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A CHARACTERIZATION OF THE WEAK CONVERGENCE OF CONVOLUTION POWERS

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Summary. Several notions of stability for sequences of convolution powers of probability distributions on the real line are discussed. A characterization theorem for the weak convergence of such sequences using Fourier seminorms is proved.

Keywords: Convolution power, weak convergence, characteristic function, Fourier seminorm.

AMS Subject Classification: 60B10, 28A33.

0. INTRODUCTION

The aim of this paper is to study the classical problem of weak convergence of convolution powers of finite nonnegative measures in Euclidean spaces. H. Bergström gave an outstanding systematic study in this field in [2] and [3]. His new approach is based on the use of suitable seminorms on the space of finite signed measures, especially Gaussian seminorms. He found necessary and sufficient conditions in the form of Bolzano-Cauchy criteria for the weak convergence of convolution products and powers and derived a representation of infinitely divisible measures by Gaussian functionals. We try to follow the traditional way in this field, making use of Fourier transforms of measures and Fourier seminorms. Many calculations seem to be easier and more illustrative in this approach than in the „Gaussian way", and practically the same results can be obtained (however, the Gassian and Fourier seminorms are not equivalent to each other – see [5], [6]).

Our theorem is motivated by its „Gaussian version" given in [3]. Though a great part of the theorem could be proved by using general methods and results from [3], we prefer a more straightforward proof based on traditional calculations with Fourier transforms.

All results are given in the one-dimensional space $\boldsymbol{R}$. This restriction should simplify the formulation of the results. On the other hand, the generalization into more dimensions does not use any new ideas.

1. NOTATION, PRELIMINARIES

Let $\bar{\mathcal{M}}$ denote the space of all finite signed measures on $\boldsymbol{R}$ (under a measure we shall always understand a σ-smooth Borel measure). With the natural addition and multiplication by real numbers, $(\bar{\mathcal{M}}, +, \cdot)$ forms a vector space. $\mathcal{M} \subseteq \bar{\mathcal{M}}$ will denote the cone of all nonnegative measures in $\bar{\mathcal{M}}$ except the zero measure. Adding the convolution we get an algebra $(\bar{\mathcal{M}}, +, \cdot, *)$ (see [3]). Obviously, the unit element associated with $*$ is the Dirac probability measure concentrated at the zero point (denoted by e).

A seminorm on $\bar{\mathcal{M}}$ is a function p from $\bar{\mathcal{M}}$ with the following properties:

$$(1) \qquad p(\mu) \geqq 0 \quad \text{for every} \quad \mu \in \bar{\mathcal{M}}\,, \qquad p(0) = 0\,,$$

$$(2) \qquad p(\mu + \nu) \leqq p(\mu) + p(\nu)\,,$$

$$(3) \qquad p(c\mu) = |c|\, p(\mu) \quad \text{for every} \quad c \in \boldsymbol{R}\,, \quad \mu, \nu \in \bar{\mathcal{M}}\,.$$

(i.e., p is a „norm'' which need not distinguish nonzero elements). We say that $\bar{\mathcal{M}}$ is seminormed under a set of seminorms $\mathcal{P}$ if the property $p(\mu) = 0$ for all $p \in \mathcal{P}$ implies $\mu = 0$. We shall deal with the following types of seminorms:

a) let $\alpha > 0$, $\mu \in \bar{\mathcal{M}}$ and put

$$(4) \qquad Q_\alpha(\mu) = \max\left\{|\mu(\boldsymbol{R})|,\ \int_{\{|x| \geqq \alpha\}} |\mu|\,(\mathrm{d}x),\ \alpha^{-1}\left|\int_{\{|x|<\alpha\}} x\, \mu(\mathrm{d}x)\right|,\ \alpha^{-2}\int_{\{|x|<\alpha\}} x^2 |\mu|\,(\mathrm{d}x)\right\}.$$

Q_α is a seminorm on $\bar{\mathcal{M}}$ for each $\alpha > 0$ and we shall call it the majorant seminorm. Obviously $\bar{\mathcal{M}}$ is seminormed under the system of majorant seminorms $\mathcal{Q} = \{Q_\alpha : \alpha > 0\}$.

b) We define the Fourier seminorms

$$(5) \qquad F_\alpha(\mu) = \sup_{|t| \leqq \alpha^{-1}} |\tilde{\mu}(t)|\,, \quad \alpha > 0\,, \quad \mu \in \bar{\mathcal{M}}\,,$$

where $\tilde{\mu}(t) = \int_{-\infty}^{\infty} \exp(itx)\, \mu(\mathrm{d}x)$ is the Fourier transform of μ. $\bar{\mathcal{M}}$ is seminormed also under the system of Fourier seminorms $\mathcal{F} = \{F_\alpha : \alpha > 0\}$.

c) H. Bergström uses most frequently the Gaussian seminorms

$$(6) \qquad G_\alpha(\mu) = \sup_{x \in \boldsymbol{R}} \left|\int_{-\infty}^{\infty} \Phi\left(\frac{x-y}{\alpha}\right) \mu(\mathrm{d}y)\right|\,, \quad \alpha > 0\,, \quad \mu \in \bar{\mathcal{M}}\,,$$

where

$$\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-t^2/2)\, \mathrm{d}t\,.$$

($G_\alpha(\mu)$ is the supremum norm of the distribution function of the convolution of μ with the centred normal probability distribution of the variance α^2.)

Any system of seminorms $\mathscr{P}$ defines naturally a convergence on $\mathscr{M}$: $\mu_n \xrightarrow{\mathscr{P}} \mu$ if $\lim_{n\to\infty} p(\mu_n - \mu) = 0$ for all $p \in \mathscr{P}$. If $\mu_n, \mu \in \mathscr{M}$, the convergence $\mu_n \xrightarrow{\mathscr{F}} \mu$ means nothing else than the locally uniform convergence of the characteristic functions $\tilde{\mu}_n(t)$ to $\tilde{\mu}(t)$. Hence, the convergence of nonnegative measures under $\mathscr{F}$ is equivalent to the weak convergence (see e.g. [7], § 3.6; $\mu_n \to \mu$ weakly if $\int f \, d\mu_n \to \int f \, d\mu$ for every bounded continuous function f). The same is true for the convergence under Gaussian seminorms (see [3], § 4.5).

However, the seminorms F_α and G_α are not equivalent on $\mathscr{M}$ (two different countreexamples are given in [5] and [6]). The next lemma states their equivalence for measures of a special form.

Lemma 1. *The seminorms F_α, G_α and Q_α are equivalent to each other and uniformly in α on the space*

$$\mathscr{M} - e = \{\mu - e\colon \mu \in \mathscr{M}\}.$$

(Seminorms p and q are equivalent on X if there are constants $c_1, c_2 > 0$ such that

$$c_1 \, p(x) \leqq q(x) \leqq c_2 \, p(x) \quad \text{for each} \quad x \in X.)$$

The equivalence of G_α and Q_α follows from [3], § 4.4; for the equivalence of F_α and Q_α see [9] and [10].

2. STABILITY CONDITIONS FOR CONVOLUTION PRODUCTS

Let $\{\mu_{nj}, 1 \leqq j \leqq k_n, n \in N\}$ be a double sequence of measures from $\mathscr{M}$ with $k_n \uparrow \infty$. Following the classical approach we are interested in the weak convergence of the sequence of convolution products

$$\{\prod_{j=1}^{k_n} \mu_{nj}, \; n \in N\}.$$

First we shall state some technical conditions useful in the study of weak convergence.

Definition. We say that $\{\mu_{nj}\}$ (or the sequence of convolution products $\prod_{j=1}^{k_n} \mu_{nj}$) is
a) *uniformly assymptotically negligible* (*u.a.n.*) if

$$\lim_{n\to\infty} \max_{1 \leqq j \leqq k_n} Q_\alpha(\mu_{nj} - e) = 0 \quad \text{for each} \quad \alpha > 0, \tag{7}$$

b) *stable*, if

$$\limsup_{n\to\infty} \sum_{j=1}^{k_n} Q_\alpha(\mu_{nj} - e) < \infty \quad \text{for each} \quad \alpha > 0. \tag{8}$$

Comments.

1) According to Lemma 1, in (7) and (8) we can replace equivalently Q_α by F_α or G_α.

2) The notion of stability was defined in a more general sense, for arbitrary classes of seminorms, in [3].

3) Consider, as a particular case, a sequence of convolution powers $\{\mu_n^{k_n}, n \in \boldsymbol{N}\}$ (i.e. $\mu_{nj} = \mu_n$ for each j). Then the stability obviously implies the *u.a.n.* condition.

The next lemma shows a connection beteen stability and weak convergence.

Lemma 2. *Let $\{\mu_{nj}, 1 \leqq j \leqq k_n, n \in \boldsymbol{N}\}$ be a double sequence of probability measures on $\boldsymbol{R}$, $k_n \uparrow \infty$. Suppose $\prod_{j=1}^{k_n} \mu_{nj}$ converges weakly. Then there are constants $a_{nj} \in \boldsymbol{R}$ such that for all $a > 0$*

$$\limsup_{n\to\infty} \sum_{j=1}^{k_n} \int_{\{|x| \geqq a\}} \mu_{nj}(a_{nj} + \mathrm{d}x) \begin{cases} < \infty \\ \to 0 \quad \text{if} \quad a \to \infty\,, \end{cases} \tag{9}$$

$$\limsup_{n\to\infty} \sum_{j=1}^{k_n} \left| \int_{\{|x| < a\}} x\, \mu_{nj}(a_{nj} + \mathrm{d}x) \right| < \infty\,, \tag{10}$$

$$\limsup_{n\to\infty} \sum_{j=1}^{k_n} \int_{\{|x| < a\}} x^2\, \mu_{nj}(a_{nj} + \mathrm{d}x) < \infty\,. \tag{11}$$

If μ_{nj} are symmetrical, (9)–(11) hold with $a_{nj} = 0$.

For the proof see [3], § 4.6.

In general, Lemma 2 does not hold with $a_{nj} = 0$ even if the *u.a.n.* condition is fulfilled: consider the example

$$k_n = 2n\,, \quad \mu_{nj} = \delta((-1)^j\, n^{-1/2})\,, \quad 1 \leqq j \leqq k_n\,,$$

where $\delta(x)$ denotes the Dirac probability measure concentrated at $x \in \boldsymbol{R}$.

According to (4), the conditions (9)–(11) without the limit condition in (9) are equivalent to the stability of the shifted measures $\nu_{nj}(\cdot) = \mu_{nj}(a_{nj} + \cdot)$. If we now take arbitrary $\alpha > \beta > 0$ we see that

$$\limsup_{n\to\infty} \sum_{j=1}^{k_n} \alpha^{-1} \left| \int_{\{|x| < \alpha\}} x\, \nu_{nj}(\mathrm{d}x) \right| \leqq \tag{12}$$

$$\leqq \alpha^{-1} \limsup_{n\to\infty} \sum_{j=1}^{k_n} \left| \int_{\{|x| < \beta\}} x\, \nu_{nj}(\mathrm{d}x) \right| + \limsup_{n\to\infty} \sum_{j=1}^{k_n} \int_{\{|x| \geqq \beta\}} \nu_{nj}(\mathrm{d}x)\,.$$

Assuming that (9)–(11) hold, we can make the second term on the right hand side arbitrarily small taking a great $\beta > 0$, and the first term tends to zero with α tending to infinity. Thus the expression on the left hand side in (12) tends to zero when $\alpha \to \infty$. In quite a similar way we can see that

$$\limsup_{n\to\infty} \alpha^{-2} \sum_{j=1}^{k_n} \int_{\{|x| < \alpha\}} x^2\, \nu_{nj}(\mathrm{d}x) \to 0 \quad (\alpha \to \infty)\,.$$

These conditions lead us to the following definition. We shall say that $\{\mu_{nj}\}$ is *strongly stable* if it is stable and

$$\lim_{n\to\infty} \sup \sum_{j=1}^{k_n} Q_\alpha(\mu_{nj} - e) \to 0 \quad (\alpha \to \infty) \tag{13}$$

(this condition can be again expressed equivalently using F_α or G_α instead of Q_α).

Now we can reformulate Lemma 2.

Corollary 1. *If the assumptions of Lemma 2 are satisfied, there exist $a_{nj} \in \boldsymbol{R}$ such that the double sequence of shifted measures $\{\mu_{nj}(a_{nj} + \cdot)\}$ is strongly stable.*

On the other hand, we shall show later that the strong stability implies the „absolute" tightness for convolution products.

In the rest of this section we shall consider only probability measures. If μ is a probability measure, the characteristic function $\tilde{\mu}(t)$ need not have in general nice properties to make taking logarithms possible. Under suitable conditions, however, we avoid difficulties.

Lemma 3. *Let μ be a probability measure such that $\tilde{\mu}(t) \neq 0$ for all $|t| \leqq T$. Then there exists a unique continuous branch of logarithm $\log \tilde{\mu}(t)$ defined on $[-T, T]$ such that $\log \tilde{\mu}(0) = 0$.*

Lemma 3 is a consequence of a classical result of complex analysis (see [1]).

The *u.a.n.* condition in the „Fourier expression" implies that $\max_j |\tilde{\mu}_{nj}(t) - 1| \to 0$, $n \to \infty$, locally uniformly in t (abbreviation *l.u.* $[t]$). Thus, given any $T > 0$, the functions $\tilde{\mu}_{nj}(t)$ are nonzero on $[-T, T]$ for sufficiently large n and all $1 \leqq j \leqq k_n$ and we can consider their logarithms, according to Lemma 3.

Lemma 4. *Let $\{\mu_{nj}\}$ be a double sequence of probability measures and suppose $\{\mu_{nj}\}$ is stable and u.a.n.. Then*

$$\lim_{n\to\infty} \sup_{|t| \leqq T} \Big|\log \prod_{j=1}^{k_n} \tilde{\mu}_{nj}(t) - \sum_{j=1}^{k_n} (\tilde{\mu}_{nj}(t) - 1)\Big| = 0 \tag{14}$$

for each $T > 0$.

Proof. The Taylor expansion of the complex logarithm gives

$$\log \prod_{j=1}^{k_n} \tilde{\mu}_{nj}(t) = \sum_{j=1}^{k_n} \log \tilde{\mu}_{nj}(t) = \sum_{j=1}^{k_n} ((\tilde{\mu}_{nj}(t) - 1) + r(\tilde{\mu}_{nj}(t) - 1)), \tag{15}$$

where $|r(z)| = o(|z|)$, $z \to 0$.

Given any $T > 0$, the *u.a.n.* condition enables us to make $|\tilde{\mu}_{nj}(t) - 1|$ arbitrarily small for all $|t| \leqq T$ and sufficiently large n. Hence we have $|r(\tilde{\mu}_{nj}(t) - 1)| \leqq \leqq \eta |\tilde{\mu}_{nj}(t) - 1|$ for any given $\eta > 0$, all $|t| \leqq T$ and sufficiently large n. Consequently,

$$\Big|\sum_{j=1}^{k_n} r(\tilde{\mu}_{nj}(t) - 1)\Big| \leqq \eta \sum_{j=1}^{k_n} |\tilde{\mu}_{nj}(t) - 1|,$$

where the right hand side can be made arbitrarily small by the choice of $\eta > 0$, due to the stability condition. Now the result follows from (15).

Lemma 5. *Let $\{\mu_{nj}\}$ be u.a.n. and strongly stable and let $J_n \subseteq \{1, 2, \ldots, k_n\}$ be arbitrary nonempty subsets. Then the sequence $\{\prod_{j\in J_n} \mu_{nj},\ n \in \mathbf{N}\}$ is tight.*

Proof. The well-known Doob's inequality (see e.g. [4], § 8.3) implies

$$(16)\qquad \Big(\prod_{j\in J_n} \mu_{nj}\Big)\{|z| \geqq \alpha\} \leqq (1 + 2\pi)^2\, \alpha \int_0^{\alpha^{-1}} \operatorname{Re}\Big(1 - \prod_{j\in J_n} \tilde{\mu}_{nj}(t)\Big)\, dt \leqq$$
$$\leqq (1 + 2\pi)^2 \sup_{|t|\leqq 1/\alpha} \Big|1 - \prod_{j\in J_n} \tilde{\mu}_{nj}(t)\Big| \,.$$

The complex exponential is Lipschitzian on the half-plane $\{\operatorname{Re}(z) \leqq 0\}$, hence there is a constant $\gamma > 0$ such that $|\exp(z) - \exp(u)| \leqq \gamma |z - u|$ for $\operatorname{Re}(z) \leqq 0$, $\operatorname{Re}(u) \leqq 0$. Hence we have

$$\Big|1 - \prod_{j\in J_n} \tilde{\mu}_{nj}(t)\Big| = \Big|\exp(0) - \exp\Big(\log \prod_{j\in J_n} \tilde{\mu}_{nj}(t)\Big)\Big| \leqq$$
$$\leqq \gamma \Big|\log \prod_{j\in J_n} \tilde{\mu}_{nj}(t)\Big| = \gamma \Big|\sum_{j\in J_n} (\tilde{\mu}_{nj}(t) - 1)\Big| + o(1)\,,$$

$n \to \infty$, *l.u.* $[t]$ (we have used Lemma 4).

From (16) we obtain

$$\Big(\prod_{j\in J_n} \mu_{nj}\Big)\{|z| \geqq \alpha\} \leqq (1 + 2\pi)^2\, \gamma \sum_{j\in J_n} F_\alpha(\mu_{nj} - e) + o(1)\,,$$

$n \to \infty$. The strong stability now guarantees the assertion of the lemma.

Finally we show that the unpleasant constants a_{nj} from Lemma 2 can be avoided in the case of convolution powers.

Lemma 6. *Let μ_n be probability measures and $k_n \uparrow \infty$ natural numbers. Suppose $\mu_n^{k_n}$ converge weakly. Then $\{\mu_n^{k_n}\}$ is strongly stable.*

Proof. By Lemma 2 there are constants $a_n \in \mathbf{R}$ such that $\{\nu_n^{k_n}\}$ is strongly stable, where $\nu_n(\cdot) = \mu_n(a_n + \cdot)$. We shall show that

$$(17)\qquad \sup_n k_n |a_n| < \infty\,.$$

Suppose (17) is not true. Turning to a suitable subsequence we can assume, say, $k_n a_n \to \infty$ (the case with $-\infty$ would be analogous). Lemma 5 now implies that $\{\nu_n^{k_n}\}$ is tight (see also Comment 3). However, we have $\nu_n^{k_n}(\cdot) = \mu_n^{k_n}(k_n a_n + \cdot)$, and since $k_n a_n \to \infty$ it is not possible for both sequences $\{\mu_n^{k_n}\}$ and $\{\nu_n^{k_n}\}$ to be tight. This contradiction proves (17).

To show the strong stability of $\{\mu_n^{k_n}\}$ we take some $a > 0$ and verify (9)–(11) with $a_n = 0$. For n so large that $|a_n| < a/2$ we have

$$(18)\qquad k_n \int_{\{|x|\geqq a\}} \mu_n(dx) = k_n \int_{\{|x+a_n|\geqq a\}} \nu_n(dx) \leqq k_n \int_{\{|x|\geqq a/2\}} \nu_n(dx)\,,$$

$$(19)\qquad k_n\left|\int_{\{|x|<a\}} x\,\mu_n(\mathrm{d}x)\right| = k_n\left|\int_{\{|x+a_n|<a\}} (x+a_n)\,\nu_n(\mathrm{d}x)\right| \leqq$$
$$\leqq k_n\left|\int_{\{|x|<a\}} x\,\nu_n(\mathrm{d}x)\right| + k_n(3a/2)\int_{\{|x|\geqq a/2\}} \nu_n(\mathrm{d}x) + k_n|a_n|\,,$$
$$(20)\qquad k_n\int_{\{|x|<a\}} x^2\,\mu_n(\mathrm{d}x) = k_n\int_{\{|x+a_n|<a\}} (x+a_n)^2\,\nu_n(\mathrm{d}x) \leqq$$
$$\leqq k_n\int_{\{|x|<3a/2\}} x^2\,\nu_n(\mathrm{d}x) + 2k_n|a_n|\left|\int_{\{|x|<a\}} x\,\nu_n(\mathrm{d}x)\right| +$$
$$+\, 2k_n|a_n|\,(3a/2)\int_{\{|x|\geqq a/2\}} \nu_n(\mathrm{d}x) + k_n^2 a_n^2\,.$$

The last three inequalities together with the strong stability of $\{\nu_n^{k_n}\}$ yield the strong stability of $\{\mu_n^{k_n}\}$.

3. INFINITELY DIVISIBLE MEASURES AND LÉVY MEASURES

If a measure $\mu \in \mathcal{M}$ satisfies

$$(21)\qquad \mu = \nu_r * \ldots * \nu_r = \nu_r^r$$

for each $r \in N$ and suitable $\nu_r \in \mathcal{M}$, we call it an *infinitely divisible measure* (*i.d.* measure). Of course, the roots ν_r are then *i.d.*, too .

It is well-known that the characteristic function of an *i.d.* probability measure is non-zero everywhere and thus we can consider its logarithm. (21) implies $\log \tilde{\nu}_r(t) = r^{-1} \log \tilde{\mu}(t)$, $r \in N$, which guarantees the uniqueness of the roots in (21).

Lemma 7. *Any i.d. probability measure is uniquely infinitely divisible.*

For the representation of *i.d.* measures we shall use Lévy measures.

Definition. Let λ be a σ-finite nonnegative Borel measure on $\boldsymbol{R} \setminus \{0\}$. If λ satisfies

$$(22)\qquad \int_{\{|x|\geqq\beta\}} \lambda(\mathrm{d}x) < \infty\,,$$
$$(23)\qquad \int_{\{|x|<\beta\}} x^2\,\lambda(\mathrm{d}x) < \infty$$

for all $\beta > 0$, we call it a *Lévy measure.*

Let now λ_n, λ be Lévy measures, $n \in N$. We say that $\lambda_n \to \lambda$ *weakly* $(n \to \infty)$ if $\lim_n \int f\,\mathrm{d}\lambda_n = \int f\,\mathrm{d}\lambda$ for every bounded continuous function f on $\boldsymbol{R}$ which vanishes in some neighbourhood of the zero point.

4. THE MAIN THEOREM

Let $\{\mu_n\} \subseteq \mathcal{M}$ *be a sequence of measures and* $k_n \uparrow \infty$ *be a sequence of natural numbers. The following conditions are equivalent:*

(i) $F_\alpha(k_n(\mu_n - e) - k_m(\mu_m - e)) \to 0 \quad \begin{pmatrix} n \to \infty \\ m \to \infty \end{pmatrix}$

for each $\alpha > 0$;

(ii) *there is a Lévy measure* λ *such that*

a) $k_n(\mu_n - e) \to \lambda$ *weakly* $(n \to \infty)$,

b) *there exist finite limits*

$$\lim_n k_n \int_{\{|x|<\beta\}} x^i \mu_n(\mathrm{d}x) = M_\beta^{(i)}, \quad i = 1, 2,$$

for all $\beta > 0$ *such that* $-\beta$ *and* β *are* λ*-continuity points,*

c) $\lim_n k_n(\mu_n(R) - 1) = c$, *where* c *is some real number;*

(iii) *there is a (infinitely divisible) measure* $\mu \in \mathcal{M}$ *such that* $\mu_n^{k_n} \to \mu$ *weakly* $(n \to \infty)$;

(iv) *for each* $r \in N$ *there is* $\nu_r \in \mathcal{M}$ *such that* $\mu_n^{[k_n/r]} \to \nu_r$ *weakly* $(n \to \infty)$ *and* $\nu_r^r = \nu_1$ *([·] denotes the integral part).*

Comments. 1) It is shown in [3] that the Bolzano-Cauchy criterion (i) implies (iii) and (iv) for a more general class of seminorms. The inverse implication is shown to be true in this general case under the additional assumption of stability. Our proof does not refer to these results because our aim is to show the power of the classical technique of Fourier transforms.

2) In [8] the implication (i) ⇒ (ii) was shown; it was formulated using functions of bounded variations instead of measures.

Proof. 1) (i) implies (ii) and (iii).

First let us consider only probability measures μ_n. Denote $f_n(t) = k_n(\tilde{\mu}_n(t) - 1)$. By virtue of the completeness of continuous functions with the supremum norm, the assumption (i) implies the existence of a complex function f such that $f_n(t) \to f(t)$, $n \to \infty$, *l.u.* $[t]$. Now we use the fact (see [7], § 5.4) that for each n

$$\exp f_n = \exp k_n(\tilde{\mu}_n - 1)$$

is a characteristic function of an infinitely divisible probability measure, say ζ_n. We find now the Lévy - Khinchine representation of ζ_n

$$\begin{aligned}
(24) \qquad f_n(t) &= k_n \int_{-\infty}^{\infty} (\mathrm{e}^{itx} - 1)\, \mu_n(\mathrm{d}x) = \\
&= k_n\, \mathrm{i}t \int_{-\infty}^{\infty} \frac{x}{1 + x^2}\, \mu_n(\mathrm{d}x) + k_n \int_{-\infty}^{\infty} \left(\mathrm{e}^{itx} - 1 - \frac{itx}{1 + x^2}\right) \mu_n(\mathrm{d}x) = \\
&= \mathrm{i}\gamma_n t + \int_{-\infty}^{\infty} \left(\mathrm{e}^{itx} - 1 - \frac{itx}{1 + x^2}\right) \frac{1 + x^2}{x^2}\, G_n(\mathrm{d}x),
\end{aligned}$$

where

$$(25) \qquad \gamma_n = k_n \int_{-\infty}^{\infty} \frac{x}{1 + x^2}\, \mu_n(\mathrm{d}x),$$

$$G_n(\mathrm{d}x) = k_n \frac{x^2}{1 + x^2} \mu_n(\mathrm{d}x)$$

(G_n is a finite measure on $\boldsymbol{R}$ with the density $x^2/(1 + x^2)$ with respect to $k_n\mu_n$).

Observe that $\zeta_n(t) = \exp f_n(t) \to \exp f(t)$, $n \to \infty$, *l.u.* $[t]$, hence (see [7], § 5.5) $\exp f(t)$ must be a characteristic function of an *i.d.* probability measure, say μ, with the Lévy - Khinchine representation

$$\text{(26)} \qquad \log \tilde{\mu}(t) = f(t) = \mathrm{i}\gamma t + \int_{-\infty}^{\infty} \left(\mathrm{e}^{itx} - 1 - \frac{itx}{1 + x^2}\right) \frac{1 + x^2}{x^2} G(\mathrm{d}x)$$

with

$$\gamma_n \to \gamma \quad (n \to \infty),$$
$$G_n \to G \quad \text{weakly} \quad (n \to \infty).$$

Define a Lévy measure λ on $\boldsymbol{R} \setminus \{0\}$ as

$$\lambda(\mathrm{d}x) = \frac{1 + x^2}{x^2} G(\mathrm{d}x).$$

By (25) and (26), $k_n\mu_n \to \lambda$ weakly $(n \to \infty)$ (note that $k_n\mu_n$ considered as a Lévy measure coincides with $k_n(\mu_n - e)$). We can find the limit $(n \to \infty)$

$$k_n \int_{\{|x|<\beta\}} x^2 \mu_n(\mathrm{d}x) = \int_{\{|x|<\beta\}} (1 + x^2)\, G_n(\mathrm{d}x) \to$$
$$\to \int_{\{|x|<\beta\}} (1 + x^2)\, G(\mathrm{d}x) = M_\beta^{(2)},$$

provided $-\beta$ and β are λ-continuity points (and hence G-continuity points). Moreover, we have

$$\text{(27)} \qquad k_n \int_{\{|x|\geqq\beta\}} \sin x\, \mu_n(\mathrm{d}x) \to \int_{\{|x|\geqq\beta\}} \sin x\, \lambda(\mathrm{d}x) \quad (n \to \infty),$$

$$\text{(28)} \qquad k_n \int_{-\infty}^{\infty} \sin x\, \mu_n(\mathrm{d}x) = k_n(\tilde{\mu}_n(1) - \tilde{\mu}_n(-1))/2 \to$$
$$\to (f(1) - f(-1))/2 \quad (n \to \infty),$$

$$\text{(29)} \qquad k_n \int_{\{|x|<\beta\}} (x - \sin x)\, \mu_n(\mathrm{d}x) = \int_{\{|x|<\beta\}} (1 + x^2) \frac{x - \sin x}{x^2} G_n(\mathrm{d}x) \to$$
$$\to \int_{\{|x|<\beta\}} (1 + x^2) \frac{x - \sin x}{x^2} G(\mathrm{d}x)$$

(the last function under integration is bounded).

Using (27)–(29), we can find the limit

$$\lim_n k_n \int_{\{|x|<\beta\}} x\, \mu_n(\mathrm{d}x) = M_\beta^{(1)}.$$

Condition c) in (ii) holds automatically when considering probability measures, hence (ii) is proved.

We have already found the limit measure μ for (iii) (see (26)); it remains to show the weak convergence $\mu_n^{k_n} \to \mu$. The assumption (i) implies the boundedness of

$F(k_n(\mu_n - e))$, hence the stability of $\{\mu_n^{k_n}\}$. The u.a.n. condition is fulfiled, too. Thus we can apply Lemma 4 obtaining

$$|\log \tilde{\mu}_n^{k_n}(t) - k_n(\tilde{\mu}_n(t) - 1)| \to 0\,, \quad n \to \infty\,, \quad l.u.\ [t]\,,$$

consequently $\log \tilde{\mu}_n^{k_n}(t) \to f(t)$, $n \to \infty$, $l.u.$ $[t]$, and (iii) is proved.

Now let μ_n be arbitrary positive measures and let us write them in the form $\mu_n = c_n \nu_n$, c_n being positive numbers and ν_n probability measures. Suppose (i) holds for μ_n. Since $c_n = \tilde{\mu}_n(0)$. putting $t = 0$ in (i) expressed by characteristic functions we get

$$|k_n(c_n - 1) - k_m(c_m - 1)| \to 0 \quad \begin{pmatrix} n \to \infty \\ m \to \infty \end{pmatrix}.$$

Hence we have

$$\lim_n k_n(c_n - 1) = c \quad \text{for some} \quad c \in \boldsymbol{R} \tag{30}$$

(consequently, $\lim\limits_n c_n = 1$).

Now we find the Cauchy convergence

$$\begin{aligned} &|c_n k_n(\tilde{\nu}_n(t) - 1) - c_m k_m(\tilde{\nu}_m(t) - 1)| \leqq \\ &\leqq |k_n(\tilde{\mu}_n(t) - 1) - k_m(\tilde{\mu}_m(t) - 1)| + \\ &+ |k_n(c_n - 1) - k_m(c_m - 1)| \to 0 \quad \begin{pmatrix} n \to \infty \\ m \to \infty \end{pmatrix} \ l.u.\ [t]\,. \end{aligned}$$

Since $c_n \to 1$, the $l.u.$ convergence of $k_n(\tilde{\nu}_n(t) - 1)$ follows. Thus the sequence $\{\nu_n\}$ fulfils (i) and, consequently, it fulfils (ii) and (iii), too. To prove (ii) for μ_n, we take the same Lévy measure λ as we have for ν_n and realize that

$$k_n(\mu_n - e) = c_n k_n(\nu_n - e) + k_n(c_n - 1)\, e\,.$$

The measure $k_n(c_n - 1)\, e$ vanishes on $\boldsymbol{R} \setminus \{0\}$ and, using (30), we see that a) holds. The proof of b) and c) is immediate.

To prove (iii), use the fact that $\lim\limits_n c_n^{k_n} = \mathrm{e}^c$, and hence $\mu_n^{k_n} \to \mathrm{e}^c \mu$ weakly if $\nu_n^{k_n} \to \mu$ $(n \to \infty)$.

2) (ii) implies (i).

The Taylor expansion gives the equalities

$$\begin{aligned} &k_n(\tilde{\mu}_n(t) - 1) - k_m(\tilde{\mu}_m(t) - 1) = \\ &= (k_n(c_n - 1) - k_m(c_m - 1)) + (k_n(\tilde{\mu}_n(t) - c_n) - k_m(\tilde{\mu}_m(t) - c_m)) = \\ &= (k_n(c_n - 1) - k_m(c_m - 1)) + \textstyle\int_{-\infty}^{\infty} (\mathrm{e}^{itx} - 1)\,(k_n\mu_n - k_m\mu_m)\,(\mathrm{d}x) = \\ &= (k_n(c_n - 1) - k_m(c_m - 1)) + \textstyle\int_{\{|x| \geqq \beta\}} (\mathrm{e}^{itx} - 1)\,(k_n\mu_n - k_m\mu_m)\,(\mathrm{d}x) + \\ &+ \textstyle\int_{\{|x| < \beta\}} itx\,(k_n\mu_n - k_m\mu_m)\,(\mathrm{d}x) + \end{aligned} \tag{31}$$

$$+ \int_{\{|x|<\beta\}} (-t^2x^2/2)\,(k_n\mu_n - k_m\mu_m)\,(\mathrm{d}x) +$$
$$+ \int_{\{|x|<\beta\}} r(tx)\,(k_n\mu_n - k_m\mu_m)\,(\mathrm{d}x)\,,$$

where $|r(u)| = o(u^2)$, $u \to 0$.

Denote by I_0, I_1, I_2, I_3, I_4 in the given order the five summands in the last expression of (31). Fix arbitrary $\varepsilon > 0$ and $T > 0$; we want to show the expression in (31) to be less than ε for all $|t| \leqq T$ and sufficiently large m, n. I_0 tends to zero with m, n tending to infinity (see c)). Taking $\beta > 0$ sufficiently small we can guarantee $|r(tx)| \leqq \leqq \eta T^2 x^2$ for all $|x| < \beta$, $|t| \leqq T$ and any given $\eta > 0$. Then

$$|I_4| \leqq \eta T^2\, 2 \sup_n k_n \int_{\{|x|<\beta\}} x^2\, \mu_n(\mathrm{d}x) < \infty$$

(see (ii) b)) and, hence, we can make $|I_4| < \varepsilon/2$ for all $|t| \leqq T$ by prescribing sufficiently small $\eta > 0$. The Cauchy criterion applied to the convergent sequences in (ii) b) yields $I_2 \to 0$ and $I_3 \to 0$ *l.u.* $[t]$ when $m, n \to \infty$. In a similar way we see that $I_1 \to 0$ *l.u.* $[t]$ $(m, n \to \infty)$ as a consequence of (ii) a). Thus (i) is proved.

3) (iii) implies (i).

First we supose μ_n to be probability measures. Supposing (iii) holds we know by Lemma 6 that $\{\mu_n^{k_n}\}$ is strongly stable. Lemma 4 implies

$$\sup_{|t|\leqq T} \left|\log \tilde{\mu}_n(t)^{k_n} - k_n(\tilde{\mu}_n(t) - 1)\right| \to 0\,, \quad n \to \infty\,,$$

and (iii) gives

$$\sup_{|t|\leqq T} \left|\log \tilde{\mu}_n(t)^{k_n} - \log \tilde{\mu}(t)\right| \to 0\,, \quad n \to \infty\,.$$

From these two tacts we derive

$$\sup_{|t|\leqq T} \left|k_n(\tilde{\mu}_n(t) - 1) - \log \tilde{\mu}(t)\right| \to 0\,, \quad n \to \infty\,.$$

The Bolzano-Cauchy criterion (i) immediately follows.

In the general case consider again measures in the form $\mu_n = c_n\nu_n$ with $c_n > 0$ and probability measures ν_n. The assumption $\mu_n^{k_n} \to \mu$ weakly gives $c_n^{k_n} \to d = \mu(\boldsymbol{R}) > 0$, thus $k_n(c_n - 1) \to c = \log d$ and $\nu_n^{k_n} \to d^{-1}\mu$ weakly. Now the proof of (i) for $\{\mu_n\}$ follows from its validity for $\{\nu_n\}$.

4. (iii) implies (iv).

We again suppose μ_n to be probability measures. Supposing that (iii) holds we know by Lemma 6 that $\{\mu_n^{k_n}\}$ is strongly stable. Then, for any fixed $r \in \boldsymbol{N}$, Lemma 5 guarantees the tightness of the sequence $\{\mu_n^{[k_n/r]}, n \in \boldsymbol{N}\}$. Let ν_r be a weak limit of some subsequence $\mu_{n_j}^{[k_{n_j}/r]}$, $j \to \infty$. Then

$$(32) \qquad \tilde{\mu}_{n_j}(t)^{[k_{n_j}/r]} \to \tilde{\nu}_r(t)\,, \quad j \to \infty\,, \quad l.u.\ [t]\,.$$

Simultaneously we have

$$(33) \qquad \tilde{\mu}_{n_j}(t)^{k_{n_j}} = (\tilde{\mu}_{n_j}(t)^{[k_{n_j}/r]})^r \, \tilde{\mu}_{n_j}(t)^{\beta_{n_j}}$$

for some $\beta_{n_j} \leqq r$, $j \in N$.

By the assumption, $\tilde{\mu}_{n_j}(t)^{k_{n_j}}$ converges to $\tilde{\mu}(t)$ *l.u.* $[t]$. The *u.a.n.* condition (which is here a consequence of stability) implies $\tilde{\mu}_{n_j}(t) \to 1$ *l.u.* $[t]$, hence also $\tilde{\mu}_{n_j}(t)^{\beta_{n_j}} \to 1$ *l.u.* $[t]$. Thus from (32) and (33) we conclude $\tilde{v}_r^r = \tilde{\mu}$.

Suppose now that v_r is not the weak limit of the whole sequence $\mu_n^{[k_n/r]}$. Then there must exist another weak limit $\zeta_r \neq v_r$ of another subsequence. By the same argument we get $\tilde{\zeta}^r = \tilde{\mu}$. However, Lemma 7 implies $\zeta_r = v_r$. This is a contradiction which completes the proof of (iv).

The case of nonnegative measures μ_n can be again easily modified to the „probability case".

Since the implication (iv) $\Rightarrow$ (iii) is trivial, the proof of the theorem is complete.

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Souhrn

CHARAKTERIZACE SLABÉ KONVERGENCE KONVOLUČNÍCH MOCNIN

JAN RATAJ

V článku je zkoumáno několik pojmů stability posloupností konvolučních mocnin pravděpodobnostních měr na přímce. Za použití fourierovských seminorem se dokazuje věta o charakterizaci slabé konvergence takových posloupností.

Резюме

ХАРАКТЕРИЗАЦИЯ СЛАБОЙ СХОДИМОСТИ СТЕПЕНЕЙ СВЕРТКИ

JAN RATAJ

В работе разбираюится некоторые понятия устойчивости последовательностей степеней свертки вероятностных мер на прямой. Используя семинормы Фурье, автор доказывает теорему о характеризации слабой сходимости таких последовательностей.

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