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CONTRACTION DISTANCE BETWEEN ISOMORPHISM CLASSES OF GRAPHS

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Summary. A distance between isomorphism classes of connected graphs with a given number n of vertices is introduced. Its definition is based on the maximum number of vertices of a graph onto which two given graphs can be contracted. The properties of this distance are studied; they are compared with the properties of other distances between isomorphism classes of graphs.

Keywords: contraction of a graph, distance, Hadwiger number, tree.

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An isomorphism class of graphs is the class of all graphs which are isomorphic to a given graph. Various distances between isomorphism classes have been introduced.

The subgraph distance δ was introduced in [6]. The distance $\delta(\mathfrak{G}_1, \mathfrak{G}_2)$ of two isomorphism classes $\mathfrak{G}_1$, $\mathfrak{G}_2$ of graphs with a given number n of vertices is equal to n minus the maximum number of vertices of a graph which is isomorphic simultaneously to an induced subgraph of a graph from $\mathfrak{G}_1$ and to an induced subgraph of a graph from $\mathfrak{G}_2$. This distance was modified by F. Kaden [3] and F. Sobik [5] for pairs of isomorphism classes of graphs which need not have the same number of vertices. For trees this distance was modified in [7]; instead of an induced subgraph a subtree is taken. This distance is denoted by δ_T. The edge distance was introduced by V. Baláž, J. Koča, V. Kvasnička and M. Sekanina [1]. Let $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$, let V_1, V_2 be the vertex sets and E_1, E_2 the edge sets of G_1, G_2, respectively. Let E_{12} be the edge set of a graph which is isomorphic to subgraphs of both G_1, G_2 and has the maximum number of edges. Then the edge distance is $\delta_E(\mathfrak{G}_1, \mathfrak{G}_2) = |E_1| + |E_2| - 2|E_{12}| + ||V_1| - |V_2||$. The edge rotation distance was introduced by G. Chartrand, F. Saba and H.-B. Zou [2]. If u, v, w are three vertices of a graph G such that u is adjacent to v and not to w, then to perform the rotation of the edge uv to the position uw means to delete the edge uv from G and to add the edge uw to G. The edge rotation distance $\delta_R(\mathfrak{G}_1, \mathfrak{G}_2)$ is the minimum number of edge rotations necessary for transforming a graph from $\mathfrak{G}_1$ into a graph from $\mathfrak{G}_2$.

Various distances were compared in [8]. There are also other distances between isomorphism classes of graphs.

Here we shall introduce a new distance between isomorphism classes of graphs called the contraction distance. All graphs considered are finite connected undirected graphs without loops and multiple edges.

Let G be a connected graph, let $\mathscr{P}$ be a partition of the vertex set $V(G)$ of G with the property that each class of $\mathscr{P}$ induces a connected subgraph of G. Let G' be the graph whose vertex set is $\mathscr{P}$ and in which two vertices P_1, P_2 are adjacent if and only if at least one vertex of P_1 is adjacent to at least one vertex of P_2 in G. Then we say that G' (and every graph isomorphic to G') is a contract of G and was obtained from G by a contraction.

If one of the classes of $\mathscr{P}$ has two elements and all the others have one element each, we say that G' was obtained from G by an elementary contraction. It is easy to see that any contraction is a superposition of elementary contractions. The number of vertices of G is equal to the sum of the number of vertices of G' and the number of elementary contractions by whose superposition G' was obtained from G.

Let $\mathscr{C}_n$ be the set of all isomorphism classes of connected graphs with n vertices. Let $\mathfrak{G}_1 \in \mathscr{C}_n$, $\mathfrak{G}_2 \in \mathscr{C}_n$. Let $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$ and let k be the maximum number of vertices of a common contract G_{12} of G_1 and G_2. Then $\delta_C(\mathfrak{G}_1, \mathfrak{G}_2) = n - k$ is called the contraction distance between $\mathfrak{G}_1$ and $\mathfrak{G}_2$.

In other words, $\delta_C(\mathfrak{G}_1, \mathfrak{G}_2)$ is the minimum number of elementary contractions whose superposition is a contraction of G_1 or of G_2 onto G_{12}.

Now we shall define an auxiliary concept. Let G' be a contract of G, let e be an edge of G and let e' be an edge of G'. We say that e and e' correspond to each other, if the end vertices of e belong to the classes of $\mathscr{P}$ which are end vertices of e'.

Theorem 1. *The functional δ_C is a metric on $\mathscr{C}_n$.*

Proof. Every two connected graphs with n vertices have a common contract, namely the graph consisting of one vertex; if $n \geqq 2$, then also the complete graph with two vertices. Therefore $\delta_C(\mathfrak{G}_1, \mathfrak{G}_2)$ is well-defined. It is easy to prove that $\delta_C(\mathfrak{G}_1, \mathfrak{G}_2) \geqq 0$, that $\delta_C(\mathfrak{G}_1, \mathfrak{G}_2) = 0$ if and only if $\mathfrak{G}_1 = \mathfrak{G}_2$ and that $\delta_C(\mathfrak{G}_1, \mathfrak{G}_2) = \delta_C(\mathfrak{G}_2, \mathfrak{G}_1)$. We shall prove the triangle inequality. Let $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3$ be three isomorphism classes of connected graphs with n vertices, let $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$, $G_3 \in \mathfrak{G}_3$. Let G_{12} (or G_{23}) be a common contract of G_1 (or G_3 respectively) and G_2 with the maximum number of vertices. The graph G_{12} is obtained from G_1 by $\delta_C(\mathfrak{G}_1, \mathfrak{G}_2)$ elementary contractions, the graph G_{23} is obtained from G_2 by $\delta_C(\mathfrak{G}_2, \mathfrak{G}_3)$ elementary contractions. Let F be the set of edges of G_2 whose contractions yield G_{23}, let F_0 be the set of edges of G_{12} corresponding to the edges of F. There may be an edge of F to which no edge of F_0 corresponds, because this edge is contracted when forming G_{12}; on the other hand, obviously there may be more than one edge of F which correspond to the same edge of F_0. Therefore $|E_0| \leqq |F| = \delta_C(\mathfrak{G}_2, \mathfrak{G}_3)$. Let G_{123} be the graph obtained from G_{12} by the superposition of elementary contractions determined by edges of F_0. As G_{12} is a common contract of G_1 and G_2, so is G_{123}. As the superposition of contractions is evidently commutative, the graph G_{123} can be obtained from G_2 also by contracting the edges of F to obtain G_{23} and the edges corresponding in G_{23} to the edges which were contracted in G_2 to obtain

G_{12}. Thus G_{123} is also a contract of G_{23} and hence of G_3. It is a common contract of G_1 and G_3 and has $n - \delta_C(\mathfrak{G}_1, \mathfrak{G}_2) - |F_0|$ vertices. Thus $\delta_C(\mathfrak{G}_1, \mathfrak{G}_3) \leqq \delta_C(\mathfrak{G}_1, \mathfrak{G}_2) + |F_0| \leqq \delta_C(\mathfrak{G}_1, \mathfrak{G}_2) + \delta_C(\mathfrak{G}_2, \mathfrak{G}_3)$, which was to be proved. □

In the sequel, instead of speaking about the distance between isomorphism classes of graphs we will speak about the distance between graphs (for the sake of brevity). We will also use the corresponding notation $\delta_C(G_1, G_2)$.

The concept of contraction distance is related to the Hadwiger number (in another terminology, contraction number) which is dealt with e.g. in [4], [9]. The Hadwiger number $\eta(G)$ is the maximum number of vertices of a complete graph which is a contract of G.

Theorem 2. *Let G_1, G_2 be two connected graphs with n vertices. Then $\delta_C(G_1, G_2) \leqq n - \min(\eta(G_1), \eta(G_2))$.*

Proof. Evidently the complete graph with $\min(\eta(G_1), \eta(G_2))$ vertices is a common contract of G_1 and G_2. This implies the assertion. □

Now we shall compare $\delta_C(G_1, G_2)$ with $\delta(G_1, G_2)$.

Theorem 3. *For each $n \geqq 6$ there exist graphs G_1, G_2 with n vertices such that $\delta_C(G_1, G_2) = [(n-1)/2] - 1$, while $\delta(G_1, G_2) = 1$.*

Proof. Consider a circuit C with $n - 1$ vertices $u_1, \ldots, u_{n-1}$ and edges $u_i u_{i+}$ for $i = 1, \ldots, n - 1$, the sum $i + 1$ being taken modulo $n - 1$. The graph G_1 (or G_2) is obtained from C by adding a new vertex v and joining it by edges with the vertices u_1 and u_2 (or u_{k+1}, respectively, where $k = \lfloor (n-1)/2 \rfloor$). The circuit C is a common induced subgraph of G_1 and G_2 with $n - 1$ vertices and thus $\delta(G_1, G_2) = n - (n - 1) = 1$. Each of the graphs G_1, G_2 has exactly two vertices of degree 3 and consists of three paths connecting these two vertices. In G_1 the lengths of these paths are 1, 2 and $n - 2$, in G_2 they are 2, $\lfloor (n-1)/2 \rfloor$, $\lceil (n-1)/2 \rceil$. Hence in no contract of G_2 there may exist a path longer than $\lceil (n-1)/2 \rceil$ and connecting two vertices of degree 3. Therefore a common contract of G_1 and G_2 with the maximum number of vertices has two vertices of degree 3 and consists of three paths connecting them; their lengths are 1, 2 and $\lceil (n-1)/2 \rceil$. Such a graph has $\lceil (n-1)/2 \rceil + 2$ vertices, hence $\delta_C(G_1, G_2) = n - (\lceil (n-1)/2 \rceil + 2) = \lfloor (n-1)/2 \rfloor - 1$. □

Theorem 4. *For each positive integer p there exist graphs G_1, G_2 such that $\delta_C(G_1, G_2) = p$, $\delta(G_1, G_2) = 2p$.*

Proof. The vertex set of G_1 is the union of disjoint sets $U = \{u_1, \ldots, u_p\}$, $V = \{v_1, \ldots, v_p\}$, $X = \{x_1, \ldots, x_q\}$, $Y = \{y_1, \ldots, y_q\}$, where $q \geqq p$. Any two vertices of U and any two vertices of V are joined by an edge. Each vertex of U with each vertex of X and each vertex of V with each vertex of Y is joined by an edge. Further edges $u_i v_i$ for $i = 1, \ldots, p$ are in G_1. No other edges are in G_1. The vertex set of G_2 is the union of disjoint sets $U = \{u_1, \ldots, u_p\}$, $W = \{w_1, \ldots, w_{p+2q}\}$. Any two vertices of U are joined by an edge and each vertex of U with each vertex of W is also joined by an edge. No other edges are in G_2. If we contract all edges $u_i v_i$ for $i = 1, \ldots, p$ in G_1 and all edges $u_1 w_{2q+i}$ for $i = 1, \ldots, p$ in G_2, we obtain isomorphic graphs. This is a minimum number of edges by whose contraction this may be done, because G_1 contains two disjoint cliques with p vertices, while G_2 only one. Therefore $\delta_i(G_1, G_2) = p$. The largest graph isomorphic to isomorphic subgraphs of both G_1 and G_2 is evidently the graph consisting of $2q$ isolated vertices. As the number of vertices of G_1 as well as of G_2 is $2p + 2q$, we have $\delta(G_1, G_2) = 2p$. □

Now we turn to trees. Note that trees can be characterized as connected graphs whose Hadwiger number is equal to 2.

We shall compare δ_C with δ_T defined above.

Theorem 5. *Let T_1, T_2 be two trees with n vertices. Then*

$$\delta_C(T_1, T_2) \leqq \delta_T(T_1, T_2) .$$

Proof. Let T be a tree, let T_0 be its subtree. By $V(T)$ (or $V(T_0)$) we denote the vertex set of T (or T_0, respectively). For each vertex $x \in V(T_0)$ let $P(x)$ be the set consisting of x and of all vertices $y \in V(T) - V(T_0)$ with the property that the path joining x and y in T does not contain an inner vertex belonging to $V(T_0)$. The sets $P(x)$ for all $x \in V(T_0)$ form a partition $\mathscr{P}$ of $V(T)$ and each of them induces a connected subgraph of T. The contract of T obtained by means of $\mathscr{P}$ is then isomorphic to T_0. We have proved that each subtree of a tree is also its contract. This implies the assertion. □

Theorem 6. *The maximum of the contraction distance between trees with $n \geqq 3$ vertices is $n - 3$. The only trees having this distance are a star and a path.*

Proof. Every tree with $n \geqq 3$ vertices contains a subtree which is a path of length 2, i.e. with 3 vertices. Therefore $\delta_C(T_1, T_2) \leqq \delta_T(T_1, T_2) \leqq n - 3$ for any two trees T_1, T_2 with n vertices. Every tree which is not a star contains a path of length 3 and every tree which is not a path contains a star with 3 edges as a subtree. Thus, if one of the trees T_1, T_2 is neither a path nor a star, then $\delta_C(T_1, T_2) \leqq \delta_T(T_1, T_2) \leqq$ $\leqq n - 4$. □

Theorem 7. *Let T_1, T_2 be two trees with n vertices. Then $\delta_C(T_1, T_2) \geqq$ $\geqq |d(T_1) - d(T_2)|$, where d denotes the diameter.*

Proof. Without loss of generality let $d(T_1) \geqq d(T_2)$. In no contract of T_2 there may be a path of length greater than $d(T_2)$. Therefore in order to obtain a common contract of T_1 and T_2 we must contract at least $d(T_1) - d(T_2)$ edges, which implies the assertion. □

Theorem 8. *Let P be a path with n vertices, let T be a tree with n vertices. Then* $\delta_C(P, T) = n - 1 - d(T)$.

Proof. We have $d(P) = n - 1$ and thus $\delta_C(P, T) \geqq n - 1 - d(T)$ by Theorem 7. On the other hand, both P and T contain a subtree which is a path with $d(T) + 1$ vertices and thus $\delta_C(P, T) \leqq \delta_T(P, T) \leqq n - 1 - d(T)$, which implies the assertion. □

Theorem 9. *Let $t(T)$ denote the number of terminal vertices of T. Let T_1, T_2 be two trees with n vertices. Then* $\delta_C(T_1, T_2) \geqq |t(T_1) - t(T_2)|$.

Proof. Without loss of generality let $t(T_1) \geqq t(T_2)$. Let T_{12} be a common contract of T_1 and T_2 and let $\mathscr{P}$ be the partition of the vertex set $V(T_2)$ of T_2 by means of which the tree T_{12} is obtained. If a class of $\mathscr{P}$ contains only non-terminal vertices of T_2, then evidently this class is a non-terminal vertex of T_{12}; hence $t(T_{12}) \leqq t(T_2)$. In order to obtain T_{12} from T_1 at least $t(T_1) - t(T_2)$ terminal edges of T_1 must be contracted, which implies the assertion. □

Theorem 10. *Let S be a star with n vertices, let T be a tree with n vertices. Then* $\delta_C(S, T) = n - 1 - t(T)$.

Proof. We have $t(S) = n - 1$ and thus $\delta_C(S, T) \geqq n - 1 - t(T)$ by Theorem 9. On the other hand, a star with $t(T) + 1$ vertices is a common contract of S and T. From S it is obtained by contracting arbitrary $n - 1 - t(T)$ edges, from T by contracting all non-terminal edges. This implies the assertion. □

In [8] there are theorems analogous to Theorem 8 and Theorem 10 for the edge rotation distance. Unfortunately, Theorem 6 from [8] is false. We will correct the paper [8] by proving the following theorem.

Theorem 11. *Let P be a path with n vertices, let T be a tree with n vertices. Then* $\delta_R(P, T) = t(T) - 2$.

Proof. When the transform P into T by edge rotations, at each edge rotation the degree of exactly one vertex may decrease. As P has two vertices of degree 1 and T has $t(T)$ such vertices, we have $\delta_R(P, T) \geqq t(T) - 2$. Now denote the terminal vertices of T by $u_0, u_1, \ldots, u_{t(T)-1}$. Let P_1 be the path in T connecting u_0 with u_1. For $k \geqq 2$ let P_k be the path in T connecting u_k with the vertex v_k of the union of paths P_j for $j = 1, \ldots, k - 1$ whose distance from u_k in T is minimum. Let w_k be the vertex of P_k adjacent to v_k ($w_k = u_k$ may hold). Let R_k for $k = 1, \ldots, t(T) - 2$ be the rotation of the edge $w_{k+1}v_{k+1}$ to the position $w_{k+1}u_k$. Evidently by the edge rotations $R_1, \ldots, R_{t(T)-2}$ the tree T is transformed into P and thus $\delta_R(P, T) = t(T) - 2$.

We see that the contraction distance of a tree T from a path depends on the diameter of T and its distance from a star depends on $t(T)$, while the edge rotation distance of T from a path depends on $t(T)$ and its distance from a star depends on the maximum degree of T.

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Souhrn

KONTRAKČNÍ VZDÁLENOST MEZI TŘÍDAMI ISOMORFISMU GRAFŮ

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Zavádí se jistá vzdálenost mezi třídami isomorfismu souvislých grafů; její definice je založena na maximálním počtu uzlů grafu, na který lze dané dva grafy převést kontrakcí.

Резюме

КОНТРАКЦИОННОЕ РАССТОЯНИЕ МЕЖДУ КЛАССАМИ ИЗОМОРФИЗМА ГРАФОВ

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Вводится расстояние между классами изоморфизма связных графов; его определение основано на максимальном числе вершин графа, на который можно два заданных графа стянуть.

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