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# A THEOREM ON CONTINUOUS DEPENDENCE ON A PARAMETER

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**1. Notation and introduction.** $E_n$ denotes the set of all $x = [x_1, \ldots, x_n]$, with real $x_i$. By definition, $x \leqq y$ in $E_n$ means that $x_i \leqq y_i$ for each $i = 1, \ldots, n$. We put $|x| = \max \{|x_1|, \ldots, |x_n|\}$, and similarly in $[t, x]$-space of all $[t, x_1, \ldots, x_n]$, with $t$ real. For $a, b \in E_n$, $a = [a_1, \ldots, a_n]$, $b = [b_1, \ldots, b_n]$, $a_i < b_i$, $i = 1, \ldots, n$, $(a, b)$ denotes the set $(a_1, b_1) \times \ldots \times (a_n, b_n)$. For $\varepsilon$ real, $\vec{\varepsilon}$ denotes $[\varepsilon, \ldots, \varepsilon] \in E_n$. Measurability notions refer to the Lebesgue measure on the real line; a.e. stands for almost everywhere.

In this paper, we investigate differential equations

$$(\mathscr{E}_\mu) \qquad x' = f(t, x, \mu), \quad x(\tau) = \xi(\mu)$$

depending on a parameter $\mu$ in a metrical space, and prove that under some assumptions $\varphi_\mu$, a solution of $(\mathscr{E}_\mu)$, depends continuously on $\mu$. Here, for each fixed $\mu$, any solution of $(\mathscr{E}_\mu)$ is considered in the sense of Carathéodory, i.e. it is an absolutely continuous function $\varphi_\mu$ on an interval $I$, with $\varphi'_\mu(t) = f(t, \varphi_\mu(t), \mu)$ holding a.e. on $I$.

The aim of this note is to prove a generalization of the following theorem.

**Theorem 1.** *Let $D$ be a region of $[t, x]$-space, $[\tau, \xi] \in D$, $I_\mu = \{\mu \in E_k; |\mu - \mu_0| < c\}$, $c > 0$, $D_\mu = D \times I_\mu$. Let a function $[t, x, \mu] \to f(t, x, \mu)$ on $D_\mu$ with values in $E_n$ fulfil the following conditions:*

(1.1) *for fixed $[x, \mu]$, the function $f(., x, \mu)$ is measurable*

(1.2) *for fixed $[t, \mu]$, the function $f(t, ., \mu)$ is continuous*

(1.3) *for fixed $t$, the function $f(t, ., .)$ is continuous at $\mu_0$*

(1.4) *the differential equation*

$$x' = f(t, x, \mu_0), \quad x(\tau) = \xi$$

*has a unique solution $\varphi_0$ on the interval $\langle a, b\rangle$, where $\tau \in \langle a, b\rangle$*

(1.5) *there exists a summable function* $m$ *on* $\langle a, b\rangle$ *such that* $|f(t, x, \mu)| \leqq m(t)$ *for each* $t \in \langle a, b\rangle$, $[t, x, \mu] \in D_\mu$.

*Then there exists a* $\delta > 0$ *such that, for any fixed* $\mu$ *satisfying* $|\mu - \mu_0| < \delta$, *every solution* $\varphi_\mu$ *of*

$$x' = f(t, x, \mu), \quad x(\tau) = \xi$$

*exists over* $\langle a, b\rangle$ (*in the sense that if* $\varphi_\mu$ *is a solution of the above equation on an interval* $\langle c, d\rangle \subset \langle a, b\rangle$, *then there exists an extension of* $\varphi_\mu$ *to the whole of* $\langle a, b\rangle$ *as a solution of this equation*), *and as* $\mu \to \mu_0$, $\varphi_\mu \to \varphi_0$ *uniformly over* $\langle a, b\rangle$.

Proof. See [1], Chapter II, theorem 4.2.

**2.** We show that (1.5) in the preceding theorem may be weakened. First, we prove the following generalization of the Lebesgue convergence theorem.

**Theorem 2.** *Let* $f_m, g_m, h_m$, $m = 0, 1, 2, \ldots$, *be measurable functions defined on* $\langle a, b\rangle$, *and let them fulfil the following conditions*:

(2.1) $g_m \leqq f_m \leqq h_m$ *a.e. on* $\langle a, b\rangle$ *for* $m = 1, 2, \ldots$

(2.2) $g_m \to g_0, f_m \to f_0, h_m \to h_0$ *a.e. on* $\langle a, b\rangle$ *for* $m \to \infty$

(2.3) $g_m, h_m$, $m = 0, 1, 2, \ldots$, *are integrable on* $\langle a, b\rangle$

(2.4) $\int_a^b g_m \to \int_a^b g_0$, $\int_a^b h_m \to \int_a^b h_0$.

*Then* $\int_a^b f_m \to \int_a^b f_0$.

Proof. From Fatou's lemma we get $\int_a^b (f_0 - g_0) = \int_a^b \lim (f_m - g_m) \leqq \leqq \liminf \int_a^b (f_m - g_m) = \liminf \int_a^b f_m - \int_a^b g_0$; hence $\int_a^b f_0 \leqq \liminf \int_a^b f_m$. Passing to opposite functions, we obtain $\int_a^b f_0 \geqq \limsup \int_a^b f_m$.

Remark. The proof remains true for more general spaces and integrals.

**3.** We pass to a theorem on continuous dependence on a parameter, first on product sets in $[t, x]$-space. In what follows, $\Pi$ denotes a metrical space of parameters with distance $|\mu - \nu|$, $\mu_0 \in \Pi$, $\tau$ is a real number, $\alpha > 0$. Further, for any $\vartheta > 0$, $\Pi(\vartheta) = \{\mu \in \Pi; |\mu - \mu_0| < \vartheta\}$.

**Theorem 3.** *Let G be a region in* $E_n$, $\xi$ *be a mapping of* $\Pi$ *into G continuous* at $\mu_0$; *put* $\xi_0 = \xi(\mu_0)$. *Let* $[t, x, \mu] \to f(t, x, \mu)$ *be a function defined on* $\langle \tau, \tau + \alpha\rangle \times G \times \Pi$ *with values in* $E_n$ *such that*

(3.1) *for each* $[x, \mu] \in G \times \Pi$, *the function* $f(., x, \mu)$ *is measurable on* $\langle \tau, \tau + \alpha\rangle$

(3.2) *for each* $[t, \mu] \in \langle \tau, \tau + \alpha\rangle \times \Pi$, *the function* $f(t, ., \mu)$ *is continuous on G*

(3.3) *for each* $\mu \in \Pi$, *there exist integrable functions* $m(., \mu)$, $M(., \mu)$ *on* $\langle \tau, \tau + \alpha\rangle$ *such that* $[t, x, \mu] \in \langle \tau, \tau + \alpha\rangle \times G \times \Pi$ *implies* $m(t, \mu) \leqq f(t, x, \mu) \leqq M(t, \mu)$

(3.4) *for almost all* $t \in \langle \tau, \tau + \alpha \rangle$, $m(t, .)$, $M(t, .)$ *are continuous at* $\mu_0$

(3.5) *the systems* $\{L(., \mu)\}$, $\{U(., \mu)\}$, $\mu \in \Pi$, *where* $L(t, \mu) = \int_\tau^t m(., \mu)$, $U(t, \mu) = \int_\tau^t M(., \mu)$ *are equicontinuous on* $\langle \tau, \tau + \alpha \rangle$

(3.6) *for each* $t \in \langle \tau, \tau + \alpha \rangle$, $L(t, .)$, $U(t, .)$ *are continuous at* $\mu_0$

(3.7) *for each* $t \in \langle \tau, \tau + \alpha \rangle$ *and each* $x \in G$, *the function* $f(t, ., .)$ *is continuous at* $[x, \mu_0]$

(3.8) *the equation*

$$(\mathscr{E}_{\mu_0}) \qquad x' = f(t, x, \mu_0), \quad x(\tau) = \xi_0$$

*has a solution* $\varphi_0$ *on* $\langle \tau, \tau + \alpha \rangle$, *which is unique in the following sense: if* $\psi$ *is a solution of* $(\mathscr{E}_{\mu_0})$ *on* $\langle \tau, \tau + \beta \rangle$, $0 < \beta \leqq \alpha$, *then* $\psi(t) = \varphi_0(t)$ *for each* $t \in \langle \tau, \tau + \beta \rangle$.

*Then given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any* $\mu \in \Pi(\delta)$, *every solution* $\varphi_\mu$ *of* $(\mathscr{E}_\mu)$ *exists over* $\langle \tau, \tau + \alpha \rangle$, *and* $|\varphi_\mu(t) - \varphi_0(t)| < \varepsilon$ *for each* $t \in \langle \tau, \tau + \alpha \rangle$.

Proof. We say that the theorem holds on $\langle \tau, \tau + \beta \rangle$, if from the assumptions follows the assertion on $\langle \tau, \tau + \beta \rangle$ instead of $\langle \tau, \tau + \alpha \rangle$. First, we show the existence of such a $\beta > 0$. From (3.5) and continuity of $\xi$ at $\mu_0$ we infer that there exists $\beta > 0$ such that $\xi(\mu) + L(t, \mu) \in G$, $\xi(\mu) + U(t, \mu) \in G$ for each $t \in \langle \tau, \tau + \beta \rangle$ and, as we may suppose, each $\mu \in \Pi$; from the proof (with a slight modification) of the existence theorem for Carathéodory differential equation ([1], Chapter II, theorem 1.1) we get that all solutions $\varphi_\mu$ of $(\mathscr{E}_\mu)$, $\mu \in \Pi$, exist over $\langle \tau, \tau + \beta \rangle$. From (3.3) and (3.5) we infer that $\{\varphi_\mu\}$, $\mu \in \Pi$, are uniformly bounded and equicontinuous on $\langle \tau, \tau + \beta \rangle$. Let $\mu_k \to \mu_0$ for $k \to \infty$. Let $\varphi_k$ be a solution of $(\mathscr{E}_{\mu_k})$ on $\langle \tau, \tau + \beta \rangle$, for $k = 1, 2, \ldots$ We prove that $\varphi_k \to \varphi_0$ uniformly on $\langle \tau, \tau + \beta \rangle$. By Ascoli's lemma, there exist $k_1, k_2, \ldots$ and $\psi$ such that $\varphi_{k_l}$ converge to $\psi$ uniformly on $\langle \tau, \tau + \beta \rangle$, for $l \to \infty$. According to (3.7), $f(t, \varphi_{k_l}(t), \mu_{k_l})$ converge to $f(t, \psi(t), \mu_0)$ on $\langle \tau, \tau + \beta \rangle$ for $l \to \infty$ so that $\psi(t) = \lim \varphi_{k_l}(t) = \lim [\xi(\mu_{k_l}) + \int_\tau^t f(s, \varphi_{k_l}(s), \mu_{k_l}) = \xi_0 + \int_\tau^t f(s, \psi(s), \mu_0)$, by (3.4), (3.6) and theorem 2. We see that $\psi$ is a solution of $(\mathscr{E}_{\mu_0})$ on $\langle \tau, \tau + \beta \rangle$; hence $\varphi_0 = \psi$ on $\langle \tau, \tau + \beta \rangle$, as it follows from (3.8). Pointwise convergence of $\{\varphi_k\}$ we prove by contradiction, and uniform convergence follows now from equicontinuity; this concludes the proof, as is easy to see.

We pass to the global assertion. Let

$$t_0 = \sup \{t \in \langle \tau, \tau + \alpha \rangle; \text{ the theorem holds on } \langle \tau, t \rangle\}.$$

Suppose first that $t_0 < \tau + \alpha$. We show that there exist $\Gamma > 0$ and $\vartheta_0 > 0$ such that, for each $\mu \in \Pi(\vartheta_0)$, every solution $\varphi_\mu$ of $(\mathscr{E}_\mu)$ exists over $\langle \tau, t_0 + \Gamma \rangle$. As $\varphi_0(t_0) \in G$, we have $\Delta > 0$ such that the cube with edge $8\Delta$ and centre $\varphi_0(t_0)$ lies in $G$. By (3.5), we can choose $\eta > 0$ such that $t_1, t_2 \in \langle \tau, \tau + \alpha \rangle$, $|t_1 - t_2| \leqq \eta$, $\mu \in \Pi$ implies $|L(t_1, \mu) - L(t_2, \mu)| \leqq \Delta$, $|U(t_1, \mu) - U(t_2, \mu)| \leqq \Delta$. Thence $|\varphi_0(t_0 - \eta) - \varphi_0(t_0)| \leqq \Delta$, and for each $\mu \in \Pi$, the estimations $|L(t_0 - \eta, \mu) - L(t_0, \mu)| \leqq \Delta$, $|U(t_0 - \eta, \mu) -$

$- U(t_0, \mu)| \leqq \Delta$ hold. Now choose $\vartheta_0 > 0$ such that $\mu \in \Pi(\vartheta_0)$ implies $|\varphi_\mu(t_0 - \eta) - \varphi_0(t_0 - \eta)| \leqq \Delta$. Hence we have $|\varphi_\mu(t_0 - \eta) - \varphi_0(t_0)| \leqq 2\Delta$ for each $\mu \in \Pi(\vartheta_0)$, and it follows from the equation

$$x(t) = \varphi_\mu(t_0 - \eta) + \int_{t_0 - \eta}^{t} f(s, x(s), \mu) \tag{3.9}$$

that all solutions $\varphi_\mu$, $\mu \in \Pi(\vartheta_0)$, may be extended over $\langle \tau, t_0 \rangle$, with $|\varphi_\mu(t_0) - \varphi_0(t_0)| \leqq 3\Delta$. Indeed, to prove this, it suffices to consider the inequality $\varphi_\mu(t_0 - \eta) + \int_{t_0-\eta}^{t} M(., \mu) \leqq \varphi_0(t_0) + \overrightarrow{3\Delta}$, valid for $t \in \langle t_0 - \eta, t_0 \rangle$, $\mu \in \Pi(\vartheta_0)$, a similar estimation from below, and the existence theorem. Now, the possibility of an extension to $\langle \tau, t_0 + \Gamma \rangle$, for a suitable $\Gamma > 0$, follows from (3.5).

Suppose for simplicity that all solutions of $(\mathscr{E}_\mu)$ exist over $\langle \tau, t_0 + \Gamma \rangle$ for each $\mu \in \Pi$. We prove that the theorem holds on $\langle \tau, t_1 \rangle$, where $t_1 > t_0$. First, we prove that it is true on $\langle \tau, t_0 \rangle$. Let $\varphi_\mu$ be a solution of $(\mathscr{E}_\mu)$ on $\langle \tau, t_0 + \Gamma \rangle$. By the assumption, $\lim_{\mu \to \mu_0} \varphi_\mu(t) = \varphi_0(t)$ for each $t \in \langle \tau, t_0)$. Supposing $\lim_{\mu \to \mu_0} \varphi_\mu(t_0) = \varphi_0(t_0)$ is false, we can evidently choose a sequence $\{\mu_k\}$ converging to $\mu_0$ such that $\lim_{k \to \infty} \varphi_{\mu_k}(t_0) = \kappa \neq \varphi_0(t_0)$. However, using equicontinuity, we get from

$$|\varphi_0(t_0) - \varphi_{\mu_k}(t_0)| \leqq |\varphi_0(t_0) - \varphi_0(t)| + |\varphi_0(t) - \varphi_{\mu_k}(t)| + |\varphi_{\mu_k}(t) - \varphi_{\mu_k}(t_0)| \tag{3.10}$$

a contradiction. Hence $\lim_{\mu \to \mu_0} \varphi_\mu(t_0) = \varphi_0(t_0)$, and by equicontinuity of the solutions $\varphi_\mu$, $\{\varphi_\mu\}$ converges to $\varphi_0$ uniformly on $\langle \tau, t_0 \rangle$.

Now consider the equations

$$x(t) = \varphi_\mu(t_0) + \int_{t_0}^{t} f(s, x(s), \mu), \quad \mu \in \Pi \tag{3.11}$$

on $\langle t_0, t_0 + \Gamma \rangle$. Using the first part of this proof, as we clearly may, we get the validity of the theorem on $\langle \tau, t_1 \rangle$, $t_1 > t_0$; a contradiction. The proof of the case $t_0 = \tau + \alpha$ is included in the above considerations.

**4.** Before stating the main theorem of this article, we introduce the following notations. For $D \subset [t, x]$-space and $x \in E_n$, let $D^{[.,x]} = \{t \in E_1; [t, x] \in D\}$, and similarly for $D^{[t,.]}$. Further let $\text{proj}_t\, D = \cup\{D^{[.,x]}; x \in E_n\}$, and similarly for $\text{proj}_x\, D$.

**Theorem 4.** *Let $D$ be a region of $[t, x]$-space, $\langle a, b \rangle \subset \text{proj}_t\, D$, $\tau \in \langle a, b \rangle$, $\Pi$ a metrical space of parameters, $\mu_0 \in \Pi$. Let $\xi$ be a mapping of $\Pi$ into $\text{proj}_x\, D$ continuous at $\mu_0$; put $\xi_0 = \xi(\mu_0)$ and suppose that $[\tau, \xi_0] \in D$. Let $[t, x, \mu] \to f(t, x, \mu)$ be a function on $D \times \Pi$ with values in $E_n$ such that*

(4.1) *for each $x \in \text{proj}_x\, D$, $\mu \in \Pi$, $f(., x, \mu)$ is measurable on $D^{[.,x]}$*

(4.2) *for each $t \in \text{proj}_t\, D$, $\mu \in \Pi$, $f(t, ., \mu)$ is continuous on $D^{[t,.]}$*

(4.3) *the equation*

$$(\mathscr{E}_{\mu_0}) \qquad x' = f(t, x, \mu_0), \quad x(\tau) = \xi_0$$

*has a solution* $\varphi_0$ *over* $\langle a, b\rangle$; *this solution is unique in the sense that if* $\psi$ *is a solution of* $x' = f(t, x, \mu_0)$ *over* $\langle c, d\rangle \subset \langle a, b\rangle$ *passing through a point of the graph of* $\varphi_0$, *then* $\varphi_0(t) = \psi(t)$ *for each* $t \in \langle c, d\rangle$

(4.4) *for each* $\mu \in \Pi$, *there exist integrable functions* $m(., \mu)$, $M(., \mu)$ *on* $\langle a, b\rangle$ *such that* $t \in \langle a, b\rangle$, $[t, x, \mu] \in D \times \Pi$ *implies* $m(t, \mu) \leqq f(t, x, \mu) \leqq M(t, \mu)$

(4.5) *for almost all* $t \in \langle a, b\rangle$, $m(t, .)$, $M(t, .)$ *are continuous at* $\mu_0$

(4.6) *the systems* $\{L(., \mu)\}$, $\{U(., \mu)\}$, $\mu \in \Pi$, *defined as in* (3.5), *are equicontinuous on* $\langle a, b\rangle$

(4.7) *for each* $t \in \langle a, b\rangle$, $L(t, .)$, $U(t, .)$ *are continuous at* $\mu_0$

(4.8) *for each* $t \in \langle a, b\rangle$ *and each* $x \in G$, *the function* $f(t, ., .)$ *is continuous at* $[x, \mu_0]$.

*Then given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, for any* $\mu \in \Pi(\delta)$, *every solution* $\varphi_\mu$ *of* $(\mathscr{E}_\mu)$ *exists over* $\langle a, b\rangle$, *and* $|\varphi_\mu(t) - \varphi_0(t)| < \varepsilon$ *for each* $t \in \langle a, b\rangle$.

Proof. Let $\varepsilon > 0$ be such that $G = \{3\varepsilon\text{-neighbourhood of the graph of } \varphi_0\}$ lies in $D$. Let $\eta$ be such that $0 < \eta \leqq \varepsilon$ and $\sup \{|\varphi_0(t) - \varphi_0(t')|; t, t' \in \langle a, b\rangle, |t - t'| \leqq \eta\} < \varepsilon$. Suppose for simplicity $\tau = a$, and let $\tau = a < \tau_1 < \tau_2 < \ldots < \tau_r = b$, $\max \{\tau_i - \tau_{i-1}, i = 1, \ldots, r\} < \eta$. Put $K_i = \langle \tau_{i-1}, \tau_i\rangle \times (\varphi_0(\tau_i) - 2\vec{\varepsilon}, \varphi_0(\tau_i) + 2\vec{\varepsilon})$; then $K_i \subset G$ for $i = 1, \ldots, r$, as is easy to see.

It follows from theorem 3 that we have $\vartheta_r > 0$ and $\varepsilon_r > 0$ such that $\mu \in \Pi(\vartheta_r)$, $|\xi_r - \varphi_0(\tau_{r-1})| < \varepsilon_r$ implies: every solution $\varphi$ of $x' = f(t, x, \mu)$, $x(\tau_{r-1}) = \xi_r$, exists over $\langle \tau_{r-1}, \tau_r\rangle$, and $|\varphi(t) - \varphi_0(t)| < \varepsilon$ holds there. Similarly, there exist $\vartheta_{r-1} \leqq \vartheta_r$ and $\varepsilon_{r-1} \leqq \varepsilon_r$ such that $\mu \in \Pi(\vartheta_{r-1})$, $|\xi_{r-1} - \varphi_0(\tau_{r-2})| < \varepsilon_{r-1}$ implies that every solution $\varphi$ of $x' = f(t, x, \mu)$, $x(\tau_{r-2}) = \xi_{r-1}$ exists over $\langle \tau_{r-2}, \tau_r\rangle$, and $|\varphi(t) - \varphi_0(t)| < \varepsilon$ there; etc. Finally, there exist $\vartheta_1$ and $\varepsilon_1$ such that supposing $\mu \in \Pi(\vartheta_1)$ and $|\xi_1 - \varphi_0(\tau)| < \varepsilon_1$, it follows that every solution $\varphi$ of $x' = f(t, x, \mu)$, $x(\tau) = \xi_1$ exists over $\langle \tau, \tau_r\rangle = \langle a, b\rangle$, and $|\varphi(t) - \varphi_0(t)| < \varepsilon$ for each $t \in \langle a, b\rangle$, whence the conclusion. The case $a \neq \tau$ may be dealt with in a similar way.

Remark. It is not difficult to see that the theorems of this paper remain true for e.g. Perron or more general integrals so that generalized absolutely continuous functions may be used as solutions of differential equations. A systematic treatment of this question will be given in another paper.

### Reference

[1] *E. A. Coddington, N. Levinson:* Theory of ordinary differential equations. New York 1955.

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## Výtah

# VĚTA O SPOJITÉ ZÁVISLOSTI NA PARAMETRU

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Hlavním výsledkem této práce je následující věta o spojité závislosti na parametru z teorie diferenciálních rovnic, zobecňující větu 4.2 z [1], II. kapitola. Řešení se uvažují v Carathéodoryově smyslu, tj. jsou to absolutně spojité funkce, vyhovující vztahu $\varphi'(t) = f(t, \varphi(t))$ skoro všude.

**Věta 4.** *Nechť $D$ je oblast v $[t, x]$-prostoru všech $[t, x_1, \ldots, x_n]$, $D^{[\cdot, x]} = \{t \in E_1; [t, x] \in D\}$, kde $x \in E_n$, $\text{proj}_t D = \cup\{D^{[\cdot, x]}; x \in E_n\}$; podobně definujme $D^{[t, \cdot]}$, $\text{proj}_x D$. Nechť je dále $\langle a, b\rangle \subset \text{proj}_t D$, $\tau \in \langle a, b\rangle$, $\Pi$ metrický prostor parametrů, $\mu_0 \in \Pi$. Nechť $\xi$ je zobrazení prostoru $\Pi$ do $\text{proj}_x D$, které je spojité v bodě $\mu_0$; položme $\xi_0 = \xi(\mu_0)$ a předpokládejme, že $[\tau, \xi_0] \in D$. Nechť $[t, x, \mu] \to f(t, x, \mu)$ je funkce definovaná na $D \times \Pi$ s hodnotami v $E_n$, pro kterou platí*

(4.1) *pro každé $x \in \text{proj}_x D$ a každé $\mu \in \Pi$ je funkce $f(., x, \mu)$ měřitelná na $D^{[\cdot, x]}$*

(4.2) *pro každé $t \in \text{proj}_t D$ a každé $\mu \in \Pi$ je funkce $f(t; ., \mu)$ spojitá na $D^{[t, \cdot]}$*

(4.3) *rovnice*

$$(\mathscr{E}_{\mu_0}) \qquad x' = f(t, x, \mu_0), \quad x(\tau) = \xi_0$$

*má řešení $\varphi_0$ na intervalu $\langle a, b\rangle$, které je jednoznačné v následujícím smyslu: je-li $\psi$ řešení rovnice $x' = f(t, x, \mu_0)$ na intervalu $\langle c, d\rangle \subset \langle a, b\rangle$, procházející nějakým bodem grafu $\varphi_0$, pak $\varphi_0(t) = \psi(t)$ pro každé $t \in \langle c, d\rangle$*

(4.4) *pro každé $\mu \in \Pi$ existují integrovatelné funkce $m(., \mu)$, $M(., \mu)$ na $\langle a, b\rangle$ tak, že $t \in \langle a, b\rangle$, $[t, x, \mu] \in D \times \Pi \Rightarrow m(t, \mu) \leqq f(t, x, \mu) \leqq M(t, \mu)$*

(4.5) *pro skoro všechna $t \in \langle a, b\rangle$ jsou funkce $m(t, .)$, $M(t, .)$ spojité v bodě $\mu_0$*

(4.6) *systémy $\{L(., \mu)\}$, $\{U(., \mu)\}$, $\mu \in \Pi$, kde $L(t, \mu) = \int_\tau^t m(., \mu)$, $U(t, \mu) = \int_\tau^t M(., \mu)$, jsou stejně spojité na $\langle a, b\rangle$*

(4.7) *pro každé $t \in \langle a, b\rangle$ jsou funkce $L(t, .)$, $U(t, .)$ spojité v bodě $\mu_0$*

(4.8) *pro každé $t \in \langle a, b\rangle$ a každé $x \in G$ je funkce $f(t, ., .)$ spojitá v bodě $[x, \mu_0]$*

*Potom ke každému $\varepsilon > 0$ existuje $\delta > 0$ tak, že pro každé $\mu$, pro něž platí $|\mu - \mu_0| < \delta$, všechna řešení $\varphi_\mu$ rovnice*

$$(\mathscr{E}_\mu) \qquad x' = f(t, x, \mu), \quad x(\tau) = \xi(\mu)$$

*existují na $\langle a, b\rangle$, a platí $|\varphi_\mu(t) - \varphi_0(t)| < \varepsilon$ pro každé $t \in \langle a, b\rangle$.*

# Резюме

## ТЕОРЕМА О НЕПРЕРЫВНОЙ ЗАВИСИМОСТИ ОТ ПАРАМЕТРА

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Главным результатом настоящей работы является следующая теорема о непрерывной зависимости от параметра из теории дифференциальных уравнений, обобщающая теорему 4.2 из [1], II. глава. Решения рассматриваются здесь в смысле Каратеодоры, т.е. они являются абсолютно непрерывными функциями, удовлетворяющими уравнению $\varphi'(t) = f(t, \varphi(t))$ почти всюду.

**Теорема 4.** *Пусть $D$ – область в $[t, x]$-пространстве всех $[t, x_1, \ldots, x_n]$, $D^{[\cdot, x]} = \{t \in E_1;\ [t, x] \in D\}$, где $x \in E_n$, $\text{proj}_t D = \cup \{D^{[\cdot, x]};\ x \in E_n\}$; подобно определим $D^{[t, \cdot]}$, $\text{proj}_x D$. Пусть, далее, $\langle a, b \rangle \subset \text{proj}_t D$, $\tau \in \langle a, b \rangle$, $\Pi$ – метрическое пространство параметров, $\mu_0 \in \Pi$. Пусть $\xi$ – отображение $\Pi$ в $\text{proj}_x D$, непрерывное в точке $\mu_0$; положим $\xi_0 = \xi(\mu_0)$ и предположим, что $[\tau, \xi_0] \in D$. Пусть $[t, x, \mu] \to f(t, x, \mu)$ – функция на $D \times \Pi$, отображающая в $E_n$ и такая, что* (4.1) *для каждого $x \in \text{proj}_x D$ и каждого $\mu \in \Pi$ функция $f(., x, \mu)$ измерима на $D^{[\cdot, x]}$;* (4.2) *для каждого $t \in \text{proj}_t D$ и каждого $\mu \in \Pi$ функция $f(t, ., \mu)$ непрерывна на $D^{[t, \cdot]}$;* (4.3) *уравнение*

$$(\mathscr{E}_{\mu_0}) \qquad x' = f(t, x, \mu_0), \quad x(\tau) = \xi_0$$

*имеет решение $\varphi_0$ на $\langle a, b \rangle$, однозначное в следующем смысле: если $\psi$ – решение уравнения $x' = f(t, x, \mu_0)$ на интервале $\langle c, d \rangle \subset \langle a, b \rangle$, проходящее через некоторую точку графика $\varphi_0$, то $\varphi_0(t) = \psi(t)$ для каждого $t \in \langle c, d \rangle$;* (4.4) *для каждого $\mu \in \Pi$ существуют интегрируемые функции $m(., \mu)$, $M(., \mu)$ на $\langle a, b \rangle$ так, что $t \in \langle a, b \rangle$, $[t, x, \mu] \in D \times \Pi$ влечет $m(t, \mu) \leqq f(t, x, \mu) \leqq M(t, \mu)$;* (4.5) *для почти всех $t \in \langle a, b \rangle$ функции $m(t, .)$, $M(t, .)$ непрерывны в точке $\mu_0$;* (4.6) *системы $\{L(., \mu)\}$, $\{U(., \mu)\}$, $\mu \in \Pi$, где $L(t, \mu) = \int_\tau^t m(., \mu)$, $U(t, \mu) = \int_\tau^t M(., \mu)$, равностепенно непрерывны на $\langle a, b \rangle$;* (4.7) *для каждого $t \in \langle a, b \rangle$ функции $L(t, .)$, $U(t, .)$ непрерывны в точке $\mu_0$;* (4.8) *для каждого $t \in \langle a, b \rangle$ и каждого $x \in G$ функция $f(t, ., .)$ непрерывна в точке $[x, \mu_0]$.*

*Тогда для всякого $\varepsilon > 0$ существует $\delta > 0$ так, что для всякого $\mu$, выполняющего $|\mu - \mu_0| < \delta$, все решения $\varphi_\mu$ уравнения,*

$$(\mathscr{E}_\mu) \qquad x' = f(t, x, \mu), \quad x(\tau) = \xi(\mu)$$

*существуют на $\langle a, b \rangle$, и $|\varphi_\mu(t) - \varphi_0(t)| < \varepsilon$ для всякого $t \in \langle a, b \rangle$.*