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Výpočet součtu řad

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VÝPOČET SOUČTU ŘAD

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V článku je podána metoda pro numerický výpočet součtu jistých řad a odhad chyby při této metodě (§ 1 a 2). V § 3 jsou jako ukázka propočteny tři příklady.

1.

Budiž $s = \alpha + i\beta$ ($\alpha > 1$, β reálné); $\varrho \geqq 0$.

Řada

$$f(z) = a_0 z^s + a_1 z^{s-1} + a_2 z^{s-2} + \ldots \quad (a_0 \neq 0,\ z = x + iy) \tag{1}$$

budiž konvergentní a $f(z) \neq 0$ pro $|z| > \varrho$. Pro $|z| > \varrho$ máme tedy konvergentní rozvoj

$$\frac{1}{f(z)} = b_0 z^{-s} + b_1 z^{-s-1} + b_2 z^{-s-2} + \ldots, \quad b_0 = \frac{1}{a_0} \neq 0\,. \tag{2}$$

Je-li $k_0 > \varrho$ přirozené číslo, je řada $\sum_{k=k_0}^{\infty} \frac{1}{f(k)}$ absolutně konvergentní (ježto $\frac{1}{f(k)} = O\left(\frac{1}{k^\alpha}\right)$); jde o stanovení jejího součtu.

Přitom $k^s = e^{s \log k}$, kde $\log k$ značí hlavní hodnotu logaritmu čísla k.

$\frac{1}{f(k)}$ můžeme rozvinout ve tvaru

$$\frac{1}{f(k)} = A_{-1}\left[\frac{1}{(k-\frac{1}{2})^{s-1}} - \frac{1}{(k+\frac{1}{2})^{s-1}}\right] + A_0\left[\frac{1}{(k-\frac{1}{2})^{s}} - \frac{1}{(k+\frac{1}{2})^{s}}\right] + \ldots$$
$$\ldots + A_r\left[\frac{1}{(k-\frac{1}{2})^{s+r}} - \frac{1}{(k+\frac{1}{2})^{s+r}}\right] + \ldots + A_d\left[\frac{1}{(k-\frac{1}{2})^{s+d}} - \frac{1}{(k+\frac{1}{2})^{s+d}}\right] +$$
$$+ \varphi(k)\,. \tag{3}$$

Rozvoje

$$\left(k - \frac{1}{2}\right)^{-(s+r)} = \sum_{i=0}^{\infty} (-1)^i \binom{-s-r}{i} \frac{1}{2^i} k^{-(s+r+i)}\,,$$

$$\left(k + \frac{1}{2}\right)^{-(s+r)} = \sum_{i=0}^{\infty} \binom{-s-r}{i} \frac{1}{2^i} k^{-(s+r+i)}$$

jsou pro všechna r a $k \geqq 1$ absolutně konvergentní.

Rovnici (3) můžeme pak psáti ve tvaru (3a):

$$b_0 k^{-s} + b_1 k^{-s-1} + b_2 k^{-s-2} + \ldots + b_n k^{-s-n} + \ldots =$$

$$= -A_{-1}\left[(-s+1)k^{-s} + \binom{-s+1}{3}\frac{1}{2^2}k^{-s-2} + \ldots + \binom{-s+1}{2r+1}\frac{1}{2^{2r}}k^{-s-2r} + \ldots\right] -$$

$$- A_0\left[-sk^{-s-1} + \binom{-s}{3}\frac{1}{2^2}k^{-s-3} + \ldots + \binom{-s}{2r+1}\frac{1}{2^{2r}}k^{-s-2r-1} + \ldots\right] -$$

$$- A_1\left[(-s-1)k^{-s-2} + \binom{-s-1}{3}\frac{1}{2^2}k^{-s-4} + \ldots + \binom{-s-1}{2r+1}\frac{1}{2^{2r}}k^{-s-2r-2} + \ldots\right] - \ldots$$

$$\ldots - A_n\left[(-s-n)k^{-s-n-1} + \binom{-s-n}{3}\frac{1}{2^2}k^{-s-n-3} + \ldots\right.$$

$$\left.\ldots + \binom{-s-n}{2r+1}\frac{1}{2^{2r}}k^{-s-n-2r-1} + \ldots\right] - \ldots$$

$$\ldots - A_d\left[(-s-d)k^{-s-d-1} + \binom{-s-d}{3}\frac{1}{2^2}k^{-s-d-3} + \ldots\right.$$

$$\left.\ldots + \binom{-s-d}{2r+1}\frac{1}{2^{2r}}k^{-s-d-2r-1} + \ldots\right] + \varphi(k)\,.$$

Srovnáním koeficientů vyjde $\varphi(k)$ ve tvaru konvergentní řady

$$\varphi(k) = \alpha_0 k^{-s} + \alpha_1 k^{-s-1} + \alpha_2 k^{-s-2} + \ldots$$

Koeficienty $A_{-1}, A_0, A_1, \ldots, A_d$ lze stanoviti jednoznačně tak, že $\alpha_0 = \alpha_1 = \ldots = \alpha_{d+1} = 0$, načež

$$\varphi(k) = \alpha_{d+2} k^{-s-d-2} + \alpha_{d+3} k^{-s-d-3} + \ldots,$$

a máme

$$\sum_{k=k_0}^{n}\frac{1}{f(k)} = A_{-1}\left[\frac{1}{(k_0-\frac{1}{2})^{s-1}} - \frac{1}{(n+\frac{1}{2})^{s-1}}\right] + A_0\left[\frac{1}{(k_0-\frac{1}{2})^{s}} - \frac{1}{(n+\frac{1}{2})^{s}}\right] + \ldots$$

$$\ldots + A_r\left[\frac{1}{(k_0-\frac{1}{2})^{s+r}} - \frac{1}{(n+\frac{1}{2})^{s+r}}\right] + \ldots + A_d\left[\frac{1}{(k_0-\frac{1}{2})^{s+d}} - \frac{1}{(n+\frac{1}{2})^{s+d}}\right] + R;$$

$$R = \sum_{k=k_0}^{n}\frac{\alpha_{d+2}}{k^{s+d+2}} + \frac{\alpha_{d+3}}{k^{s+d+3}} + \ldots \tag{4}$$

Je vidět, že koeficienty $A_{-1}, A_0, \ldots, A_d$ nezávisí na volbě d. Rovnice (4) můžeme použít též pro $\alpha < 1$, pokud $\alpha + d + 2 > 1$.

Pro $n \to \infty$, $\alpha > 1$ obdržíme

$$\sum_{k=k_0}^{\infty}\frac{1}{f(k)} = A_{-1}\frac{1}{(k_0-\frac{1}{2})^{s-1}} + A_0\frac{1}{(k_0-\frac{1}{2})^{s}} + \ldots + A_r\frac{1}{(k-\frac{1}{2})^{s+r}} + \ldots$$

$$\ldots + A_d\frac{1}{(k_0-\frac{1}{2})^{s+d}} + R;$$

$$R = \sum_{k=k_0}^{\infty}\frac{\alpha_{d+2}}{k^{s+d+2}} + \frac{\alpha_{d+3}}{k^{s+d+3}} + \ldots \tag{5}$$

Touto rovnicí je funkce $\Phi(s)$ definovaná pro $\alpha > 1$ řadou

$$\sum_{k=1}^{\infty} \frac{1}{f(k)} = \sum_{k=1}^{k_0-1} \frac{1}{f(k)} + \sum_{k=k_0}^{\infty} \frac{1}{f(k)}$$

analyticky prodloužena pro $\alpha < 1$, pokud $\alpha + d + 2 > 1$.

Z rovnice (3a) plyne:

$$A_{-1} = \frac{1}{a_0(s-1)}, \quad A_0 = \frac{b_1}{s}, \quad A_1 = \frac{12b_2 - \binom{s+1}{2} b_0}{12(s+1)},$$

$$A_2 = \frac{12b_3 - \binom{s+2}{2} b_1}{12(s+2)}, \quad A_3 = \frac{3\,.\,5\,.\,2^4 b_4 + 7b_0 \binom{s+3}{4} - 20 \binom{s+3}{2} b_2}{3\,.\,5\,.\,2^4(s+3)}, \text{ atd.}$$

Pro $r = 1, 2, 3, \dots$ je

$$\alpha_{d+2r} = b_{d+2r} + A_{d-1} \binom{-s-d+1}{2r+1} \frac{1}{2^{2r}} + A_{d-3} \binom{-s-d+3}{2r+3} \frac{1}{2^{2r+2}} + \dots$$

$$\dots + A_\nu \binom{-s-\nu}{d+2r-\nu} \frac{1}{2^{d+2r-\nu-1}} + \dots + \begin{cases} A_0 \binom{-s}{d+2r} \dfrac{1}{2^{d+2r-1}} & \text{(pro lichá } d\text{)};\\[2mm] A_{-1} \binom{-s+1}{d+2r+1} \dfrac{1}{2^{d+2r}} & \text{(pro sudá } d\text{)}; \end{cases}$$

$$\alpha_{d+2r+1} = b_{d+2r+1} + A_d \binom{-s-d}{2r+1} \frac{1}{2^{2r}} + A_{d-2} \binom{-s-d+2}{2r+3} \frac{1}{2^{2r+2}} + \dots$$

$$\dots + A_\nu \binom{-s-\nu}{d+2r-\nu+1} \frac{1}{2^{d+2r-\nu}} + \dots + \begin{cases} A_{-1} \binom{-s+1}{d+2r+2} \dfrac{1}{2^{d+2r+1}} & \text{(pro lichá } d\text{)}\\[2mm] A_0 \binom{-s}{d+2r+1} \dfrac{1}{2^{d+2r}} & \text{(pro sudá } d\text{)}. \end{cases} \tag{6}$$

Pro zbytek R obdržíme odhad, platný pro každé $k_0 > \varrho$.

$$|R| < \sum_{k=k_0}^{\infty} \frac{1}{|k^{s+d+2}|} \Big\{ |b_{d+2}| \left[1 + \frac{|b_{d+4}|}{|b_{d+2}|\, k_0^2} + \frac{|b_{d+6}|}{|b_{d+2}|\, k_0^4} + \dots \right] +$$

$$+ |A_{d-1}| \frac{1}{2^2} \left[\left| \binom{-s-d+1}{3} \right| + \left| \binom{-s-d+1}{5} \right| \frac{1}{4k_0^2} + \left| \binom{-s-d+1}{7} \right| \frac{1}{4^2 k_0^4} + \dots \right] +$$

$$+ |A_{d-3}| \frac{1}{2^4} \left[\left| \binom{-s-d+3}{5} \right| + \left| \binom{-s-d+3}{7} \right| \frac{1}{4k_0^2} + \left| \binom{-s-d+3}{9} \right| \frac{1}{4^2 k_0^4} + \dots \right] + \dots$$

$$\dots + |A_\nu| \frac{1}{2^{d+1-\nu}} \left[\left| \binom{-s-\nu}{d+2-\nu} \right| + \right.$$

$$\left. + \left| \binom{-s-\nu}{d+4-\nu} \right| \frac{1}{4k_0^2} + \left| \binom{-s-\nu}{d+6-\nu} \right| \frac{1}{4^2 k_0^4} + \dots \right] + \dots$$

$$\dots + \left\langle \begin{aligned} &|A_0| \frac{1}{2^{d+1}} \left[\left| \binom{-s}{d+2} \right| + \left| \binom{-s}{d+4} \right| \frac{1}{4k_0^2} + \left| \binom{-s}{d+6} \right| \frac{1}{4^2 k_0^4} + \dots \right] \text{(pro lichá } d) + \\ &|A_{-1}| \frac{1}{2^{d+2}} \left[\left| \binom{-s+1}{d+3} \right| + \left| \binom{-s+1}{d+5} \right| \frac{1}{4k_0^2} + \left| \binom{-s+1}{d+7} \right| \frac{1}{4^2 k_0^4} + \dots \right] \end{aligned} \right.$$

(pro sudá d) +

$$+ \sum_{k=k_0}^{\infty} \frac{1}{|k^{s+d+3}|} \left\{ |b_{d+3}| \left[1 + \frac{|b_{d+5}|}{|b_{d+3}| k_0^2} + \frac{|b_{d+7}|}{|b_{d+3}| k_0^4} + \dots \right] + \right.$$

$$+ |A_d| \frac{1}{2^2} \left[\left| \binom{-s-d}{3} \right| + \left| \binom{-s-d}{5} \right| \frac{1}{4k_0^2} + \left| \binom{-s-d}{7} \right| \frac{1}{4^2 k_0^4} + \dots \right] +$$

$$+ |A_{d-2}| \frac{1}{2^4} \left[\left| \binom{-s-d+2}{5} \right| + \left| \binom{-s-d+2}{7} \right| \frac{1}{4k_0^2} + \left| \binom{-s-d+2}{9} \right| \frac{1}{4^2 k_0^4} + \dots \right] + \dots$$

$$\dots + |A_\nu| \frac{1}{2^{d-\nu+2}} \left[\left| \binom{-s-\nu}{d+3-\nu} \right| + \right.$$

$$\left. + \left| \binom{-s-\nu}{d+5-\nu} \right| \frac{1}{4k_0^2} + \left| \binom{-s-\nu}{d+7-\nu} \right| \frac{1}{4^2 k_0^4} + \dots \right] + \dots$$

$$\dots + \left\langle \begin{aligned} &|A_{-1}| \frac{1}{2^{d+3}} \left[\left| \binom{-s+1}{d+4} \right| + \left| \binom{-s+1}{d+6} \right| \frac{1}{4k_0^2} + \left| \binom{-s+1}{d+8} \right| \frac{1}{4^2 k_0^4} + \dots \right] \text{(pro lichá } d). \\ &|A_0| \frac{1}{2^{d+2}} \left[\left| \binom{-s}{d+3} \right| + \left| \binom{-s}{d+5} \right| \frac{1}{4k_0^2} + \left| \binom{-s}{d+7} \right| \frac{1}{4^2 k_0^4} + \dots \right] \end{aligned} \right.$$

(pro sudá d) .

Je-li n_0 nejmenší přirozené číslo takové, že $n_0 \geqq \varrho$, plyne z konvergence řady (2)

$$\left| \frac{b_{d+\nu+2}}{b_{d+2} n_0^\nu} \right| \leqq 1 \,.^{*)}$$

Je-li tudíž $k_0 > n_0$ přirozené číslo, platí

$$\left| \frac{b_{d+\nu+2}}{b_{d+2} k_0^\nu} \right| \leqq \sigma^\nu \,, \quad \text{kde} \quad \sigma = \frac{n_0}{k_0} < 1 \,. \tag{7}$$

Provádějme tedy odhad zbytku pro $k_0 > n_0$. Potom

$$1 + \frac{|b_{d+3}|}{|b_{d+2}| k_0} + \frac{|b_{d+4}|}{|b_{d+2}| k_0^2} + \dots < \frac{1}{1-\sigma} \,.$$

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*) Upozorňujeme čtenáře, že pro jednoduchost předpokládáme, že platí $\left| \frac{b_{n+1}}{b_n k_0} \right| \leqq 1$ pro všechna přirozená n. Obecně platí $\left| \frac{b_{n+1}}{b_n k_0} \right| \leqq 1$ jen od jistého n počínaje, a tedy je nutno v dalším vzorec (9) příslušným způsobem pozměnit, což si čtenář lehce provede sám.

Zbývající řady, vyskytující se při odhadu zbytku R, jsou tvaru:

$$S = \left|\binom{-(s+r)}{m}\right| + \left|\binom{-(s+r)}{m+2}\right| \frac{1}{4k_0^2} + \left|\binom{-(s+r)}{m+4}\right| \frac{1}{4^2k_0^4} + \ldots \leqq S_1\,,$$

kde

$$S_1 = \left|\binom{-|\alpha+r|-|\beta|}{m}\right| + \left|\binom{-|\alpha+r|-|\beta|}{m+2}\right| \frac{1}{4k_0^2} +$$

$$+ \left|\binom{-|\alpha+r|-|\beta|}{m+4}\right| \frac{1}{4^2k_0^4} + \ldots;$$

přitom $s = \alpha + i\beta$, $m = 1; 3; 5; \ldots; r = -1; 0; 1; 2; \ldots$.

Označme

$$\frac{\left|\binom{-|\alpha+r|-|\beta|}{m+2}\right|}{\left|\binom{-|\alpha+r|-|\beta|}{m}\right| 4k_0^2} = \omega_{m,r}\,. \tag{8}$$

Pokud

$$|\alpha + r| + |\beta| > 1\,,$$

klesá $\omega_{r,m}$ s rostoucím m a $S_1 < \left|\binom{-|\alpha+r|-|\beta|}{m}\right| \frac{1}{1-\omega_{r,m}}$, je-li k_0 tak zvoleno, že $\omega_{r,m} < 1$.

Je-li $|\alpha + r| + |\beta| = 1$, je $S_1 = \dfrac{1}{1 - \dfrac{1}{4k_0^2}}$, je-li $|\alpha + r| + |\beta| < 1$, je $S_1 < \dfrac{1}{1 - \dfrac{1}{4k_0^2}}$.

Při odhadu zbytku R vyskytnou se čísla $\omega_{d-1,3}$; $\omega_{d-3,5}$; …; $\omega_{0,d+2}$; $\omega_{-1,d+3}$; $\omega_{d,3}$; $\omega_{d-2,5}$; …; $\omega_{0,d+3}$; $\omega_{-1,d+4}$. Ze všech těchto čísel je největší $\omega_{d,3}$ (jak lze snadno ukázat).

Pro R obdržíme tedy odhad:

$$|R| < \sum_{k=k_0}^{\infty} \frac{1}{|k^{s+d+2}|} \Bigg(|b_{d+2}| \frac{1}{1-\sigma} + \frac{1}{1-\omega_{d,3}} \Bigg\{ \frac{1}{2^2} \Bigg[\frac{|A_d|}{k_0} \left|\binom{-|\alpha+d|-|\beta|}{3}\right| +$$

$$+ \left|\binom{-|\alpha+d-1|-|\beta|}{3}\right| . |A_{d-1}| \Bigg] +$$

$$+ \frac{1}{2^4} \Bigg[\frac{|A_{d-2}|}{k_0} \left|\binom{-|\alpha+d-2|-|\beta|}{5}\right| + |A_{d-3}| . \left|\binom{-|\alpha+d-3|-|\beta|}{5}\right| \Bigg] + \ldots$$

$$\ldots + \frac{1}{2^{2r}} \Bigg[\frac{|A_{d-2r+2}|}{k_0} . \left|\binom{-|\alpha+d-2r+2|-|\beta|}{2r+1}\right| +$$

$$+ |A_{d-2r+1}| . \left|\binom{-|\alpha+d-2r+1|-|\beta|}{2r+1}\right| \Bigg] + \ldots$$

$$\ldots + \begin{cases} \frac{1}{2^{d+1}}\left[\frac{|A_1|}{k_0}\left|\binom{-|\alpha+1|-|\beta|}{d+2}\right| + |A_0| \cdot \left|\binom{-|\alpha|-|\beta|}{d+2}\right|\right] + \\ \qquad + \frac{1}{2^{d+3}}\frac{|A_{-1}|}{k_0} \cdot \left|\binom{-|\alpha-1|-|\beta|}{d+4}\right|\Bigg]\Bigg\} \text{(pro lichá } d), \\ \frac{1}{2^{d+2}}\left[\frac{|A_0|}{k_0}\left|\binom{-|\alpha|-|\beta|}{d+3}\right| + |A_{-1}| \cdot \left|\binom{-|\alpha-1|-|\beta|}{d+3}\right|\right]\Bigg\} \text{(pro sudá } d), \end{cases} \quad (9)$$

který platí pro k_0 taková, že $\omega_{d,3} < 1$.

2.

Budiž $s = \alpha + i\beta$ ($\alpha > 0$, $\beta = 0$); $0 < x < 1$; $\varrho \geqq 0$. $\frac{1}{f(k)}$ je dáno řadou (2), konvergentní pro $k_0 > \varrho$. Máme stanovit $\sum_{k=k_0}^{\infty} \frac{e^{2k\pi xi}}{f(k)}$. Uvažujme následující rozvoj $\frac{1}{f(k)}$:

$$\frac{1}{f(k)} = A_0\left[\frac{e^{-\pi xi}}{(k-\frac{1}{2})^s} - \frac{e^{\pi xi}}{(k+\frac{1}{2})^s}\right] + A_1\left[\frac{e^{-\pi xi}}{(k-\frac{1}{2})^{s+1}} - \frac{e^{\pi xi}}{(k+\frac{1}{2})^{s+1}}\right] + \ldots$$
$$\ldots + A_\nu\left[\frac{e^{-\pi xi}}{(k-\frac{1}{2})^{s+\nu}} - \frac{e^{\pi xi}}{(k+\frac{1}{2})^{s+\nu}}\right] + \ldots$$
$$\ldots + A_d\left[\frac{e^{-\pi xi}}{(k-\frac{1}{2})^{s+d}} - \frac{e^{\pi xi}}{(k+\frac{1}{2})^{s+d}}\right] + \varphi(k). \quad (10)$$

Rozvineme-li $(k-\frac{1}{2})^{-(s+r)}$ a $(k+\frac{1}{2})^{-(s+r)}$ jako v § 1, obdržíme:

$$b_0k^{-s} + b_1k^{-s-1} + b_2k^{-s-2} + \ldots =$$
$$= -2A_0 i \sin \pi x\left[k^{-s} + \binom{-s}{2}k^{-s-2}\frac{1}{2^2} + \binom{-s}{4}\frac{1}{2^4}k^{-s-4} + \ldots\right] -$$
$$- A_0 \cos \pi x\left[-sk^{-s-1} + \binom{-s}{3}\frac{1}{2^2}k^{-s-3} + \binom{-s}{5}\frac{1}{2^4}k^{-s-5} + \ldots\right] - \ldots$$
$$\ldots - 2A_\nu i \sin \pi x\left[k^{-s-\nu} + \binom{-s-\nu}{2}\frac{1}{2^2}k^{-s-\nu-2} + \binom{-s-\nu}{4}\frac{1}{2^4}k^{-s-\nu-4} + \ldots\right] -$$
$$- A_\nu \cos \pi x\left[-(s+\nu)k^{-s-\nu-1} + \binom{-s-\nu}{3}\frac{1}{2^2}k^{-s-\nu-3} + \right.$$
$$\left. \binom{-s-\nu}{5}\frac{1}{2^4}k^{-s-\nu-5} + \ldots\right] - \ldots$$
$$\ldots - 2A_d i \sin \pi x\left[k^{-s-d} + \binom{-s-d}{2}\frac{1}{2^2}k^{-s-d-2} + \binom{-s-d}{4}\frac{1}{2^4}k^{-s-d-4} + \ldots\right] -$$
$$- A_d \cos \pi x\left[-(s+d)k^{-s-d-1} + \binom{-s-d}{3}\frac{1}{2^2}k^{-s-d-3} + \right.$$
$$\left. + \binom{-s-d}{5}\frac{1}{2^4}k^{-s-d-5} + \ldots\right] + \varphi(k).$$

Koeficienty $A_0, A_1, \ldots, A_d$ vypočteme zcela obdobně jako v § 1 při výpočtu $\Sigma \frac{1}{f(k)}$. Pro $\varphi(k)$ obdržíme konvergentní rozvoj tvaru

$$\varphi(k) = \alpha_{d+1} k^{-s-d-1} + \alpha_{d+2} k^{-s-d-2} + \ldots$$

Obdržíme:

$$A_0 = \frac{b_0 \mathrm{i}}{2 \sin \pi x}, \quad A_1 = \frac{b_0 s \cos \pi x + 2 \mathrm{i} b_1 \sin \pi x}{4 \sin^2 \pi x},$$

$$A_2 = \frac{b_1(s+1) \sin 2\pi x - \mathrm{i}\left[b_0 \binom{s+1}{2}(1 + \cos^2 \pi x) - 4 b_2 \sin^2 \pi x\right]}{8 \sin^3 \pi x}, \text{ atd.}$$

Pro $r = 0, 1, 2, \ldots$ je

$$\alpha_{d+2r+1} = b_{d+2r+1} - A_d \cos \pi x \cdot \frac{1}{2^{2r}} \binom{-s-d}{2r+1} -$$

$$- 2A_{d-1} \mathrm{i} \sin \pi x \cdot \frac{1}{2^{2r+2}} \binom{-s-d+1}{2r+2} - A_{d-2} \cos \pi x \cdot \frac{1}{2^{2r+2}} \binom{-s-d+2}{2r+3} -$$

$$- 2A_{d-3} \mathrm{i} \sin \pi x \cdot \frac{1}{2^{2r+4}} \binom{-s-d+3}{2r+4} - \ldots$$

$$\ldots - \begin{cases} 2A_0 \mathrm{i} \sin \pi x \cdot \dfrac{1}{2^{d+2r+1}} \dbinom{-s}{d+2r+1} \text{ (pro lichá } d\text{)}; \\ A_0 \cos \pi x \cdot \dfrac{1}{2^{d+2r}} \dbinom{-s}{d+2r+1} \text{ (pro sudá } d\text{)}; \end{cases} \tag{11}$$

$$\alpha_{d+2r+2} = b_{d+2r+2} - 2A_d \mathrm{i} \sin \pi x \cdot \frac{1}{2^{2r+2}} \binom{-s-d}{2r+2} -$$

$$- A_{d-1} \cos \pi x \cdot \frac{1}{2^{2r+2}} \binom{-s-d+1}{2r+3} - 2A_{d-2} \mathrm{i} \sin \pi x \cdot \frac{1}{2^{2r+4}} \binom{-s-d+2}{2r+4} -$$

$$- A_{d-3} \cos \pi x \cdot \frac{1}{2^{2r+4}} \binom{-s-d+3}{2r+5} - \ldots$$

$$\ldots - \begin{cases} A_0 \cos \pi x \cdot \dfrac{1}{2^{d+2r+1}} \dbinom{-s}{d+2r+2} \text{ (pro lichá } d\text{)}. \\ 2A_0 \mathrm{i} \sin \pi x \cdot \dfrac{1}{2^{d+2r+2}} \dbinom{-s}{d+2r+2} \text{ (pro sudá } d\text{)}. \end{cases} \tag{11a}$$

Z rovnice (10) plyne

$$\sum_{k=k_0}^{\infty} \frac{\mathrm{e}^{2k\pi x \mathrm{i}}}{f(k)} = \mathrm{e}^{(2k_0-1)\pi x \mathrm{i}} \left[\frac{A_0}{(k_0 - \frac{1}{2})^s} + \frac{A_1}{(k_0 - \frac{1}{2})^{s+1}} + \ldots + \frac{A_d}{(k_0 - \frac{1}{2})^{s+d}}\right] + R;$$

$$R = \sum_{k=k_0}^{\infty} \left(\frac{\alpha_{d+1}}{k^{s+d+1}} + \frac{\alpha_{d+2}}{k^{s+d+2}} + \ldots\right). \tag{12}$$

Pro $\alpha + d > 1$ obdržíme obdobně jako v § 1 následující odhad zbytku R:

$$|R| < \sum_{k=k_0}^{\infty} \frac{1}{|k^{s+d+1}|} \left(\frac{|b_{d+1}|}{1-\sigma} + \frac{1}{1-\omega_{d,1}} \left\{ |A_d| \left[|s+d| + \frac{1}{2k_0} \left| \binom{-s-d}{2} \right| \right] + \right.\right.$$

$$+ \frac{1}{2} |A_{d-1}| \left[\left| \binom{-s-d+1}{2} \right| + \frac{1}{2k_0} \left| \binom{-s-d+1}{3} \right| \right] + \dots$$

$$+ \frac{1}{2^{d-\nu}} |A_\nu| \left[\left| \binom{-s-\nu}{d-\nu+1} \right| + \frac{1}{2k_0} \left| \binom{-s-\nu}{d-\nu+2} \right| \right] + \dots$$

$$\left.\left. \dots + \frac{1}{2^d} |A_0| \left[\left| \binom{-s}{d+1} \right| + \frac{1}{2k_0} \left| \binom{-s}{d+2} \right| \right] \right\} \right) . \tag{13}$$

Oddělíme-li ve (12) část reálnou a imaginární, obdržíme vzorce pro součty řad

$$\sum_{k=k_0}^{\infty} \frac{\cos 2k\pi x}{f(k)} \quad \text{a} \quad \sum_{k=k_0}^{\infty} \frac{\sin 2k\pi x}{f(k)} .$$

3. Příklady.

1. *Výpočet funkce* $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$, *kde* $s = \alpha + i\beta$ ($\alpha > 1$, β reálné).

Podle (3) obdržíme

$$k^{-s} = A_{-1} [(k - \tfrac{1}{2})^{-s+1} - (k + \tfrac{1}{2})^{-s+1}] + A_0[(k - \tfrac{1}{2})^{-s} - (k + \tfrac{1}{2})^{-s}] + \dots$$

$$\dots + A_\nu[(k - \tfrac{1}{2})^{-s-\nu} - (k + \tfrac{1}{2})^{-s-\nu}] + \dots$$

$$\dots + A_d[(k - \tfrac{1}{2})^{-s-d} - (k + \tfrac{1}{2})^{-s-d}] + \varphi(k) ;$$

$$\varphi(k) = \alpha_{d+2} k^{-s-d-2} + \alpha_{d+3} k^{-s-d-3} + \alpha_{d+4} k^{-s-d-4} + \dots .$$

Srovnáním koeficientů dostaneme pro lichá d:

$$A_{2r} = 0 \left(r = 0, 1, \dots, \frac{d-1}{2} \right); \quad \alpha_{d+2r} = 0 \ (r = 1, 2, \dots) ;$$

$$A_{-1} = \frac{1}{s-1} ; \quad A_1 = -\frac{s}{24} ; \quad A_3 = \frac{7}{3 \,.\, 5 \,.\, 2^6} \binom{s+2}{3} = \binom{s+2}{3} 7{,}29 \,.\, 10^{-3} ;$$

$$A_5 = -\frac{31}{2^7 \,.\, 3^2 \,.\, 7} \binom{s+4}{5} = -\binom{s+4}{5} 3{,}85 \,.\, 10^{-3} ;$$

$$A_7 = \binom{s+6}{7} \frac{127}{3 \,.\, 5 \,.\, 2^{11}} = \binom{s+6}{7} 4{,}14 \,.\, 10^{-3} ;$$

$$A_9 = -\binom{s+8}{9} \frac{7 \,.\, 73}{3 \,.\, 11 \,.\, 2^{11}} = -\binom{s+8}{9} 7{,}55 \,.\, 10^{-3} ;$$

$$A_{11} = \frac{1\,414\,477}{5 \,.\, 7 \,.\, 13 \,.\, 3^3 \,.\, 2^{14}} \binom{s+10}{11} = \binom{s+10}{11} 2{,}10 \,.\, 10^{-2} ;$$

$$A_{13} = -\binom{s+12}{13} \frac{8191}{3 \,.\, 2^{15}} = -\binom{s+12}{13} 8{,}33 \,.\, 10^{-2} ;$$

$$A_{15} = \binom{s+14}{15} \frac{118\,518\,239}{3 \,.\, 5 \,.\, 17 \,.\, 2^{20}} = \binom{s+14}{15} 4{,}43 \,.\, 10^{-1};$$

$$A_{17} = -\binom{s+16}{17} \frac{5\,749\,691\,557}{7 \,.\, 19 \,.\, 3^3 \,.\, 2^{19}} = -\binom{s+16}{17} 3{,}054; \text{ atd.}$$

Z (5) plyne pro výpočet $\zeta(2)$:

$$\sum_{k=20}^{\infty} \frac{1}{k^2} = \frac{1}{19{,}5} - \frac{1}{12 \,.\, 19{,}5^3} + \frac{7}{3 \,.\, 5 \,.\, 2^4 \,.\, 19{,}5^5} - \frac{31}{3 \,.\, 7 \,.\, 2^6 \,.\, 19{,}5^7} +$$

$$+ \frac{127}{3 \,.\, 5 \,.\, 2^8 \,.\, 19{,}5^9} - \frac{5 \,.\, 7 \,.\, 73}{3 \,.\, 11 \,.\, 2^{10} \,.\, 19{,}5^{11}} + \frac{1\,414\,477}{3 \,.\, 5 \,.\, 7 \,.\, 13 \,.\, 2^{12} \,.\, 19{,}5^{13}} -$$

$$- \frac{7 \,.\, 8191}{3 \,.\, 2^{14} \,.\, 19{,}5^{15}} + \frac{118\,518\,239}{3 \,.\, 5 \,.\, 17 \,.\, 2^{16} \,.\, 19{,}5^{17}} - \frac{5\,749\,691\,557}{3 \,.\, 7 \,.\, 19 \,.\, 2^{18} \,.\, 19{,}5^{19}} + R \,.$$

Podle (9) je

$$|R| < \sum_{k=20}^{\infty} \frac{1}{k^{22}} \Bigg[|A_{17}| \,.\, \left|\binom{-19}{3}\right| \frac{1}{2^2} + |A_{15}| \,.\, \left|\binom{-17}{5}\right| \frac{1}{2^4} + |A_{13}| \,.\, \left|\binom{-15}{7}\right| \frac{1}{2^6} +$$

$$+ |A_{11}| \,.\, \left|\binom{-13}{9}\right| \frac{1}{2^8} + |A_9| \,.\, \left|\binom{-11}{11}\right| \frac{1}{2^{10}} + |A_7| \left|\binom{-9}{13}\right| \frac{1}{2^{12}} + |A_5| \,.\, \left|\binom{-7}{15}\right| \frac{1}{2^{14}} +$$

$$+ |A_3| \left|\binom{-5}{17}\right| \frac{1}{2^{16}} + |A_1| \left|\binom{-3}{19}\right| \frac{1}{2^{18}} + |A_{-1}| \left|\binom{-1}{21}\right| \frac{1}{2^{20}} \Bigg] 1{,}02 <$$

$$< 2{,}6 \,.\, 10^4 \sum_{k=20}^{\infty} \frac{1}{k^{22}} < 2{,}6 \,.\, 10^4 \,.\, 3{,}8 \,.\, 10^{-29} < 1 \,.\, 10^{-24} \,.$$

$$\sum_{k=1}^{19} \frac{1}{k^2} = 1{,}593\,663\,243\,913\,023\,316\,640\,878\,87\ldots;$$

$$\frac{1}{19{,}5} - \frac{1}{12} \cdot \frac{1}{19{,}5^3} + \frac{7}{3 \,.\, 5 \,.\, 2^4 \,.\, 19{,}5^5} - \frac{31}{3 \,.\, 7 \,.\, 2^6 \,.\, 19{,}5^7} + \frac{127}{3 \,.\, 5 \,.\, 2^8 \,.\, 19{,}5^9} -$$

$$- \frac{5 \,.\, 7 \,.\, 73}{3 \,.\, 11 \,.\, 2^{10} \,.\, 19{,}5^{11}} + \frac{1\,414\,477}{3 \,.\, 5 \,.\, 7 \,.\, 13 \,.\, 2^{12} \,.\, 19{,}5^{13}} =$$

$$= 0{,}051\,270\,822\,935\,203\,119\,882\,765\,21;$$

$$- \frac{7 \,.\, 8191}{3 \,.\, 2^{14} \,.\, 19{,}5^{15}} + \frac{118\,518\,239}{3 \,.\, 5 \,.\, 17 \,.\, 2^{16} \,.\, 19{,}5^{17}} - \frac{5\,749\,691\,557}{3 \,.\, 7 \,.\, 19 \,.\, 2^{18} \,.\, 19{,}5^{19}} =$$

$$= -5{,}122\,934 \,.\, 10^{-20} \,.$$

Tedy $\zeta(2) = 1{,}644\,934\,066\,848\,226\,436\,472\,414\,74\ldots + R$; $|R| < 1 \,.\, 10^{-24}$. Správná hodnota $\zeta(2) = \frac{1}{6}\pi^2 = 1{,}644\,934\,066\,848\,226\,436\,472\,415\,17\ldots$; tedy $R \doteq 4{,}3 \,.\, 10^{-25}$.

2. *Výpočet* $\sum_{k=10}^{n} \sqrt{4k^4 - k^2 + 1}$.

Protože platí $\alpha + d + 2 > 1$, lze použít rovnice (4) a také odhadu z § **1.** Z rov. (3) dostaneme

$$\sqrt{4k^4 - k^2 + 1} = 2k^2 - \tfrac{1}{4} + \tfrac{15}{64}k^{-2} + \tfrac{15}{512}k^{-4} - 1{,}0071 \,.\, 10^{-2}k^{-6} -$$
$$- 4{,}69 \,.\, 10^{-3}k^{-8} + 3{,}79 \,.\, 10^{-4}k^{-10} + 7{,}45 \,.\, 10^{-4}k^{-12} + 9{,}2 \,.\, 10^{-5}k^{-14} + \ldots =$$
$$= A_{-1}[(k - \tfrac{1}{2})^3 - (k + \tfrac{1}{2})^3] + A_1[(k - \tfrac{1}{2}) - (k + \tfrac{1}{2})] +$$
$$+ A_3\left[\frac{1}{k - \frac{1}{2}} - \frac{1}{k + \frac{1}{2}}\right] + A_5\left[\frac{1}{(k - \frac{1}{2})^3} - \frac{1}{(k + \frac{1}{2})^3}\right] + \varphi(k)\,.$$

V tomto případě pišme $\varphi(k)$ ve tvaru

$$\varphi(k) = \frac{M_8 + M_{10}k^{-2} + M_{12}k^{-4} + \ldots}{(k^2 - \frac{1}{4})^3}\,.$$

$$A_0 = A_2 = \ldots = A_{2r} = 0 \;\; (r = 0, 1, 2, \ldots);\;\; M_7 = M_9 = \ldots$$
$$\ldots = M_{2r+1} = 0\,;\; (r = 3, 4, \ldots)\,.$$

Vynásobíme-li hořejší rovnici $(k^2 - \frac{1}{4})^3$, dostaneme

$$2k^8 - \tfrac{7}{4}k^6 + \tfrac{51}{64}k^4 - \tfrac{115}{512}k^2 + 1{,}5808 \,.\, 10^{-2} + 4{,}69 \,.\, 10^{-3}k^{-2} + 1{,}55 \,.\, 10^{-3}k^{-4} -$$
$$- 2{,}61 \,.\, 10^{-4}k^{-6} - 3{,}23 \,.\, 10^{-4}k^{-8} + \ldots = - [A_{-1}(3k^2 + \tfrac{1}{4}) + A_1](k^2 - \tfrac{1}{4})^3 +$$
$$+ A_3(k^2 - \tfrac{1}{4})^2 + A_5(3k^2 + \tfrac{1}{4}) + M_8 + M_{10}k^{-2} + M_{12}k^{-4} + \ldots\,.$$

Srovnáním koeficientů vypočteme:

$$A_{-1} = - \tfrac{2}{3},\; A_1 = \tfrac{5}{12},\; A_3 = \tfrac{15}{64},\; A_5 = - \tfrac{5}{512},\; M_8 = - 3{,}05 \,.\, 10^{-4}\,,$$
$$M_{10} = 4{,}69 \,.\, 10^{-3},\; M_{12} = 1{,}55 \,.\, 10^{-3},\; M_{14} = -2{,}61 \,.\, 10^{-4},\; M_{16} = -3{,}23 \,.\, 10^{-4}, \ldots$$

Tedy podle (4)

$$\sum_{k=10}^{n} \sqrt{4k^4 - k^2 + 1} = - \tfrac{2}{3}[9{,}5^3 - (n + \tfrac{1}{2})^3] + \tfrac{5}{12}[9{,}5 - (n + \tfrac{1}{2})] +$$
$$+ \frac{15}{64}\left[\frac{1}{9{,}5} - \frac{1}{n + \frac{1}{2}}\right] - \frac{5}{512}\left[\frac{1}{9{,}5^3} - \frac{1}{(n + \frac{1}{2})^3}\right] + R$$

$$|R| < \sum_{k=10}^{n} \frac{|- 3{,}05 \,.\, 10^{-4} + 4{,}69 \,.\, 10^{-3}k^{-2} + \ldots|}{(k^2 - \frac{1}{4})^3} < 4 \,.\, 10^{-4} \sum_{k=10}^{n} \frac{1}{(k^2 - \frac{1}{4})^3} <$$
$$< 4 \,.\, 10^{-4} \,.\, 1{,}01 \sum_{k=10}^{\infty} \frac{1}{k^6} < 4 \,.\, 10^{-4} \,.\, 1{,}01 \,.\, 2{,}6 \,.\, 10^{-6} < 1{,}1 \,.\, 10^{-9}\,.\text{*)}$$

3. *Výpočet* $\displaystyle\sum_{k=20}^{\infty} \frac{\cos\frac{k\pi}{3}}{\sqrt{4k^4 - k^2 + 1}}$ a $\displaystyle\sum_{k=20}^{\infty} \frac{\sin\frac{k\pi}{3}}{\sqrt{4k^4 - k^2 + 1}}$.

Nejprve vypočteme $\displaystyle\sum_{k=20}^{\infty} \frac{e^{\frac{k\pi i}{3}}}{\sqrt{4k^4 - k^2 + 1}}$. Z rovnice (10) plyne:

$$\frac{1}{2}k^{-2} + \frac{1}{16}k^{-4} - \frac{13}{256}k^{-6} - 2{,}10 \,.\, 10^{-2}k^{-8} + 4{,}93 \,.\, 10^{-3}k^{-10} + 5{,}31 \,.\, 10^{-3}k^{-12} +$$
$$+ 7{,}1 \,.\, 10^{-4}k^{-14} - 1{,}03 \,.\, 10^{-3}k^{-16} - 3{,}5 \,.\, 10^{-4}k^{-18} + \ldots =$$

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*) Z vlastností koeficientů rozvoje $(k^2 - \frac{1}{4})^3\sqrt{4k^4 - k^2 + 1}$ plyne totiž pro $k \geqq 10$, že $|- 3{,}05 \,.\, 10^{-4} + 4{,}69 \,.\, 10^{-3}k^{-2} + 1{,}55 \,.\, 10^{-3}k^{-4} \ldots| < 3{,}05 \,.\, 10^{-4} - 4{,}69 \,.\, 10^{-3}k^{-2} + (10^{-4} + 10^{-6} + 10^{-8} + \ldots) < 4 \,.\, 10^{-4}$.

$$= -A_0\mathrm{i}\left[k^{-2} + \binom{-2}{2}\frac{1}{2^2}k^{-4} + \binom{-2}{4}\frac{1}{2^4}k^{-6} + \dots\right] -$$

$$-A_0\frac{\sqrt{3}}{2}\left[-2k^{-3} + \binom{-2}{3}\frac{1}{2^2}k^{-5} + \binom{-2}{5}\frac{1}{2^4}k^{-7} + \dots\right]\dots$$

$$\dots - A_\nu\mathrm{i}\left[k^{-2-\nu} + \binom{-2-\nu}{2}\frac{1}{2^2}k^{-4-\nu} + \binom{-2-\nu}{4}\frac{1}{2^4}k^{-6-\nu} + \dots\right] -$$

$$-A_\nu\frac{\sqrt{3}}{2}\left[-(2+\nu)k^{-3-\nu} + \binom{-2-\nu}{3}\frac{1}{2^2}k^{-5-\nu} + \right.$$

$$\left. + \binom{-2-\nu}{5}\frac{1}{2^4}k^{-7-\nu} + \dots\right] - \dots + \varphi(k).$$

$$A_0 = \frac{\mathrm{i}}{2}, \quad A_1 = \frac{\sqrt{3}}{2}, \quad A_2 = -\frac{41\mathrm{i}}{16}, \quad A_3 = -\frac{45}{8}\sqrt{3}.$$

Z rovnice (12) vypočteme:

$$\sum_{k=20}^{\infty}\frac{\mathrm{e}^{\frac{k\pi\mathrm{i}}{3}}}{\sqrt{4k^4 - k^2 + 1}} =$$

$$= \mathrm{e}^{\frac{13}{2}\pi\mathrm{i}}\left[\frac{\mathrm{i}}{2}\left(\frac{2}{39}\right)^2 + \frac{\sqrt{3}}{2}\left(\frac{2}{39}\right)^3 - \frac{41\mathrm{i}}{16}\left(\frac{2}{39}\right)^4 - \frac{45\sqrt{3}}{8}\left(\frac{2}{39}\right)^5\right] + R. \qquad \text{(a)}$$

Ze vzorce (13) odhadneme zbytek R:

$$|R| < \sum_{k=20}^{\infty}\frac{1}{k^6}\,1{,}01\left[\frac{13}{256} + \frac{45\,.\,\sqrt{3}\,.\,5\,.\,1{,}08}{8} + \frac{1}{2}\,\frac{41}{16}\,.\,10\,.\,1{,}05 + \right.$$

$$\left. + \frac{\sqrt{3}}{2}\,\frac{5}{2}\,1{,}04 + \frac{5}{2}\,.\,\frac{1{,}03}{2^3}\right] < 69{,}4\sum_{k=20}^{\infty}\frac{1}{k^6} < 69{,}4\,.\,7{,}46\,.\,10^{-8} < 5{,}2\,.\,10^{-6}.$$

Oddělíme-li v (a) část reálnou a imaginární, obdržíme:

$$\sum_{k=20}^{\infty}\frac{\cos\frac{k\pi}{3}}{\sqrt{4k^4 - k^2 + 1}} = -\frac{1}{2}\left(\frac{2}{39}\right)^2 + \frac{41}{16}\left(\frac{2}{39}\right)^4 + R_1 =$$

$$= -0{,}001\,30 + R_1, \; |R_1| < |R| < 5{,}2\,.\,10^{-6};$$

$$\sum_{k=20}^{\infty}\frac{\sin\frac{k\pi}{3}}{\sqrt{4k^4 - k^2 + 1}} = \frac{\sqrt{3}}{2}\left(\frac{2}{39}\right)^3 - \frac{45\sqrt{3}}{8}\left(\frac{2}{39}\right)^5 + R_2 =$$

$$= 0{,}000\,11 + R_2, \; |R_2| < |R| < 5{,}2\,.\,10^{-6}.$$