphic to $\mathfrak{u}(4)$. It can be easily verified that the decomposition $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$ is reductive.

We obtain by the direct computations the relations for the Lie bracket on $\mathfrak{m}'$:

$$\begin{aligned}
-[E_1, E_2] &= [E_3, E_4] = A\,, \\
[E_1, E_3] &= [E_2, E_4] = B\,, \\
[E_1, E_4] &= qZ_3 + (2-q^2)(B_{12} + A_{23}) + (4-q^2)A_{14}\,, \\
[E_2, E_3] &= qZ_3 + (2-q^2)(B_{12} + A_{14}) + (4-q^2)A_{23}\,, \\
[Z_1, Z_2] &= qZ_3 + (2-q^2)(A_{14} + A_{23}) + (4-q^2)B_{12}\,, \\
[Z_2, Z_3] &= qZ_1\,, \quad [Z_3, Z_1] = qZ_2\,, \\
[Z_1, E_1] &= -C_1\,, \quad [Z_2, E_1] = -C_2\,, \quad [Z_3, E_1] = qE_4\,, \\
[Z_1, E_2] &= -C_3\,, \quad [Z_2, E_2] = -C_4\,, \quad [Z_3, E_2] = qE_3\,, \\
[Z_1, E_3] &= C_2\,, \quad [Z_2, E_3] = -C_3\,, \quad [Z_3, E_3] = -qE_2\,, \\
[Z_1, E_4] &= C_4\,, \quad [Z_2, E_4] = -C_1\,, \quad [Z_3, E_4] = -qE_1\,.
\end{aligned} \tag{42}$$

Now let us consider the invariant scalar product on $\mathfrak{m}'$ defined by the orthonormal basis $\{E_1, E_2, E_3, E_4, Z_1, Z_2, Z_3\}$ and consider the invariant metric $g$ on $G'/H' = \mathrm{U}(4)/\mathrm{U}(3)$ which comes from the identification of $T_e(G'/H') \simeq \mathfrak{m}'$. We are going to verify that all the vectors $X \in \mathfrak{m}'$ are geodesic.
According to the equation (16), the vector $X \in \mathfrak{m}'$ is geodesic if the equation

$$\langle [X, Y]_{\mathfrak{m}'}, X \rangle = 0 \tag{43}$$

is satisfied for all $Y$ from $\mathfrak{m}'$. In relations (42), the projections of the bracket $[X, Y]$ to $\mathfrak{m}'$ are nonzero only in 9 cases. If we write the vector $X \in \mathfrak{m}'$ in the form $X = x_1E_1 + \cdots + x_4E_4 + z_1Z_1 + \cdots + z_3Z_3$, it is easily seen that the equation (43) is satisfied for every $Y$ from the basis of $\mathfrak{m}'$. It follows that the space $(G'/H', g)$ is naturally reductive.
By the direct computation (using the computer and the same method as in [3]) it can be verified that the curvature tensor of the homogeneous space $G'/H' =$

U(4)/U(3) is the same as the curvature tensor given by the formulas (9) for the homogeneous space $G/H = [\mathrm{SO}(5) \times \mathrm{SO}(2)]/\mathrm{U}(2)$ (for $p = 2$). The equality holds also for the first covariant derivatives of the curvature tensors. In particular, for both spaces $\nabla R \neq 0$, unless $q = 2$. Hence the spaces are not locally symmetric for any $q > 0, q \neq 2$. We conjecture that, for each $q > 0$, these Riemannian homogeneous spaces are locally isometric and the manifold $M = G/H$ is naturally reductive.

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