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THE DECOMPOSITION OF A GENERALIZED GRAPH INTO ISOMORPHIC SUBGRAPHS

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1. THE DECOMPOSITION OF A COMPLETE GENERALIZED GRAPH INTO TWO SUBGRAPHS ISOMORPHIC TO EACH OTHER

First, we shall state the definition of the generalized graph (see [1], [2], [3]).

The generalized graph of the dimension d or the d-graph (without loops) is by definition the union of two sets, the set U whose elements are called vertices and the set H whose elements are called edges of the dimension d, while between vertices and edges a relation of incidence is given such that each edge is incident exactly with d different vertices and each d vertices are incident together at most with one edge.

If $d = 2$, we obtain an indirected graph in the usual sense.

The complete d-graph is by definition a d-graph such that for each d of its vertices an edge exists which is incident with all these vertices (we say that these d vertices are joined by an edge). Analogously as in the case of 2-graphs we define also the complement of a d-graph and the isomorphism between two d-graphs.

We are going to study the isomorphism between a d-graph and its complement. This problem for the case of 2-graphs is studied in the papers [4], [5], [6], [7]. If a d-graph is isomorphic with its complement, we call it a self-complementary d-graph.

Theorem 1. *If n is the number of vertices of a self-complementary d-graph G, the number $\binom{n}{d}$ is even.*

Proof. Obviously the number of edges of a complete d-graph with n vertices is $\binom{n}{d}$. If G is a self-complementary d-graph, then G and its complement $\bar{G}$ are isomorphic; consequently they have the same number of edges. At the same time they have no edge in common and their union is a complete d-graph which is a d-graph with $\binom{n}{d}$ edges. Therefore each of the graphs G, $\bar{G}$ contains $\binom{n}{d}\Big/2$ edges and this number must be an integer. Thus $\binom{n}{d}$ must be even.

Theorem 2. *Let G be a self-complementary d-graph with the vertex set U, let f be an isomorphic mapping of the d-graph G onto its complement $\bar{G}$. Let $\mathscr{C}_1, \ldots, \mathscr{C}_q$ be the cycles of the permutation p induced by the mapping f on the set U, let their numbers of vertices be $c_1, \ldots, c_q$. Furthermore, for every integer b let $C_b(x_1, \ldots, x_q) = \sum_{i=1}^{q} (c_i/(b, c_i))\, x_i$ be a linear form with q indefinites $x_1, \ldots, x_q$. Then for no odd positive integer b the equation*

$$C_b(x_1, \ldots, x_q) = d \tag{1}$$

has a solution $a_1, \ldots, a_q$ such that a_i would be non-negative integers and $a_i \leqq (b, c_i)$ for all $i = 1, \ldots, q$.

By a pair of numbers in brackets their largest common divisor is denoted.

Proof. Prove this theorem by contradiction. Let $a_1, \ldots, a_q$ be a solution of the equation (1) such that a_i are non-negative integers and $a_i \leqq (b, c_i)$ for $i = 1, \ldots, q$. Let $u_i, f(u_i), \ldots, f^{c_i-1}(u_i)$ be the vertices of the cycle $\mathscr{C}_i$ for $i = 1, \ldots, q$. Take vertices $f^k(u_i)$ in the cycle $\mathscr{C}_i$ such that k is congruent with some of the numbers $1, \ldots, a_i$ modulo (b, c_i). The total number of such vertices in the cycle $\mathscr{C}_i$ is $(c_i/(b, c_i))\, a_i$, because $a_i \leqq (b, c_i)$. If we do this in each of the cycles $\mathscr{C}_1, \ldots, \mathscr{C}_q$ we get $C_b(a_1, \ldots, a_q)$ vertices, i.e. d vertices; the set of those vertices will be denoted by A. Either in G, or in $\bar{G}$ there exists and edge h incident with all vertices of A and only with them; without the loss of generality we may assume that it is contained in G. Now consider an image $f^b(h)$ of the edge h; as b is odd, the edge $f^b(h)$ is contained in $\bar{G}$. Let v be a vertex of the cycle $\mathscr{C}_i$ $(1 \leqq i \leqq q)$ belonging to A; so $v = f^k(u)$, where k is congruent with some of the numbers $1, \ldots, a_i$ modulo (b, c_i).

We have $f^b(v) = f^{k+b}(u)$. As b is a multiple of the number (b, c_i), the number $k + b$ is again congruent with k and therefore also with some of the numbers $1, \ldots, a_i$ modulo (b, c_i). So $f^b(v) \in A$. We have chosen the vertex v quite arbitrarily; this means that the mapping f^b sends the set A and also the edge h into itself. However, then the edge h would be contained at the same time in G and in $\bar{G}$, which is a contradiction.

Corollary 1. *If G is a self-complementary d-graph and f an isomorphic mapping of the d-graph G onto its complement $\bar{G}$, then f has at most $d - 1$ fixed vertices.*

Proof. Let $\mathscr{C}_1, \ldots, \mathscr{C}_q$ be the cycles of the permutation p induced by the mapping f on the set of vertices of the d-graph G and (without the loss of generality) let each of the cycles $\mathscr{C}_1, \ldots, \mathscr{C}_d$ consist of one fixed vertex $(d \leqq q)$. Therefore there are at least d

fixed vertices and $c_i = 1$ for $i \leqq d$. Then it suffices to put $a_i = 1$ for $1 \leqq i \leqq d$ and $a_i = 0$ for $d < i \leqq q$, and for $b = 1$ we get

$$C_1(a_1, \ldots, a_q) = \sum_{i=1}^{q} \frac{c_i}{(b, c_i)} a_i = d .$$

Theorem 3. *Let a finite set U and a permutation p on it with the cycles $\mathscr{C}_1, \ldots, \mathscr{C}_q$ be given such that the numbers of vertices of those cycles are $c_1, \ldots, c_q$ and the equation (1) has, for no odd positive integer b, a solution $a_1, \ldots, a_q$ such that a_i would be non-negative integers and $a_i \leqq (b, c_i)$ for $i = 1, \ldots, q$. Then a self-complementary d-graph G exists, whose vertex set is U and the permutation p is induced on U by the isomorphic mapping f of the d-graph G onto its complement $\bar{G}$.*

Proof. Let the condition of the theorem be satisfied. Choose an edge h incident with the vertices $v_1, \ldots, v_d$ and include it into G. Then for each odd (or even respectively) k include the edge incident with the vertices $p^k(v_1), \ldots, p^k(v_d)$ into $\bar{G}$ (or into G respectively). We shall verify that no edge can be included at the same time into G and into $\bar{G}$ by this manner, i.e. that no integers k, l exist such that k would be odd, l would be even and the set of vertices $\{p^k(v_1), \ldots, p^k(v_d)\}$ would coincide with the set $\{p^l(v_1), \ldots, p^l(v_d)\}$. Assume that such k, l exist and let $B = \{p^k(v_1), \ldots, p^k(v_d)\} = \{p^l(v_1), \ldots, p^l(v_d)\}$. Then the permutation p^{l-k} transforms B again into B (we assume that $l > k$; in the opposite case the proof would be analogous). Let u_i be a vertex of the cycle $\mathscr{C}_i$ $(1 \leqq i \leqq q)$ belonging to B. Together with it all vertices $p^s(u_i)$, where s is congruent with some integral multiple of the number $l - k$ modulo c_i (and thus congruent with some integral multiple of the largest common divisor of $l - k$ and c_i), belong to B. The total number of such vertices in the cycle $\mathscr{C}_i$ is $c_i/(l - k, c_i)$. If $\mathscr{C}_i$ contains some other vertex u_i' of B, which does not belong to the above mentioned ones, then it contains again all vertices $p^s(u_i')$, where s is congruent with some integral multiple of the number $(l - k, c_i)$ modulo c_i. The number of vertices of B in $\mathscr{C}_i$ is therefore $(c_i/(l - k, c_i))\, a_i$, where $a_i \leqq (l - k, c_i)$, because in the opposite case $(c_i/(l - k, c_i))\, a_i > c_i$ would hold and the number of the elements of B in C_i would be larger than the total number of vertices of the cycle $\mathscr{C}_i$. Therefore the set B contains totally $\sum_{i=1}^{q} (c_i/(l - k, c_i))\, a_i = C_{l-k}(a_1, \ldots, a_q)$ elements, $l - k$ being an odd number, $a_i \leqq (l - k, c_i)$ and a_i are non-negative integers. But we know that the set B has d elements and therefore $C_{l-k}(a_1, \ldots, a_q) = d$, which is a contradiction with the assumption of the theorem.

Then we may choose again an edge h_2 which still has not been included into G or into $\bar{G}$ and we continue this process until each edge is included either into G or into $\bar{G}$. This procedure is analogous to the construction from [3] and [4].

An analogous theorem holds for infinite d-graphs (the needed contribution to the proof of Theorem 3 would be simple).

Theorem 4. *Let a set U and a permutation p on it be given such that the cycles of p are C_ι for $\iota < \lambda$ (λ is some ordinal number) and let the cycles $\mathscr{C}_\iota$ for $\iota < \mu \leqq \lambda$ contain a finite number of elements, the cycles $\mathscr{C}_\iota$ for $\mu \leqq \iota < \lambda$ contain an infinite number of elements. Let the numbers of elements of the cycles $\mathscr{C}_\iota$ be c_ι and let the equation*

$$\sum_{\iota<\mu} \frac{c_\iota}{(b, c_\iota)} a_\iota = d$$

have, for no odd positive integer b, a solution a_ι ($\iota < \mu$) such that a_ι for $\iota < \mu$ would be non-negative integers, only finite number of those numbers would be different from zero and $a_\iota \leqq (b, c_\iota)$ for $\iota < \mu$. Then a self-complementary d-graph G exists, whose vertex set is U and the permutation p is induced on U by an isomorphic mapping of the d-graph G onto its complement $\bar{G}$.

Corollary 2. *Let a set U and a permutation p on it be given such that each cycle of p has an even number of elements. Let d be an odd positive integer. Then there exists a self-complementary d-graph G whose vertex set is U and the permutation p is induced on U by an isomorphic mapping of the d-graph G onto its complement $\bar{G}$.*

Proof. If c_i is even, b is odd, then the largest common divisor (b, c_i) is odd and the quotient $c_i/(b, c_i)$ is even. Therefore for an arbitrary integer a_i the number $(c_i/(b, c_i))\, a_i$ is even and thus also $C_b(a_1, \ldots, a_q)$ is even and cannot be equal to the odd number d. (The case when U is infinite is analogous.)

2. THE DECOMPOSITION OF A COMMON d-GRAPH INTO TWO SUBGRAPHS ISOMORPHIC TO EACH OTHER

Here we shall prove some existence theorem for generalized R_2-graphs or R_2-d-graphs. The definition of an R_2-d-graph is quite analogous to the definition of an R_2-graph in [5]. An R_2-d-graph is by definition a d-graph G which can be decomposed into two edge-disjoint subgraphs, each of them containing all vertices of G, which are isomorphic to each other and the isomorphic mapping of one of them onto the other is an automorphism of G.

Theorem 5. *Let d, m, n be positive integers, n even, d odd, m even, $m \leqq \binom{n}{d}$. Then there exists an R_2-d-graph G with n vertices and m edges.*

Proof. Take a complete d-graph G' with n vertices. Decompose its vertex set U into pairwise disjoint pairs $U_1, \ldots, U_{n/2}$. Now define an automorphism f of the graph G so that if $U_i = \{u_i, v_i\}$ for $i = 1, \ldots, u/2$, then $f(u_i) = v_i$, $f(v_i) = u_i$. Thus the pairs U_i form cycles of the permutation induced by the mapping f on the set U. According to the Corollary 2 the complete d-graph G' is an R_2-d-graph, therefore

$f(h) \neq h$ for each edge h from G'. As $f^2(u) = u$ for each vertex $u \in U$, we have also $f^2(h) = h$ for each edge of the d-graph G'. Therefore the edge set of the d-graph G' is also decomposed into pairwise disjoint pairs of edges such that each edge of a pair is the image of the other edge of this pair. Let $m' = \binom{n}{d} - m$; it is an even number. So omit $m'/2$ mentioned pairs of edges from the d-graph G'; we obtain a d-graph G, which is evidently an R_2-d-graph with m edges and n vertices.

Theorem 6. *Let d, m, n be positive integers, d even, m even, $m \leqq \binom{n}{d} - \binom{n/2}{d/2}$ in the case of n even, $m \leqq \binom{n}{d} - \binom{(n-1)/2}{d/2}$ in the case of n odd. Then there exists an R_2-d-graph G with n vertices and m edges.*

Proof. First let us have n even. We construct a d-graph G' in the following manner. Take a vertex set U with n elements and decompose it into $n/2$ pairwise disjoint pairs $U_1, \ldots, U_{n/2}$. Join by edges all d-tuples of vertices except for those which consist of $d/2$ pairs U_i $(1 \leqq i \leqq n/2)$; there are exactly $\binom{n/2}{d/2}$ such d-tuples. So we obtain the graph G' with $\binom{n}{d} - \binom{n/2}{d/2}$ vertices. Define again the mapping f so that if $U_i = \{u_i, v_i\}$, then $f(u_i) = v_i$, $f(v_i) = u_i$. An arbitrary d-tuple of vertices from U is sent by the mapping f into itself if and only if with every vertex from any pair U_i it contains also its other vertex; this is possible if and only if it consists of $d/2$ pairs U_i and so it is not joined by an edge. Therefore no edge from G' is fixed in the mapping f. And as $f^2(u) = u$ for each vertex of G', also $f^2(h) = h$ holds for each edge from G'; thus, if we construct d-subgraphs G'_1, G'_2 so that $f(h)$ is in G'_2 if and only if h is in G'_1 (see [3] and [4]), the d-graphs G'_1, G'_2 form a decomposition of the d-graph G' into two isomorphic d-subgraphs. The edge set of the d-graph G' is decomposed — similarly to the proof of the Theorem 5 — into pairwise disjoint pairs of edges such that each edge of a pair is the image of the other edge of that pair under the mapping f. Now if $m' = \binom{n}{d} - \binom{n/2}{d/2} - m$, then it suffices to omit $m'/2$ such involutory pairs of edges and we obtain the sought d-graph. If n is odd, by the above described manner we construct a graph G' whose vertex set U has $n - 1$ elements. To it we adjoin a vertex w and all edges incident with d-tuples of vertices consisting of the vertex w and $d - 1$ vertices of U; denote the resulting d-graph by G^* and define the automorphism f^* of the d-graph G^* so that $f^*(u) = f(u)$ for $u \in U$ and $f^*(w) = w$. No edge which is contained in G^* and not contained in G' can be fixed, because it is incident with an odd number of vertices of U and so there exists at least one pair U_i $(1 \leqq i \leqq (n-1)/2)$ such that this edge is incident exactly with one vertex of this pair. Evidently again $f^2(u) = u$ for all vertices u of G^* and therefore also $f^2(h) = h$ for all edges h of G^*. The d-graph G^* contains $\binom{n}{d} - \binom{(n-1)/2}{d/2}$ edges. We continue as in the case of n even taking G^* instead of G'.

Theorem 7. *Let d, m, n be positive integers, d odd, m even, n odd, $m \leqq \binom{n}{d} - \binom{(n-1)/2}{(d-1)/2}$. Then there exists an R_2-d-graph G with n vertices and m edges.*

Proof. Construct again the d-graph G' whose vertex set U contains n elements. Decompose the set U into disjoint subsets $U_1, \ldots, U_{(n-1)/2}, W$ so that each of the sets U_i $(1 \leqq i \leqq (n-1)/2)$ might contain exactly two elements u_i, v_i, the set W might contain a unique element w. By edges we join exactly all d-tuples except for those which consist of $(d-1)/2$ pairs U_i and of the vertex w. The total number of edges of the constructed d-graph is $\binom{n}{d} - \binom{(n-1)/2}{(d-1)/2}$. Define $f(u_i) = v_i, f(v_i) = u_i$ for $i = 1, \ldots, (n-1)/2$, $f(w) = w$. The continuation of the proof is similar to the proofs of preceding theorems.

We exclude the case of m odd, because an R_2-d-graph must evidently contain an even number of edges.

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