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LIAPUNOV'S DIRECT METHOD IN ABSTRACT CONTROL PROCESSES

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INTRODUCTION

Since 1893, when the famous Liapunov's fundamental memoir [2] concerning stability of motion has been first published, the analysis of stability problems forms the center of interest of many mathematicians and technicians dealing with the analysis or synthesis of dynamical systems, control systems, etc. A majority of papers devoted to this theme is concerned with physical systems, whose behaviour may be described by means of systems of ordinary differential equations. In the last decades, especially in connection with the study of optimal control processes, systems of partial differential equations and functional differential equations are very often used to describe a behaviour of such systems.

In connection with the above mentioned variety of descriptions it appears desirable to make the analysis of stability properties of these systems as much as possible independent on the special type of equations describing their behaviour.

One of the last attempts to solve this problem in a full generality is the concept of the abstract process introduced by HÁJEK in his lecture on the second EQUADIFF-Symposium in Bratislava in 1966 (see [1]). His abstract process seems to be the last step in the process of generalization of the concept of ordinary differential equation. (It is not surprising that this concept enables one to describe a behaviour of systems, described usually in the classical formulations in the terms of partial differential equations, difference-differential equations, functional differential equations, etc.) The definition of this abstract process is recalled and its basic properties are described in the first chapter of the paper. Using this concept, we introduce two basic concepts — an abstract control system and an abstract control process. The abstract control process is not, however, an abstract process, although it is shown to share many basic properties with an abstract process. In the second and the third chapter several stability properties of certain sets, m, with respect to an abstract control process, p, are discussed. In this discussion we do not use any special continuity structure on the domain of the relation p (except a very simple structure induced by a certain non-negative function g assuming the zero value precisely on the set m). To illustrate the analogy between our problem and the classical stability problem for a system of ordinary differential equations in R^n, we set up a correspondence between our control process p and the system of differential equations, between the set m and the trivial solution of the differential system, and finally between the function g and the distance (in $\mathsf{R}^n \times \mathsf{R}^1$) of a point $(x, \alpha) \in \mathsf{R}^n \times \mathsf{R}^1$ from the trivial solution. In analogy with the situation in differential equation theory we introduce the concept of a solution of a control process. Since in our theory we do not assume a uniqueness of solutions (of the intial value problems for a given initial value) it seems to be useful to divide the analysis of the stability problem into two parts.

In the second chapter we occupy ourselves with the problem of the so called strong stability of m. Intuitively and loosely speaking, the set m is strongly stable with respect to a control process p if each solution of the initial value problem with an initial value near the set m (where the measure of this nearness is the corresponding value of the function g) remains near this set m also in the further time instants. In this chapter Liapunov's functions will be used to set up necessary and sufficient conditions of the strong stability, the strong asymptotic stability, and the corresponding uniform modifications of these concepts.

In the third chapter similar investigations are carried out for concepts of a weak stability, an asymptotic weak stability and their uniform modifications, where the set m is said to be weakly stable if for each initial value problem with the initial value near the set m there is at least one its solution remaining near m in the following time instants.

In the last chapter several possibilities of analysis of control processes on abstract set using control processes on the set of non-negative reals are shown (see also [3]).

STANDING NOTATION

The symbols R^1, R^0, R^+ and $\mathsf{R}^\#$ denote the sets $(-\infty, +\infty)$, $\langle 0, +\infty)$, $(0, +\infty)$ and $(-\infty, +\infty\rangle$, respectively. R denotes a given non-void subset of R^1, P and U denote given abstract sets, I and I^0 denote the intervals $(0, 1\rangle$ and $\langle 0, 1\rangle$, respectively. For a given $\alpha \in \mathsf{R}^1$ let $[\alpha]$ denote the integer such that $[\alpha] \leqq \alpha < [\alpha] + 1$. To simplify the writing of some formulas, we shall use the following notation: $Q = P \times U$, $W = U \times P \times U$, $T = P \times U \times P \times U$. Since one of the most important concepts used in the paper is is the concept of the relation, we introduce some notation and conventions concerning the relations. A relation r between sets X and Y (in this order) is a subset of the cartesian product $Y \times X$. If a pair $(y, x) \in Y \times X$ belongs to the relation r, we prefer to write $y\ r\ x$ instead of $(y, x) \in r$. If r is a relation between sets X and Y, then the relation inverse to the relation r is the relation between Y and X denoted by r^{-1} such that $x\ r^{-1}\ y$ iff $y\ r\ x$. A relation r between X and Y is called also a relation on X. The identity relation on X is the relation 1_X such that $y\, 1_X\, x$ iff $y = x$. If r is a relation between X and Y, s a relation between Y and Z, then $s \circ r$ denotes a relation between X and Z such that $z\, s \circ r\, x$ iff there is a $y \in Y$ such that $z\, s\, y$ and $y\, r\, x$. A relation r between X and Y is called a partial map out of X into Y (and is denoted by $r: X \to Y$) iff $r \circ r^{-1} \subset 1_Y$. Given a relation r between X and Y, let domain r denote the set of all $x \in X$ such that there is at least one $y \in Y$ such that $y\, r\, x$. A partial map $r: X \to Y$ is termed a map iff domain $r = X$. In what follows, we use often the projection maps defined as follows. Let there be given a system of sets X_j for $j = 1, 2, \ldots, n$. For each ordered set $(i_1, i_2, \ldots, i_k)$ of integers i_s such that $1 \leqq i_s < i_{s+1} \leqq n$ with $1 \leqq s \leqq k - 1$ define

$$\mathrm{proj}_{i_1, i_2, \ldots, i_k} : X_1 \times X_2 \times \ldots \times X_n \to X_{i_1} \times X_{i_2} \times \ldots \times X_{i_k}$$

so that for $a_j \in X_j$, $j = 1, 2, \ldots, n$ it holds

$$\mathrm{proj}_{i_1, i_2, \ldots, i_k}(a_1, a_2, \ldots, a_n) = (a_{i_1}, a_{i_2}, \ldots, a_{i_k}) .$$

In the paper we shall analyse several properties of some relations between sets of the type $X \times R$ and $Y \times R$ where X and Y are abstract sets. We suppose that the relation r between $X \times R$ and $Y \times R$ fulfils always the following condition

$$(1) \qquad (y, \beta) \in Y \times R\,, \quad (x, \alpha) \in X \times R\,, \quad (y, \beta)\, r\, (x, \alpha) \Rightarrow \beta \geqq \alpha\,.$$

Each such relation r defines a system

$$(2) \qquad \{{}_\beta r_\alpha : \beta \geqq \alpha \text{ in } R\}$$

of relations ${}_\beta r_\alpha$ between X and Y so that

$$(3) \qquad y\; {}_\beta r_\alpha\, x \Leftrightarrow (y, \beta)\, r\, (x, \alpha)\,;$$

and conversely, each system (2) defines by (3) a relation r between $X \times R$ and $Y \times R$. With each relation r between sets $X \times R$ and $Y \times R$ we associate the following sets:

$$E_r = \text{domain } r\,;$$

$$D_r = \{(\theta, x, \alpha) \in R \times X \times R : (y, \theta)\, r\, (x, \alpha) \quad \text{for some} \quad y \in Y\}\,;$$

$${}_\theta r_\alpha x = \{y \in Y : (y, \theta)\, r\, (x, \alpha) \quad \text{for a given} \quad (\theta, x, \alpha) \in D_r\}\,.$$

Finally, given a relation r between $X \times R$ and $Y \times R$, we define a very important partial map

$$\varepsilon_r : E_r \to \mathrm{R}^\# : \varepsilon_r(x, \alpha) = \sup\{\beta \in R : (\beta, x, \alpha) \in D_r\}$$

(with the l.u.b. taken in the extended real line).

1. ABSTRACT CONTROL PROCESS

1.1. In this chapter we remember a notion of a process [1] and we use this concept to define a control system, a control process and their several basic properties. The notation and conventions introduced in the preceding part will be used throughout.

1.2. Definition. Let X be a set, $R \subset \mathrm{R}^1$, h a relation on $X \times R$ satisfying the condition

$$(1) \qquad {}_\beta h_\alpha \text{ implies } \beta \geqq \alpha \text{ for all } \alpha, \beta \in R .$$

A relation h is said to be *a process* on X over R iff

(i) ${}_\alpha h_\alpha \subset 1_X$ for all α in R;

(ii) ${}_\gamma h_\beta \circ {}_\beta h_\alpha = {}_\gamma h_\alpha$ for all $\gamma \geqq \beta \geqq \alpha$ in R.

A process h is termed *local* (*global*) iff $\varepsilon_h(x, \alpha) > \alpha$ ($\varepsilon_h(x, \alpha) = +\infty$) holds for all $(x, \alpha) \in E_h$.

Before illustrating the concept of a process let us introduce several new concepts related to the process.

1.3. Definition. Let h be a process on X over R. s is said to be *a solution* of a process h iff

(i) $s : R \to X$;

(ii) domain s is an interval in R;

(iii) $(s(\theta), \theta)\ h\ (s(\alpha), \alpha)$ holds for each $\theta \geqq \alpha$ in domain s.

1.4. Definition. Let h be a process on X over R. The process h is said to be *solution-complete* iff, corresponding to each couple $(x, \alpha), (y, \beta)$ in E_h with $(y, \beta)\ h\ (x, \alpha)$ there exists a solution s such that $s(\alpha) = x$, $s(\beta) = y$.

1.5. Lemma. *Let h be a process on X over R and let s_1, s_2 be solutions of h such that* domain $s_1 \cap$ domain $s_2 \neq \emptyset$ *and* $s_1(\theta) = s_2(\theta)$ *for all* $\theta \in$ domain $s_1 \cap$ domain s_2. *Then $s_1 \cap s_2$ and $s_1 \cup s_2$ are also solutions of h and there hold* domain $(s_1 \cap \cap s_2) =$ domain $s_1 \cap$ domain s_2 *and* domain $s_1 \cup$ domain $s_2 =$ domain $(s_1 \cup s_2)$.

Now let us introduce a special type of process, which generalizes the known concept of a differential equation with a periodic right hand side.

1.6. Definition. A process h on X over R is said *to admit the period* τ iff $\tau \in \mathrm{R}^1$ and

$${}_{\beta-\tau}h_{\alpha-\tau} = {}_\beta h_\alpha = {}_{\beta+\tau}h_{\alpha+\tau}$$

holds for all $\beta \geqq \alpha$ in R.

1.7. Example. In this paragraph we shall describe an example exhibiting the basic interpretation of the concept of process.

Consider an initial value problem for an ordinary differential equation

$$\frac{dz}{d\theta} = f(z, \theta)$$

in euclidean n-space R^n, formulated in the following classical manner. Let there be given an open subset E in R^{n+1} and a continuous mapping $f : E \to R^n$. Given $(x, \alpha) \in \in E$, we have to find a partial map $s : R^1 \to R^n$ (which need not to be uniquely determined by the point (x, α)) such that

(i) domain s is an interval in R^1 (either degenerate or nondegenerate);
(ii) $s(\alpha) = x$;
(iii) if the interval domain s is nondegenerate, then $ds(\theta)/d\theta = f(s(\theta), \theta)$ holds for all $\theta \in$ domain s (with the corresponding modifications in the end-points of the interval domain s).

A partial map s is then called a solution of the initial value problem with the initial value (x, α). With the initial value problem one may associate a process h on R^n in the following way: $(y, \beta)\, h\, (x, \alpha)$ iff there exists a solution s with the initial value (x, α) such that $s(\beta) = y$.

1.8. Definition. Let P, U be abstract sets, $Q = P \times U$. A pair (q, s) is said to be *an r-process* on Q iff q and s are processes on Q satisfying the following conditions:

(i) $(y, v, \beta)\, q\, (x, u, \alpha)$ implies $v = u$;
(ii) $(y, v, \beta)\, s\, (x, u, \alpha)$ implies $y = x$.

1.9. Remark. From the preceding definition there follows directly that for each r-process (q, s) on $P \times U$ the pair (s, q) is an r-process on $U \times P$.

1.10. Example. Consider the initial value problems for the systems of ordinary differential equations

$$(1) \qquad \frac{dz}{d\theta} = f(z, w, \theta), \qquad \frac{dw}{d\theta} = 0,$$

$$(2) \qquad \frac{dz}{d\theta} = 0, \qquad \frac{dw}{d\theta} = g(z, w, \theta),$$

with $f : R^{n+m+1} \to R^n$, $g : R^{n+m+1} \to R^m$ continuous. The method described in 1.7. enables us to associate with each of these initial value problems a process on $R^n \times R^m$. Denoting q and s the process associated with the system (1) and (2) respectively, one has an r-process (q, s) on $R^n \times R^m$.

Now, let us formulate one of the basic definitions of the paper.

1.11. Definition. Let there be given an r-process (q, s) on Q. *A control system t, generated by the r-process* (q, s), is the relation t on $Q \times Q \times R$, defined in the following manner:

(i) domain $t = \{(x, u, y, v, \alpha) : (x, u, \alpha) \in \text{domain } q, (y, v, \alpha) \in \text{domain } s\}$;

(ii) let $(x, u, y, v, \alpha) \in$ domain t be given;

if $\theta \in \langle \alpha, [\alpha] + 1)$, then define

$$_\theta t_\alpha(x, u, y, v) = \{(x_1, u, y, v_1) : (x_1, u) \, _\theta q_\alpha (x, u), (y, v_1) \, _\theta s_\alpha (y, v)\} ;$$

if $\theta = [\alpha] + 1$, then define

$$_\theta t_\alpha(x, u, y, v) = \{(x_1, v_1, x_1, v_1) : (x_1, u) \, _{[\alpha]+1} q_\alpha (x, u), (y, v_1) \, _{[\alpha]+1} s_\alpha (y, v)\} ;$$

if $k \geqq 1$ and $_\theta t_\alpha$ is defined for all $\theta \in \langle \alpha, [\alpha] + k \rangle$, then for $\theta \in \langle [\alpha] + k, [\alpha] + k + 1 \rangle$ define

$$_\theta t_\alpha = {_\theta t_{[\alpha]+k}} \circ {_{[\alpha]+k} t_\alpha}$$

whenever the right hand side is defined.

1.12. Lemma. *A control system t, generated by the r-process* (q, s) *on* Q, *is a process on* $Q \times Q \times R$.

Proof. According to 1.11., t has the properties 1.2. (1) and 1.2. (i). It remains therefore to prove that t has also the property 1.2. (ii), i.e. we have to prove that

$$1) \qquad _\theta t_\alpha = {_\theta t_\beta} \circ {_\beta t_\alpha}$$

holds for each $\alpha \leqq \beta \leqq \theta$ such that at least one side of the relation (1) is defined. To prove this assertion it suffices to show that for each positive integer k there holds the following proposition $T(k)$:
Let reals α, β, θ are such that $\alpha \leqq \beta \leqq \theta \leqq [\alpha] + k$. Then the relation (1) holds with these α, β, θ.

First we prove the proposition $T(1)$. Let $\alpha \leqq \beta \leqq \theta \leqq [\alpha] + 1$ are given and let $(x_2, u, y, v_2) \, _\theta t_\alpha (x, u, y, v)$. Then $(x_2, u) \, _\theta q_\alpha (x, u)$, $(y, v_2) \, _\theta s_\alpha (y, v)$ and since q and s are processes, corresponding to each $\beta \in \langle \alpha, \theta \rangle$ there exists a point (x_1, u, y, v_1) so that $(x_2, u) \, _\theta q_\beta (x_1, u)$, $(x_1, u) \, _\beta q_\alpha (x, u)$, $(y, v_2) \, _\theta s_\beta (y, v_1)$, $(y, v_1) \, _\beta s_\alpha (y, v)$, hence $(x_2, u, y, v) \, _\theta t_\beta (x_1, u, y, v_1)$ and $(x_1, u, y, v_1) \, _\beta t_\alpha (x, u, y, v)$ and according to the compositivity property $(x_2, u, y, v_2) \, _\theta t_\beta \circ {_\beta t_\alpha} (x, u, y, v)$. Hence the inclusion $_\theta t_\alpha \subset {_\theta t_\beta} \circ {_\beta t_\alpha}$ follows.

Now, let there be given $\beta \in \langle \alpha, \theta \rangle$ and let there be points (x, u, y, v), (x_1, u, y, v_1), (x_2, u, y, v_2) such that $(x_2, u, y, v_2) \, _\theta t_\beta (x_1, u, y, v_1)$ and $(x_1, u, y, v_1) \, _\beta t_\alpha (x, u, y, v)$. Then there hold also $(x_1, u) \, _\beta q_\alpha (x, u)$, $(x_2, u) \, _\theta q_\beta (x_1, u)$, $(y, v_1) \, _\beta s_\alpha (y, v)$, $(y, v_2) \, _\theta s_\beta (y, v_1)$ so that $(x_2, u) \, _\theta q_\alpha (x, u)$, $(y_2, v) \, _\theta s_\alpha (y, v)$, which is equivalent with $(x_2, u, y, v_2) \, _\theta t_\alpha (x, u, y, v)$, hence $_\theta t_\beta \circ {_\beta t_\alpha} \subset {_\theta t_\alpha}$ and proposition $T(1)$ follows.

Suppose now that proposition $T(k)$ holds for some $k \geqq 1$ and let $\alpha \leqq \beta \leqq \theta \leqq \leqq [\alpha] + k + 1$ be given. If $\theta \leqq [\alpha] + k$, then according to the assumption the relation (1) holds. Suppose therefore that $[\alpha] + k < \theta \leqq [\alpha] + k + 1$. According to 1.7. (ii) there holds ${}_\theta t_\alpha = {}_\theta t_{[\alpha]+k} \circ {}_{[\alpha]+k} t_\alpha$. Now, if $\beta \in \langle \alpha, [\alpha] + k)$, then according to the assumption there holds

$${}_\theta t_\alpha = {}_\theta t_{[\alpha]+k} \circ {}_{[\alpha]+k} t_\beta \circ {}_\beta t_\alpha = {}_\theta t_\beta \circ {}_\beta t_\alpha \,;$$

if $[\alpha] + k < \beta \leqq \theta \leqq [\alpha] + k + 1$, then proposition $T(1)$ gives ${}_\theta t_{[\alpha]+k} = {}_\theta t_\beta \circ \circ {}_\beta t_{[\alpha]+k}$ so that

$${}_\theta t_\alpha = {}_\theta t_\beta \circ {}_\beta t_{[\alpha]+k} \circ {}_{[\alpha]+k} t_\alpha = {}_\theta t_\beta \circ {}_\beta t_\alpha \,,$$

hence $T(k + 1)$ follows, which finishes the proof of Lemma 1.12.

1.13. Remark. Considering that the whole theory of the paper is built up in abstract concepts, it seems to be convenient to show at least on an example a motivation of the introduced concepts, borrowed from the theory of control systems. Thus we return to the concept of r-process (q, s) described in 1.10. As a physical model of this r-process we may take an arbitrary couple of physical systems Q and S, whose behaviour is described by the equations 1.10. (1) and 1.10. (2), evidently, generally non-stationary. Then z is an n-dimensional phase coordinate of the system Q and an n-dimensional parameter of the system S, and similarly, w is an m-dimensional phase coordinate of the system S and an m-dimensional parameter of the system Q. For the sake of simplicity we suppose systems 1.10. (1) and 1.10. (2) to fulfil the conditions of the existence and the uniqueness of solutions, i.e. given an initial condition, the corresponding initial value problem has a uniquelly determined solution, which is supposed to be defined on a sufficiently large interval.

Consider now a new system P, created from the two systems Q and S, whose behaviour may be formally described in the following way.

Let (x, u, α), (y, v, α) be points in $\mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^1$. According to the assumption there exist two uniquelly determined solutions

$$k[x, u, \alpha] : \langle \alpha, [\alpha] + 1 \rangle \to \mathrm{R}^n \times \mathrm{R}^m \,,$$

and

$$l[y, v, \alpha] : \langle \alpha, [\alpha] + 1 \rangle \to \mathrm{R}^n \times \mathrm{R}^m$$

of the initial value problems 1.10. (1) and 1.10. (2) with the initial values (x, u, α) and (y, v, α), respectively, i.e. such that $k[x, u, \alpha](\alpha) = (x, u)$, $l[y, v, \alpha](\alpha) = (y, v)$. Now, the behaviour of the system P may be described by the system of partial maps

$${}_\theta t_\alpha : \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^n \times \mathrm{R}^m \to \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^n \times \mathrm{R}^m$$

defined as follows:

for $(x, u, y, v, \alpha) \in \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^1$,

if $\theta \in \langle\alpha, [\alpha] + 1)$, then $(x_1, u, y, v_1) = {}_\theta t_\alpha (x, u, y, v)$ iff
$(x_1, u) = k[x, u, \alpha](\theta), (y, v_1) = l[y, v, \alpha](\theta)$;

if $\theta = [\alpha] + 1$, then $(x_1, v_1, x_1, v_1) = {}_{[\alpha]+1}t_\alpha(x, u, y, v)$ iff
$(x_1, u) = k[x, u, \alpha]([\alpha] + 1), (y, v_1) = l[y, v, \alpha]([\alpha] + 1)$;

if $\theta \in \langle[\alpha] + k, [\alpha] + k + 1\rangle$ and if ${}_{[\alpha]+k}t_\alpha(x, u, y, v)$ is defined, then
$(x_1, u_1, y_1, v_1) = {}_\theta t_\alpha(x, u, y, v)$ iff

$$(x_1, u_1) = k[\mathrm{proj}_1 \circ {}_{[\alpha]+k}t_\alpha(x, u, y, v), \mathrm{proj}_4 \circ {}_{[\alpha]+k}t_\alpha(x, u, y, v), [\alpha] + k](\theta),$$
$$(y_1, v_1) = l[\mathrm{proj}_1 \circ {}_{[\alpha]+k}t_\alpha(x, u, y, v), \mathrm{proj}_4 \circ {}_{[\alpha]+k}t_\alpha(x, u, y, v), [\alpha] + k](\theta).$$

In the above introduced physical interpretation the control system p may be described thus: In the initial instant the corresponding initial values of phase variables and parameters of both systems Q and S are arranged. Since the instant α to the instant $[\alpha] + 1$ the behaviour of each of the both processes is uniquely determined, independently of the behaviour of the another process, by the physical laws given by the inner structure of the system and the initial values of the phase variables and parameters. The values of the parameters of the both systems remain in this whole time interval constant. In the instant $[\alpha] + 1$ the values of the parameters are changed so that the parameter of the system Q is adjusted to the instantaneous value of the phase variable of the system S and vice versa. The values of the phase variables of the both systems remain unchanged. Now, with the "initial values" adjusted in the described manner, the behaviour of the both systems is again uniquely determined until the time instant $[\alpha] + 2$, when the initial values of the parameters are changed again and this situation is periodically repeated.

In analyzing control systems we are concerned very often with situations in which a behaviour of the system Q is simulated on the system S. Then the action of the system S on the system Q, described above, effects that the phase variables of the system Q vary in some prescribed domain. In this situation the system S is termed "controller" and the system Q "control process". Since in the problems of this type we are interested mainly in the behaviour of phase variables of the control process, it appears convenient to introduce the following definition.

1.14. Definition. Let there be given a control system t on $P \times U$ over R. *A control process* p on P over R is a relation p between $P \times U \times P \times U \times R$ and $P \times R$ defined as follows:

$$(z, \theta)\, p\, (x, u, y, v, \alpha) \quad \text{iff there exists} \quad (u_1, y_1, r_1) \in U \times P \times U$$
$$\text{so that} \quad (z, u_1, y_1, v_1, \theta)\, t\, (x, u, y, v, \alpha)\,.$$

A control process p is termed *local* (*global*) iff t is a local (global) process.

1.15. Remark. Using the projection map a control process may be characterized as $p = \mathrm{proj}_{1,5} \circ t$, or, if $\{{}_\theta t_\alpha\}$ is the system of the relations associated with the relation t, then the system $\{_\theta p_\alpha\}$, characterizing the relation p, may be defined as $_\theta p_\alpha = \text{proj}_1 \circ \circ\, _\theta t_\alpha$.

If p is a control process of a control system generated by an r-process (q, s), then we say also that the process p is generated by the r-process (q, s).

Consider in some details the situation arising when the control system t is generated by an r-process (q, s) on $Q \times R$ such that domain $s \subset$ domain q and for each $\theta \geqq \alpha$ there holds $_\theta s_\alpha \subset 1_Q$. Let us try to find a correspondence between the behaviour of the control system t and that one of the process p.

Let there be given $(x, u, y, v, \alpha) \in$ domain t. According to 1.11. and the properties of the process s, for each $\theta \in \langle\alpha, [\alpha] + 1)$ there hold

$$_\theta t_\alpha(x, u, y, v) = \{(z', u, y, v) : (z', u)\, _\theta q_\alpha\, (x, u)\}\,,$$

and

$$_{[\alpha]+1}t_\alpha(x, u, y, v) = \{(z_1, v, z_1, v) : (z_1, u)_{[\alpha]+1}q_\alpha(x, u)\}\,.$$

Further, for each $\theta \in ([\alpha] + 1, [\alpha] + 2)$ one obtains in the same way

$$_\theta t_\alpha(x, u, y, v) = {_\theta t_{[\alpha]+1}} \circ {_{[\alpha]+1}t_\alpha}(x, u, y, v)\,,$$

hence

$$_\theta t_\alpha(x, u, y, v) = \{(z_1', v, z_1, v) : (z_1', v)\, _\theta q_{[\alpha]+1}\, (z_1, v)$$
$$\text{for some} \quad (z_1, v, z_1, v) \in {_{[\alpha]+1}t_\alpha}(x, u, y, v)\}\,;$$

$$_{[\alpha]+2}t_\alpha(x, u, y, v) = \{(z_2, v, z_2, v) : (z_2, v)\, _{[\alpha]+2}q_{[\alpha]+1}\, (z_1, v)$$
$$\text{for some} \quad (z_1, v, z_1, v) \in {_{[\alpha]+1}t_\alpha}(x, u, y, v)\}\,.$$

Now it is seen that one may easily by induction prove the following proposition.

Let there be given $(x, u, y, v, \alpha) \in$ domain t and an integer $k \geqq 1$.

If $\theta \in ([\alpha] + k, [\alpha] + k + 1)$ and $_\theta t_\alpha(x, u, y, v)$ is defined then

$$_\theta t_\alpha(x, u, y, v) = \{(z_k', v, z_k, v) : (z_k', v)\, _\theta q_{[\alpha]+k}\, (z_k, v)$$
$$\text{for some} \quad (z_k, v, z_k, v) \in {_{[\alpha]+k}t_\alpha}(x, u, y, v)\}\,;$$

if $\theta = [\alpha] + k + 1$ and $_{[\alpha]+k+1}t_\alpha(x, u, y, v)$ is defined then

$$_{[\alpha]+k+1}t_\alpha(x, u, y, v) = \{(z_{k+1}, v, z_{k+1}, v) : (z_{k+1}, v)\, _{[\alpha]+k+1}q_{[\alpha]+k}\, (z_k, v)$$
$$\text{for some} \quad (z_k, v, z_k, v) \in {_{[\alpha]+k}t_\alpha}(x, u, y, v)\}\,.$$

This remark shows that for each initial value $(x, u, y, v) \in$ domain t and for any $\theta \geqq [\alpha] + 1$ the behaviour of the system t is fully described by the behaviour of the process q. Especially, if $u = v$ then this assertion holds evidently for each $\theta \geqq \alpha$. Hence there follows directly the following proposition.

1.16. Lemma. *Let a control process p be generated by an r-process* (q, s) *on* $Q \times R$ *such that* domain $s \subset$ domain q *and for each* $\theta \geqq \alpha$ *it holds* ${}_\theta s_\alpha \subset 1_Q$. *Then*

$$(z, \theta)\, p\,(x, u, y, u, \alpha) \quad \textit{iff} \quad (z, \theta)\, \mathrm{proj}_{1,3} \circ q \circ \mathrm{proj}_{1,2,5}\,(x, u, y, u, \alpha)\,.$$

1.17. Remark. In what follows, we use certain generalizations of methods of the Liapunov's stability theory for differential equations to study several stability and boundedness properties of control processes. It is seen that each control system t on $P \times U \times P \times U \times R = T \times R$ defines a partial order U on $T \times R$ as follows

$$((x^1, u^1, y^1, v^1, \alpha^1), (x, u, y, v, \alpha)) \in \mathsf{U} \quad \text{iff} \quad (x^1, u^1, y^1, v^1, \alpha^1)\, t\,(x, u, y, v, \alpha)\,.$$

It is therefore natural to set up the following definition.

1.18. Definition. Let p be a control process of a control system t on $P \times U \times P \times \times U$ over R. A partial map

$$V: E_p \to I^0$$

is said to be *a Liapunov function* of the control process p iff V is non-increasing along p, i.e. iff $(x^j, u^j, y^j, v^j, \alpha^j) \in$ domain V, $j = 1, 2$, $(x^2, u^2, y^2, v^2, \alpha^2)\, t\,(x^1, u^1, y^1, v^1, \alpha^1)$ implies $V(x^2, u^2, y^2, v^2, \alpha^2) \leqq V(x^1, u^1, y^1, v^1, \alpha^1)$.

1.19. Definition. Let p be a control process of a control system t on $P \times U \times P \times \times U = P \times W$. A partial map $s : R \to P$ is called *a solution of the control process p* iff there exists a solution $\sigma : R \to P \times W$ of t such that $s = \mathrm{proj}_1 \circ \sigma$.

1.20. Definition. Let p be a control process of a control system t on $P \times W$ over R. A partial map

$$V: E_p \to I^0$$

is said to be *a weak Liapunov function* of the control process p iff corresponding to each $(x, w, \alpha) \in$ domain V there exists a solution σ of the control process t such that domain $\sigma \supset \langle \alpha, \varepsilon_p(x, w, \alpha))$ and $V(\sigma(\theta), \theta) \leqq V(\sigma(\beta), \beta)$ hold for all $\alpha \leqq \beta \leqq \theta < < \varepsilon_p(x, w, \alpha)$ in R.

1.21. Definition. Let p be a control process of a control system t on $P \times W$ over R. The control process p is said to be *solution-complete* iff the control system t is solution complete. The control process p is said *to admit a period* τ iff t admits the period τ.

1.22. Lemma. *Let p be a control process of a control system t. Then*

$$(z, \theta)\, p\,(x, w, \alpha), (x, w, \alpha)\, t\,(x^1, w^1, \alpha^1) \quad \textit{imply} \quad (z, \theta)\, p\,(x^1, w^1, \alpha^1)\,.$$

Proof, see 1.14. and 1.12.

2.1. Notation. In this chapter we suppose given sets P and U, a control process p of a control system t on $P \times U \times P \times U \times R = P \times W \times R$ generated by an r-process (q, s) on $P \times U$. In what follows we shall write E and D instead of E_p and D_p, respectively. In the whole following part of the paper we are given a non-void set

(1) $$m \subset P \times R$$

and a function

(2) $$g : P \times R \to \mathrm{R}^0$$

such that there holds

(3) $$g(x, \alpha) = 0 \quad \text{iff} \quad (x, \alpha) \in m\,.$$

The sets domain p and domain V are interpreted in a twofold way, as subsets of $P \times W \times R$ or of $P \times U \times P \times U \times R$, so that both the manner of the notation $(x, w, \alpha) \in$ domain p and $(x, u, y, v, \alpha) \in$ domain p are admissible and equivalent.

2.2. Definition. m is said to be *strongly invariant* with respect to p iff

$$(\theta, x, w, \alpha) \in D\,, \quad (x, \alpha) \in m\,, \quad (z, \theta)\, p\, (x, w, \alpha) \quad \text{imply} \quad (z, \theta) \in m\,.$$

2.3. Definition. m is said to be *strongly stable* with respect to p iff there exists a map

(1) $$\omega : R \times I \to I$$

such that

(2) $$(\theta, x, w, \alpha) \in D\,, \quad g(x, \alpha) \leqq \omega(\alpha, \zeta)\,, \quad (z, \theta)\, p\, (x, w, \alpha) \quad \text{imply} \quad g(z, \theta) \leqq \zeta\,.$$

2.4. Lemma. *If m is strongly stable, it is strongly invariant.*

Proof. Suppose that the lemma does not hold. Then there exist $(\theta, x, w, \alpha) \in D$ and $z \in P$ such that $(x, \alpha) \in m$, i.e. $g(x, \alpha) = 0$, and $(z, \theta)\, p\, (x, w,)$, $(z, \theta) \notin m$, i.e. $g(z, \theta) = \varrho > 0$. Then for each $\varrho_1 \in (0, \varrho)$ there holds $g(x, \alpha) \leqq \omega(\alpha, \varrho_1)$ and $g(z, \theta) > \varrho_1$, which contradicts 2.3. (2).

2.5. Theorem. *m is strongly stable with respect to p iff there exist maps*

(1) $$V : E \to I^0\,, \quad \delta : R \to I\,, \quad \omega_0 : R \times I \to I\,, \quad a : I^0 \to I^0$$

such that

$\omega_0(\alpha, \zeta) \to 0$ as $\zeta \to 0$ and $\alpha \in R$; a increasing, $a(r) \to 0$ as $r \to 0$, with the following properties:

(i) *V is a Liapunov function;*

(ii) domain $V \supset \{(x, w, \alpha) \in E : (x, w, \alpha)\, t\, (x^1, w^1, \alpha^1)$ *for some* $(x^1, w^1, \alpha^1) \in E$ *with* $g(x^1, \alpha^1) \leqq \delta(\alpha)\}$;
(iii) $(x, w, \alpha) \in$ domain V, $g(x, \alpha) \leqq \omega_0(\alpha, \zeta)$ *imply* $V(x, w, \alpha) \leqq \zeta$;
(iv) $(x, w, \alpha) \in$ domain V *implies* $a(g(x, \alpha)) \leqq V(x, w, \alpha)$.

Proof. Let m be strongly stable. Define a map

(2) $$\delta : R \to I : \delta(\alpha) = \omega(\alpha, 1)$$

and a partial map

(3) $$V : E \to I^0 : V(x, w, \alpha) = \sup \{g(z, \theta) : (z, \theta)\, p\, (x, w, \alpha)\}$$

whenever there exists $(x^1, w^1, \alpha^1) \in E$ such that $g(x^1, \alpha^1) \leqq \delta(\alpha^1)$ and $(x, w, \alpha)\, t\, (x^1, w^1, \alpha^1)$.

Further, define maps

(4) $$\omega_0 : R \times I \to I : \omega_0(\alpha, \zeta) = \omega(\alpha, \zeta)\,;$$

(5) $$a : I^0 \to I^0 : a(r) = r\,.$$

Now we shall prove that the maps (2) to (5) have the properties (i) to (iv).

Ad (*i*): Let there be given $(x^1, w^1, \alpha^1) \in$ domain V and let (x, w, α) be such that $(x, w, \alpha)\, t\, (x^1, w^1, \alpha^1)$. Then for each (z, θ) satisfying the relation $(z, \theta)\, p\, (x, w, \alpha)$ there holds, according to 1.22., $(z, \theta)\, p\, (x^1, w^1, \alpha^1)$, hence

$$V(x, w, \alpha) = \sup \{g(z, \theta) : (z, \theta)\, p\, (x, w, \alpha)\} \leqq \leqq \sup \{g(z, \theta) : (z, \theta)\, p\, (x^1, w^1, \alpha^1)\} = V(x^1, w^1, \alpha^1)\,,$$

thus V is a Liapunov function.

Ad (ii): Let $(x^1, w^1, \alpha^1) \in E$ be such that $g(x^1, \alpha^1) \leqq \delta(\alpha^1)$. If $(x, w, \alpha)\, t\, (x^1, w^1, \alpha^1)$, then, according to 1.22., corresponding to each (z, θ) with $(z, \theta)\, p\, (x, w, \alpha)$ it holds $(z, \theta)\, p\, (x^1, w^1, \alpha^1)$. Hence using (2) and 2.3. (2) it follows $g(z, \theta) \leqq 1$, so that $V(x, w, \alpha)$ is by (3) really defined and V has property (ii).

Ad (iii): From $(x, w, \alpha) \in$ domain V, $g(x, \alpha) \leqq \omega_0(\alpha, \zeta) = \omega(\alpha, \zeta)$, $(z, \theta)\, p\, (x, w, \alpha)$ there follows $g(z, \theta) \leqq \zeta$, hence $V(x, w, \alpha) = \sup \{g(z, \theta) : (z, \theta)\, p\, (x, w, \alpha)\} \leqq \zeta$.

Ad (iv): Clearly, $g(x, \alpha) \in \{g(z, \theta) : (z, \theta)\, p\, (x, w, \alpha)\}$, hence $g(x, \alpha) \leqq V(x, w, \alpha)$.

Now, let there exist maps (1) having properties (i) to (iv). First define a map $\zeta_0 : R \to I$ with $0 < \zeta_0(\alpha) < \sup \{\zeta \in I : \omega_0(\alpha, a(\zeta)) < \delta(\alpha)\}$, and a map ω from definition 2.3. as follows

(6) $$\omega(\alpha, \zeta) = \omega_0(\alpha, a(\zeta)) \quad \text{for} \quad \zeta \in (0, \zeta_0(\alpha)\rangle\,,$$

(7) $$\omega(\alpha, \zeta) = \omega_0(\alpha, a(\zeta_0(\alpha)) \quad \text{for} \quad \zeta \in \langle \zeta_0(\alpha), 1\rangle\,.$$

Now, it may be easily shown that for each $(\theta, x, w, \alpha) \in D$, $g(x, \alpha) \leqq \omega(\alpha, \zeta)$,

$(z, \theta)\, p\, (x, w, \alpha)$ and for some $w^1 \in W$ with $(z, w^1, \theta)\, t\, (x, w, \alpha)$ there holds

$$a(g(z, \theta)) \leqq V(z, w^1, \theta) \leqq V(x, w, \alpha) \leqq a(\zeta)\,.$$

Hence $g(z, \theta) \leqq \zeta$ and m is strongly stable.

2.6. Remark. Condition (iv) in Theorem 2.5. may be replaced by the condition

(iv)′ $(x, w, \alpha) \in$ domain V, $V(x, w, \alpha) \leqq a(\zeta)$ imply $g(x, \alpha) \leqq \zeta$.

To prove this assertion it is sufficient to use the proof of Theorem 2.5. with the following two changes. The part Ad (iv) should be replaced by the following text Ad (iv)′: If $(x, w, \alpha) \in$ domain V such that $V(x, w, \alpha) \leqq a(\zeta) = \zeta$, then, using the evident relation $g(x, \alpha) \leqq V(x, w, \alpha)$ one obtains $g(x, \alpha) \leqq \zeta$, hence (iv)′ follows.

In the second part of the proof it suffices to define the map ω again by 2.5.(6) and 2.5.(7). Now, let there be given $(\theta, x, w, \alpha) \in D$, $g(x, \alpha) \leqq \omega(\alpha, \zeta)$ and (z, θ) such that $(z, w^1, \theta)\, t\, (x, w, \alpha)$ holds for some $w^1 \in W$. Then, using 2.5.(6) and 2.5.(7) one easily obtains $V(z, w^1, \theta) \leqq V(x, w, \alpha) \leqq a(\zeta)$, hence, according to (iv)′, $g(z, \theta) \leqq \zeta$.

2.7. Definition. m is said to be *uniformly strongly stable* with respect to p iff there exists a map

(1) $$\psi : I \to I$$

such that

(2) $(\theta, x, w, \alpha) \in D\,,\quad g(x, \alpha) \leqq \psi(\zeta)\,,\quad (z, \theta)\, p\, (x, w, \alpha)$ imply $g(z, \theta) \leqq \zeta\,.$

2.8. Theorem. *m is uniformly strongly stable with respect to p iff there exist maps*

(1) $V : E \to I^0$, $a, b : I^0 \to I^0$, *a increasing, b nondecreasing, $b(r) \to 0$ as $r \to 0$, and a real $\delta \in I$ with the following properties*:

(i) *V is a Liapunov function;*

(ii) domain $V = \{(x, w, \alpha) \in E : (x, w, \alpha)\, t\, (x^1, w^1, \alpha^1)$ *for some* $(x^1, w^1, \alpha^1) \in E$ *with* $g(x^1, \alpha^1) \leqq \delta\}$;

(iii) $(x, w, \alpha) \in$ domain V *implies* $a(g(x, \alpha)) \leqq V(x, w, \alpha) \leqq b(g(x, \alpha))$.

Proof. Let m be uniformly strongly stable. Define a partial map V by 2.5. (3) with $\delta(\alpha) = \psi(1)$ for all α. Clearly, V has properties (i) and (ii). Further, it is possible to show that the map ψ in 2.7. may be chosen increasing and continuous, hence the map b may be defined by the relations

$$b(r) = \psi^{-1}(r) \quad \text{for} \quad r \in \langle 0, \psi(1) \rangle\,,$$
$$b(r) = 1 \quad \text{for} \quad r \in \langle \psi(1), 1 \rangle\,.$$

The definition of V gives directly that for each $(x, w, \alpha) \in$ domain V there follows $g(x, \alpha) \leqq 1$ and $V(x, w, \alpha) \leqq 1$. Thus, it is sufficient to show that $V(x, w, \alpha) \leqq b(g(x, \alpha))$ holds for each $(x, w, \alpha) \in$ domain V with $g(x, \alpha) \leqq \psi(1)$. Given $(x, w, \alpha) \in$ domain V, there is $\zeta \in I$ such that $g(x, \alpha) = \psi(\zeta)$. Hence for each (z, θ) with $(z, \theta)\, p\, (x, w, \alpha)$ it holds $g(z, \theta) \leqq \zeta = \psi^{-1}(g(x, \alpha)) = b(g(x, \alpha))$, hence $V(x, w, \alpha) \leqq b(g(x, \alpha))$. Defining $a(r) = r$ for each $r \in I$, we may see that the maps V, a, b have also property (iii).

Let now exist maps (1) with properties (i) to (iii). Define the map ψ from 2.7.(1) so that the relations

$$(2) \qquad 0 < b(\psi(\zeta)) \leqq a(\zeta) \quad \text{for} \quad \zeta \in (0, \delta\rangle\,; \quad \psi(\zeta) = \psi(\delta) \quad \text{for} \quad \zeta \in \langle \delta, 1\rangle$$

hold. Then, for each $(x, w, \alpha) \in E$ with $g(x, \alpha) \leqq \psi(\zeta)$, and for each (z, θ) with $(z, w^1, \theta)\, t\, (x, w, \alpha)$ for some $w^1 \in W$, there hold

$$a(g(z, \theta)) \leqq V(z, w^1, \theta) \leqq V(x, w, \alpha) \leqq b(g(x, \alpha)) \leqq b(\psi(\zeta)) \leqq a(\zeta)\,,$$

hence $g(z, \theta) \leqq \zeta$, i.e. m is uniformly strongly stable.

2.9. Remark. Condition (iii) in Theorem 2.8. may be replaced by the following two conditions:

(iii)′ $(x, w, \alpha) \in$ domain V *implies* $a(g(x, \alpha)) \leqq V(x, w, \alpha)$;

(iv) $(x, w, \alpha) \in$ domain V, $g(x, \alpha) \leqq b(\zeta)$ *imply* $V(x, w, \alpha) \leqq \zeta$.

Proof. In the proof of the first implication it suffices to prove only property (iv), as property (iii)′ is contained in 2.8. (iii). Property (iv) follows, of course, directly from the definitions of the uniform strong stability and the partial map V; since, defining $b(\zeta) = \psi(\zeta)$ for each $\zeta \in I$, from $(x, w, \alpha) \in$ domain V, $g(x, \alpha) \leqq b(\zeta) \leqq \psi(\zeta)$ and $(z, w^1, \theta)\, t\, (x, w, \alpha)$ it follows $g(z, \theta) \leqq \zeta$, hence $V(x, w, \alpha) \leqq \zeta$.

Let now there exist maps 2.8. (1) with properties 2.8. (i), 2.8. (ii), (iii)′ and (iv). Taking $\zeta_0 \in (0, \sup \{\zeta \in I : b(a(\zeta)) < \delta\})$, define the map ψ from 2.7. (1) as follows:

$$\psi(\zeta) = b(a(\zeta)) \quad \text{for} \quad \zeta \in \langle 0, \zeta_0\rangle\,,$$
$$\psi(\zeta) = b(a(\zeta_0)) \quad \text{for} \quad \zeta \in \langle \zeta_0, 1\rangle\,.$$

Then, according to (iv), the relations $(x, w, \alpha) \in$ domain V and $g(x, \alpha) \leqq \psi(\zeta)$ give $V(x, w, \alpha) \leqq a(\zeta)$, hence for each (z, θ) with $(z, w^1, \theta)\, t\, (x, w, \alpha)$ it holds

$$a(g(z, \theta)) \leqq V(z, w^1, \theta) \leqq V(x, w, \alpha) \leqq a(\zeta)\,.$$

Thus $g(z, \theta) \leqq \zeta$ takes place and m is uniformly strongly stable.

2.10. Remark. Condition (iii)′ in the preceding Remark may be replaced by the following condition:

(iii)″ $(x, w, \alpha) \in$ domain V, $V(x, w, \alpha) \leqq a(\zeta)$ *imply* $g(x, \alpha) \leqq \zeta$.

2.11. Theorem. *Let a control process p admit a period $\tau > 0$ and let the map g from* 2.1. (2) *be periodic in the second variable with the period τ. If m is strongly stable and if there are maps $\mu : I \to I$ and $\omega : R \times I \to I$ such that ω satisfies Definition* 2.3. *and $\mu(\zeta) \leqq \omega(\alpha, \zeta)$ for each $\alpha \in R$ and $\zeta \in I$, then m is uniformly strongly stable and there exists a Liapunov function V periodic with respect to the last variable with the period τ, having properties* 2.8. (ii) *to* (iii), 2.9 (iii)′ *to* (iv) *and* 2.10. (iii)″.

Proof is trivial.

2.12. Definition. m is said to be *asymptotically strongly stable* with respect to p iff there exist maps

(1) $$\omega : R \times I \to I\,, \quad \Omega : R \to I\,, \quad T : P \times R \times I \to \mathsf{R}^+$$

such that

(2) $(\theta, x, w, \alpha) \in D\,, \quad g(x, \alpha) \leqq \omega(\alpha, \zeta)\,, \quad (z, \theta)\, p\, (x, w, \alpha)$ imply $g(z, \theta) \leqq \zeta\,;$

(3) $(\theta, x, w, \alpha) \in D\,, \; g(x, \alpha) \leqq \Omega(\alpha)\,, \; \theta \geqq \alpha + T(x, \alpha, \zeta)\,, \; (z, \theta)\, p\, (x, w, \alpha)$ imply $g(z, \theta) \leqq \zeta\,.$

2.13. Theorem. *m is asymptotically strongly stable with respect to p iff there exist maps* 2.5. (1) *with properties* 2.5. (i) *to* 2.5. (iv) *and a partial map $T_0 : P \times \times R \times I \to \mathsf{R}^+$ such that*

(v) $(x, w, \alpha) \in$ domain V, $g(x, \alpha) \leqq \delta(\alpha)\,, (z, w^1, \theta)\, t\, (x, w, \alpha)\,, \theta \geqq \alpha + T_0(x, \alpha, \zeta)$ *imply* $V(z, w^1, \theta) \leqq \zeta$.

Proof. Let m be asymptotically strongly stable. Then m is strongly stable and according to Theorem 2.5. there exist maps 2.5. (1) with properties 2.5. (i) to 2.5. (iv). Clearly, the map δ may be taken so that $\delta(\alpha) = \min\{\Omega(\alpha), \omega(\alpha, 1)\}$ holds. Now, define a partial map $T_0 : P \times R \times I \to \mathsf{R}^+ : T_0(x, \alpha, \zeta) = T(x, \alpha, \zeta)$. Then, according to 2.12. (3), for each $(x, w, \alpha) \in$ domain V and $(z, w^1, \theta)\, t\, (x, w, \alpha)$ with $\theta \geqq \alpha + T_0(x, \alpha, \zeta)$ there holds $g(z, \theta) \leqq \zeta$, and as for each $(z^1, \theta^1)\, p\, (z, w^1, \theta)$ there hold $\theta^1 \geqq \theta \geqq \alpha + T_0(x, \alpha, \zeta)$ and $(z^1, \theta^1)\, p\, (x, w, \alpha)$, one has also $V(z, w^1, \theta) \leqq \zeta$, hence $g(z^1, \theta^1) \leqq \zeta$.

In proving the implication in the opposite direction it is sufficient to define maps 2.12. (1) by the relations 2.5. (6) and 2.5. (7), $\Omega(\alpha) = \min\{\omega_0(\alpha, 1), \delta(\alpha)\}$, and $T(x, \alpha, \zeta) = T_0(x, \alpha, a(\zeta))$.

2.14. Theorem. *Let p be a global solution complete control process of a control system t. Let there exist maps* 2.5. (1) *with properties* 2.5. (i) *to* 2.5. (iv) *and a map $c : I^0 \to I$, nondecreasing, with the following property*:

(v) $V(s(\theta), \theta) - V(s(\alpha), \alpha) \leqq - \int_\alpha^\theta c(V(s(\sigma), \sigma))\, \mathrm{d}\sigma$ *holds for each solution s of t and each $\alpha, \theta \in$ domain s with $\alpha \leqq \theta$.*

Then m is asymptotically strongly stable.

Proof. Evidently, according to Theorem 2.5. m is strongly stable, so that it remains to prove the existence of maps Ω and T from 2.12. (1) satisfying 2.12. (3). Suppose that these maps do not exist. Then corresponding to each map $\Omega : R \to I$ there exist $(x_\Omega, w_\Omega, \alpha_\Omega)$ and ζ_Ω with $g(x_\Omega, \alpha_\Omega) \leqq \Omega(\alpha_\Omega)$ and corresponding to each $\beta > 0$ there exist $\gamma \geqq \beta$ and a solution $s_{\Omega,\beta}$ of t such that

$$s_{\Omega,\beta}(\alpha_\Omega) = (x_\Omega, w_\Omega), \quad g(\mathrm{proj}_1 \circ s_{\Omega,\beta}(\alpha_\Omega + \gamma), \quad \alpha_\Omega + \gamma) > \zeta_\Omega .$$

Now, given a map Ω, take β so that for each $\gamma \geqq \beta$ it holds

$$\gamma > \frac{V(x_\Omega, w_\Omega, \alpha_\Omega)}{c(a(\zeta_\Omega))} . \tag{1}$$

Since V is non-decresing along t, it holds for each $\theta \in \langle \alpha_\Omega, \alpha_\Omega + \gamma \rangle$

$$V(s_{\Omega,\beta}(\theta), \theta) \geqq V(s_{\Omega,\beta}(\alpha_\Omega + \gamma), \alpha_\Omega + \gamma) \geqq$$
$$\geqq a(g(\mathrm{proj}_1 \circ s_{\Omega,\beta}(\alpha_\Omega + \gamma), \alpha_\Omega + \gamma)) > a(\zeta_\Omega) .$$

Hence, using (v), one obtains

$$V(x_\Omega, w_\Omega, \alpha_\Omega) - V(s_{\Omega,\beta}(\alpha_\Omega + \gamma), \alpha_\Omega + \gamma) \geqq \int_{\alpha_\Omega}^{\alpha_\Omega + \gamma} c(V(s_{\Omega,\beta}(\sigma), \sigma))\, \mathrm{d}\sigma =$$
$$= \int_{\alpha_\Omega}^{\alpha_\Omega + \gamma} c(a(\zeta_\Omega))\, \mathrm{d}\sigma = c(a(\zeta_\Omega)) \,.\, \gamma,$$

from which there easily follows the relation

$$\gamma \leqq \frac{V(x_\Omega, w_\Omega, \alpha_\Omega)}{c(a(\zeta_\Omega))}$$

contradicting (1) and proving the asymptotic strong stability of m.

2.15. Remark. In the preceding two Theorems it is possible to replace condition 2.5. (iv) by condition 2.6. (iv)'.

2.16. Definition. m is said to be *uniformly asymptotically strongly stable* with respect to p iff there exist

$$\psi : I \to I\,, \quad T : I \to \mathrm{R}^+\,, \quad \Omega \in \mathrm{R}^+ \tag{1}$$

such that

(2) $(\theta, x, w, \alpha) \in D\,, \quad g(x, \alpha) \leqq \psi(\zeta)\,, \quad (z, \theta)\, p\, (x, w, \alpha)$ imply $g(z, \theta) \leqq \zeta$;

(3) $(\theta, x, w, \alpha) \in D\,, \quad g(x, \alpha) \leqq \Omega\,, \quad \theta \geqq \alpha + T(\zeta)\,, \quad (z, \theta)\, p\, (x, w, \alpha)$ imply $g(z, \theta) \leqq \zeta$.

2.17. Theorem. *m is uniformly asymptotically strongly stable with respect to p iff there exist maps* 2.8. (1) *with properties* 2.8. (i) *to* 2.8. (iii) *and a map* $T_0 : I \to \mathsf{R}^+$ *such that*

(iv) $(x, w, \alpha) \in \text{domain } V$, $(z, w^1, \theta)\, t\, (x, w, \alpha)$, $\theta \geqq \alpha + T_0(\zeta)$ *imply* $V(z, w^1, \theta) \leqq \zeta$.

Proof. Let m be uniformly asymptotically strongly stable. The existence of the maps 2.8. (1) with properties 2.8. (i) to 2.8. (iii) follows from 2.8. Taking $\delta \in \in (0, \min \{\Omega, \psi(1)\})$, one may choose as a map T_0 the map T from Definition 2.16. Then, clearly, $(x, w, \alpha) \in \text{domain } V$, $\theta \geqq \alpha + T_0(\zeta)$, $(z, w^1, \theta)\, t\, (x, w, \alpha)$ imply $g(z, \theta) \leqq \leqq \zeta$, whence also $V(z, w^1, \theta) \leqq \zeta$.

To prove the opposite direction implication it is sufficient to take ψ satisfying 2.8. (2), $\Omega = \min \{\delta, \psi(1)\}$ and $T : I \to \mathsf{R}^+ : T(\zeta) = T_0(a(\zeta))$.

2.18. Theorem. *Let p be a global solution complete control process of a control system t. Let there exist maps* 2.8. (1) *with properties* 2.8. (i) *to* 2.8. (iii) *and a map $c : I^0 \to I$, non-decresing, with the following property*:

(iv) $V(s(\theta), \theta) - V(s(\alpha), \alpha) \leqq -\int_\alpha^\theta c(g(\text{proj}_1 \circ s(\sigma), \sigma))\, d\sigma$, *for each solution s of t and each* $\alpha, \theta \in \text{domain } s$ *with* $\alpha \leqq \theta$.

Then m is uniformly asymptotically strongly stable.

Proof. According to Theorem 2.8., m is uniformly strongly stable. Define

$$\Omega = \delta\,; \quad T : I \to \mathsf{R}^+ : T(\zeta) = \frac{b(\Omega)}{c(\psi(\zeta))}. \tag{1}$$

Let there be given $\zeta \in I$. Suppose there exist $(\theta, x, w, \alpha) \in D$ with $g(x, \alpha) \leqq \Omega$ and $\theta \geqq \alpha + T(\zeta)$ with $g(z, \theta) > \zeta$ for some $(z, w^1, \theta)\, t\, (x, w, \alpha)$. Let s be a solution of t with $s(\alpha) = (x, w)$, $s(\theta) = (z, w^1)$ and let there exist $\gamma \in \langle \alpha, \theta \rangle$ such that $g(\text{proj}_1 \circ \circ s(\gamma), \gamma) \leqq \psi(\zeta)$. Then from $(z, w^1, \theta)\, t\, (s(\gamma), \gamma)$ there follows $g(\text{proj}_1 \circ s(\theta), \theta) \leqq \leqq g(z, \theta) \leqq \zeta$, contradicting the assumption. Thus it must hold $g(\text{proj}_1 \circ s(\gamma), \gamma) > > \psi(\zeta)$ for each $\gamma \in \langle \alpha, \theta \rangle$. Hence, using (iv), there follows

$$V(s(\theta), \theta) \leqq V(s(\alpha), \alpha) - \int_\alpha^\theta c(g(\text{proj}_1 \circ s(\sigma), \sigma))\, d\sigma < b(\Omega) - c(\psi(\zeta)) \,.\, T(\zeta) = 0\,,$$

which contradicts the non-negativeness of V. Thus Ω and T have to satisfy Definition 2.16. and m is uniformly asymptotically strongly stable.

2.19. Remark. Evidently, in both preceding Theorems condition 2.8. (iii) may be replaced by 2.9. (iii)′ and 2.9. (iv) or by 2.10. (iii)″ and 2.9. (iv).

3.1. Notation. We continue to use notation introduced in the preceding text. Especially, t will denote a control system on $P \times W$, p the corresponding control process. Also in this chapter m and g will denote the set 2.1. (1) and the map 2.1. (2) satisfying 2.1. (3). For a given $(x, w, \alpha) \in E$ let $\Sigma(x, w, \alpha)$ denote the set of all solutions σ of t such that $\sigma(\alpha) = (x, \alpha)$ and let $\Sigma_0(x, w, \alpha)$ denote the set $\{\sigma \in \Sigma(x, w, \alpha) : : \text{domain } \sigma \supset \langle \alpha, \varepsilon(x, w, \alpha))\}$. Similarly, symbols $S(x, w, \alpha)$ resp. $S_0(x, w, \alpha)$ are defined thus: $s \in S(x, w, \alpha)$ iff $s = \text{proj}_1 \circ \sigma$ for some $\sigma \in \Sigma(x, w, \alpha)$, and $s \in S_0(x, w, \alpha)$ iff $s = \text{proj}_1 \circ \sigma$ for some $\sigma \in \Sigma_0(x, w, \alpha)$.

3.2. Definition. m is said to be *weakly stable* with respect to p iff there exists a map

(1) $$\omega : R \times I \to I$$

such that

(2) $(x, w, \alpha) \in D$, $g(x, \alpha) \leqq \omega(\alpha, \zeta)$ imply $g(s(\theta), \theta) \leqq \zeta$ for some $s \in S_0(x, w, \alpha)$ and all $\theta \in \langle \alpha, \varepsilon(x, w, \alpha))$.

3.3. Remark and notation. If m is weakly stable with respect to p, then it may be shown that the map ω in 3.2. (1) may be chosen so that for each $\alpha \in R$ the map ω is continuous and increasing in the variable ζ on the interval I (i.e. for each $\alpha \in R$ the map $\omega_\alpha : I \to I : \omega_\alpha(\zeta) = \omega(\alpha, \zeta)$ has the both properties).

Now, suppose m to be weakly stable and ω to be continuous and increasing in ζ on I. Given $(x, w, \alpha) \in E$ with $0 < g(x, \alpha) \leqq \omega(\alpha, 1)$, there exists $\zeta \in I$ such that $g(x, \alpha) = \omega(\alpha, \zeta)$, and according to Definition 3.2. there exists at least one solution s such that $s(\alpha) = x$, domain $s \supset \langle \alpha, \varepsilon(x, w, \alpha))$ and $g(s(\theta), \theta) \leqq \zeta$ for each $\theta \in \in \langle \alpha, \varepsilon(x, w, \alpha))$. Denote $S_1(x, w, \alpha)$ the set of all solutions s of the control process p such that $s(\alpha) = x$, domain $s \supset \langle \alpha, \varepsilon(x, w, \alpha))$ and $g(s(\theta), \theta) \leqq \zeta$ for all $\theta \in \in \langle \alpha, \varepsilon(x, w, \alpha))$. Similarly, define $\Sigma_1(x, w, \alpha)$ as follows: $\sigma \in \Sigma_1(x, w, \alpha)$ iff $\text{proj}_1 \circ \circ \sigma \in S_1(x, w, \alpha)$.

3.4. Theorem. *m is weakly stable with respect to p iff there exist maps*

(1) $$V : E \to I^0, \quad \delta : R \to I, \quad \omega_0 : R \times I \to I, \quad a : I^0 \to I^0$$

such that $\omega_0(\alpha, \zeta) \to 0$ as $\zeta \to 0$ and $\alpha \in R$; a increasing, $a(r) \to 0$ as $r \to 0$, with the following properties:

(i) *V is a weak Liapunov function* (see 1.20);
(ii) domain $V = \{(\sigma(\theta), \theta) : \sigma \in \Sigma_0(x, w, \alpha), \theta \in \langle \alpha, \varepsilon(x, w, \alpha))$ *for all* $(x, w, \alpha) \in E$ *with* $g(x, \alpha) \leqq \delta(\alpha)\}$;
(iii) $(x, w, \alpha) \in$ domain V, $g(x, \alpha) \leqq \omega_0(\alpha, \zeta)$ *imply* $V(x, w, \alpha) \leqq \zeta$;
(iv) $(x, w, \alpha) \in$ domain V *implies* $a(g(x, \alpha)) \leqq V(x, w, \alpha)$.

Proof. Let m be weakly stable. Define a map

(2) $$\delta : R \to I : \delta(\alpha) = \omega(\alpha, 1)$$

and a partial map

(3) $$V : E \to I^0 : V(x, w, \alpha) =$$
$$= \sup \{g(s(\theta), \theta) : s \in S_1(x, w, \alpha), \theta \in \langle \alpha, \varepsilon(x, w, \alpha))\},$$

whenever there exists $(x^1, w^1, \alpha^1) \in E$ such that $g(x^1, \alpha^1) \leqq \delta(\alpha^1)$ and for some $\sigma \in \Sigma_1(x^1, w^1, \alpha^1)$ there holds $\sigma(\alpha) = (x, w)$. It is easily seen that V defined by (3) has property (ii). Defining

(4) $$\omega_0 : R \times I \to I : \omega_0(\alpha, \zeta) = \omega(\alpha, \zeta),$$

one obtains $g(s(\theta), \theta) \leqq \zeta$ for each $s \in S_1(x, w, \alpha)$ and $\theta \in \langle \alpha, \varepsilon(x, w, \alpha))$ with $(x, w, \alpha) \in \in E$ and ζ satisfying $g(x, \alpha) \leqq \delta(\alpha)$ and $g(x, \alpha) \leqq \omega_0(\alpha, \zeta)$. Hence, according to (3), $V(x, w, \alpha) \leqq \zeta$ and (iii) holds.

Define now

(5) $$a : I^0 \to I^0 : a(r) = r .$$

Cleraly, for each $(x, w, \alpha) \in$ domain V there holds

$$g(x, \alpha) \in \{g(s(\theta), \theta) : s \in S_1(x, w, \alpha), \theta \in \langle \alpha, \varepsilon(x, w, \alpha))\},$$

hence $g(x, \alpha) = a(g(x, \alpha)) \leqq V(x, w, \alpha)$, i.e. (iv) takes place.

It remains to prove that V is a weak Liapunov function of the control process p. Given $(x, w, \alpha) \in$ domain V, there exists $\sigma_0 \in \sum_1(x, w, \alpha)$ such that $(\sigma_0(\theta), \theta) \in$ domain V for each $\theta \in \langle \alpha, \varepsilon(x, w, \alpha))$. Further, for each couple β, γ such that $\alpha \leqq \beta \leqq \leqq \gamma < \varepsilon(x, w, \alpha)$ there hold the following relations:

$$\sigma_0 \in \Sigma_1(\sigma_0(\beta), \beta), \quad \sigma_0 \in \Sigma_1(\sigma_0(\gamma), \gamma),$$
$$\{(\sigma(\theta), \theta) : \sigma \in \Sigma_1(\sigma_0(\gamma), \gamma), \gamma \leqq \theta < \varepsilon(x, w, \alpha)\} \subset$$
$$\subset \{(\sigma(\theta), \theta) : \sigma \in \Sigma_1(\sigma_0(\beta), \beta), \beta \leqq \theta < \varepsilon(x, w, \alpha)\}.$$

Hence easily follows

$$V(\sigma_0(\gamma), \gamma) = \sup \{g(s(\theta), \theta) : s \in S_1(\sigma_0(\gamma), \gamma), \gamma \leqq \theta < \varepsilon(\sigma_0(\gamma), \gamma)\} \leqq$$
$$\leqq \sup \{g(s(\theta), \theta) : s \in S_1(\sigma_0(\beta), \beta), \beta \leqq \theta < \varepsilon(\sigma_0(\beta), \beta)\} = V(\sigma_0(\beta), \beta),$$

thus V is a weak Liapunov function of the control process p.

Let now exist maps (1) with properties (i) to (iv). First define a map $\zeta_0 : R \to I$ satisfying

(6) $$0 < \zeta_0(\alpha) < \sup \{\zeta \in I : \omega_0(\alpha, a(\zeta)) < \delta(\alpha)\}$$

and the map ω from Definition 2.3. by

$$(7) \qquad \omega(\alpha, \zeta) = \omega_0(\alpha, a(\zeta)) \quad \text{for} \quad \zeta \in (0, \zeta_0(\alpha)\rangle ,$$

$$(8) \qquad \omega(\alpha, \zeta) = \omega_0(\alpha, a(\zeta_0(\alpha))) \quad \text{for} \quad \zeta \in \langle\zeta_0(\alpha), 1\rangle .$$

Now, given $(x, w, \alpha) \in E$ with $g(x, \alpha) \leqq \omega(\alpha, \zeta)$, according to (6) to (8) one has $g(x, \alpha) \leqq \delta(\alpha)$, hence $(x, w, \alpha) \in$ domain V. According to (i) and Definition 1.20. there exists $\sigma \in \Sigma_0(x, w, \alpha)$ such that V is non-decreasing along σ. Now it will be shown that $s = \text{proj}_1 \circ \sigma$ satisfies Definition 3.2. Clearly, $a(g(s(\theta), \theta)) \leqq V(\sigma(\theta), \theta) \leqq \leqq V(\sigma(\alpha), \alpha)$ holds for each $\theta \in \langle\alpha, \varepsilon(x, w, \alpha))$. According to (7), (8), (iii) and $g(x, \alpha) \leqq \leqq \omega(\alpha, \zeta)$ there hold $V(\sigma(\alpha), \alpha) = V(x, w, \alpha) \leqq a(\zeta)$, hence, as a is increasing, there follows $g(s(\theta), \theta) \leqq \zeta$. Thus m is weakly stable.

3.5. Remark. Condition (iv) in Theorem 3.4. may be replaced by

(iv)′ $(x, w, \alpha) \in$ domain V, $V(x, w, \alpha) \leqq a(\zeta)$ *imply* $g(x, \alpha) \leqq \zeta$.

3.6. Definition. m is said to be *uniformly weakly stable* with respect to p iff there exists a map

$$(1) \qquad \psi : I \to I$$

such that

(2) $(x, w, \alpha) \in D$, $g(x, \alpha) \leqq \psi(\zeta)$ imply $g(s(\theta), \theta) \leqq \zeta$ for some $s \in S_0(x, w, \alpha)$ and all $\theta \in \langle\alpha, \varepsilon(x, w, \alpha))$.

3.7. Theorem. *m is uniformly weakly stable with respect to p iff there exist*

$$(1) \qquad \delta \in I , \quad V : E \to I^0 , \quad a, b : I^0 \to I^0 ;$$

a increasing, b non-decreasing, $b(r) \to 0$ *as* $r \to 0$, *with the following properties*:

(i) *V is a weak Liapunov function*;

(ii) domain $V = \{(\sigma(\theta), \theta) : \sigma \in \Sigma_0(x, w, \alpha), \theta \in \langle\alpha, \varepsilon(x, w, \alpha))$ *for each* $(x, w, \alpha) \in \in E$ *with* $g(x, \alpha) \leqq \delta\}$;

(iii) $(x, w, \alpha) \in$ domain V *implies* $a(g(x, \alpha)) \leqq V(x, w, \alpha) \leqq b(g(x, \alpha))$.

Proof. Let m be uniformly weakly stable. Set $\delta = \psi(1)$ and define a partial map V by 3.4. (3) with $\delta(\alpha) = \psi(1)$ for all $\alpha \in R$. Then V has properties (i) and (ii). If the map a is defined by 3.4. (5), then $a(g(x, \alpha)) \leqq V(x, w, \alpha)$ for all $(x, w, \alpha) \in$ domain V. To prove the remaining part of (iii), one may suppose ψ to be continuous and increasing on I, so that ψ^{-1} exists. The map b may be now defined by $b(r) = \psi^{-1}(r)$ for $r \in \in (0, \psi(1)\rangle$, $b(r) = 1$ for $r \in \langle\psi(1), 1\rangle$. Clearly, $V(x, w, \alpha) \leqq 1$ for each $(x, w, \alpha) \in \in$ domain V; thus it suffices to prove the relation $V(x, w, \alpha) \leqq b(g(x, \alpha))$ for $(x, w, \alpha) \in \in$ domain V with $g(x, \alpha) \leqq \psi(1)$. Then, of course, there exists $\zeta \in I$ such that $g(x, \alpha) =$

$= \psi(\zeta)$, so that $g(s(\theta), \theta) \leqq \zeta = \psi^{-1}(g(x, \alpha))$ holds for each $s \in S_1(x, w, \alpha)$ and $\theta \in \in \langle\alpha, \varepsilon(x, w, \alpha))$. Hence $V(x, w, \alpha) \leqq b(g(x, \alpha))$.

Now, let there exist maps (1) with properties (i) to (iii), let $\psi : I \to I$ satisfy $b(\psi(\zeta)) \leqq \leqq a(\zeta)$ for $0 < \zeta \leqq \delta$, $\psi(\zeta) = \psi(\delta)$ for $\delta \leqq \zeta \leqq 1$, and given $(x, w, \alpha) \in E$ with $g(x, \alpha) \leqq \psi(\zeta)$ let $\sigma \in \Sigma_0(x, w, \alpha)$ be such that V is non-increasing along σ. Denote $s = \text{proj}_1 \circ \sigma$. Then for each $\theta \in \langle\alpha, \varepsilon(x, w, \alpha))$ there holds

$$a(g(s(\theta), \theta)) \leqq V(\sigma(\theta), \theta) \leqq V(x, w, \alpha) \leqq b(g(x, \alpha)) \leqq b(\psi(\zeta)) \leqq a(\zeta),$$

hence $g(s(\theta), \theta) \leqq \zeta$ and m is uniformly weakly stable.

3.8. Remark. Condition (iii) in the preceding theorem may be replaced by 2.9. (iii)′ and 2.9. (iv), or by 2.10. (iii)″ and 2.9. (iv). From the proof of Theorem 3.7. it is directly seen that Definition 3.6. is satisfied by each solution along which the function V is non-increasing.

3.9. Theorem. *Let a control process p admit a period $\tau > 0$ and let the map g from* 2.1. (2) *be τ-periodic with respect to the second variable. If m is weakly stable and if there exist maps $\omega : R \times I \to I$ and $\mu : I \to I$ such that ω satisfies Definition* 3.2. *and $\mu(\zeta) \leqq \omega(\alpha, \zeta)$ holds for each $\alpha \in R$ and $\zeta \in I$, then m is uniformly weakly stable and there exists a weak Liapunov function V, τ-periodic in the last variable, with the properties* 3.4. (ii) *to* 3.4. (iv), 2.9. (iii)′, 2.9. (iv) *and* 2.10. (iii)′.

Proof is trivial.

3.10. Definition. m is said to be *asymptotically weakly stable* with respect to p iff there exist maps

$$\omega : R \times I \to I\,, \quad \Omega : R \to I\,, \quad T : P \times R \times I \to \mathsf{R}^+ \tag{1}$$

such that corresponding to each $(x, w, \alpha) \in E$ there exists $s \in S_0(x, w, \alpha)$ with the following properties:

(i) if $g(x, \alpha) \leqq \Omega(\alpha)$ and $\theta \geqq \alpha + T(x, \alpha, \zeta)$ then $g(s(\theta), \theta) \leqq \zeta$;
(ii) if $g(x, \alpha) \leqq \omega(\alpha, \zeta)$ and $\theta \geqq \alpha$ then $g(s(\theta), \theta) \leqq \zeta$.

3.11. Theorem. *m is asymptotically weakly stable with respect to p iff there exist maps* 3.4. (1) *with properties* 3.4. (i) *to* 3.4. (iv), *and a partial map*

$$T_0 : P \times R \times I \to \mathsf{R}^+ \tag{1}$$

such that

(v) $(x, w, \alpha) \in$ domain V, $g(x, \alpha) \leqq \Omega(\alpha)$ *imply* $V(\sigma(\theta), \theta) \leqq \zeta$ *for some* $\sigma \in \in \Sigma_0(x, w, \alpha)$ *and all* $\theta \geqq \alpha + T_0(x, \alpha, \zeta)$.

Proof. Let there exist maps 3.4. (1) and 3.11. (1) with properties 3.4. (i) to 3.4. (iv) and 3.11. (v). Define maps 3.10. (1) by 3.4. (7), 3.4. (8), $\Omega(\alpha) = \min \{\delta(\alpha), \omega_0(\alpha, 1)\}$, $T(x, \alpha, \zeta) = T_0(x, \alpha, a(\zeta))$, respectively. Clearly, according to Theorem 3.4., condition 3.10. (ii) is satisfied. To prove 3.10. (i), take $(x, w, \alpha) \in E$ with $g(x, \alpha) \leqq \Omega(\alpha)$ and a solution $\sigma \in \Sigma_0(x, w, \alpha)$ satisfying condition (v). Then $V(\sigma(\theta), \theta) \leqq a(\zeta)$ for each $\theta \geqq \alpha + T_0(x, \alpha, a(\zeta))$, hence, applying 3.4. (iv), one has for $s = \mathrm{proj}_1 \circ \sigma$

$$a(g(s(\theta), \theta)) \leqq V(\sigma(\theta), \theta) \leqq a(\zeta),$$

so that $g(s(\theta), \theta) \leqq \zeta$ and 3.10. (i) is fulfilled.

3.12. Theorem. *Let there be given a control process p, let there exist maps* 3.4. (1) *with properties* 3.4. (ii) *to* 3.4. (iv) *and let there exist non-decreasing map*

$$(1) \qquad c : I^0 \to I^0$$

such that

(i)′ *corresponding to each* $(x, w, \alpha) \in \text{domain}\, V$ *there exists* $\sigma \in \Sigma_0(x, w, \alpha)$ *such that for each* $\alpha \leqq \beta \leqq \gamma$ *there holds*

$$V(\sigma(\gamma), \gamma) - V(\sigma(\beta), \beta) \leqq -\int_\beta^\gamma c(V(\sigma(\theta), \theta))\, d\theta .$$

Then m is asymptotically weakly stable.

Proof. Since (i)′ gives directly $V(\sigma(\gamma), \gamma) \leqq V(\sigma(\beta), \beta)$ for each $\alpha \leqq \beta \leqq \gamma$, condition 3.4. (i) is satisfied and m is weakly stable according to Theorem 3.4. Now, let us prove that there exist maps Ω and T from 3.10. (1) such that corresponding to each $(x, w, \alpha) \in E$ with $g(x, \alpha) \leqq \Omega(\alpha)$ there exists $\sigma \in \Sigma_0(x, w, \alpha)$ satisfying (i)′ and 3.10. (2). Suppose that such maps Ω and T do not exist. Then, corresponding to each $\Omega : R \to I$ there exist $(x, w, \alpha) \in E$ and ζ with the following properties:

$$g(x, \alpha) \leqq \Omega(\alpha);$$

corresponding to each $\beta > 0$ and $\sigma \in \Sigma_0(x, w, \alpha)$ satisfying (i)′ there exists $\gamma_\sigma \geqq \beta$ such that $g(s(\alpha + \gamma_\sigma), \alpha + \gamma_\sigma) > \zeta$, where $s = \mathrm{proj}_1 \circ \sigma$.

Given Ω, take β so that

$$(2) \qquad \gamma_\sigma \geqq \beta > \frac{V(x, w, \alpha)}{c(a(\zeta))}$$

holds for each mentioned σ. Since V is non-decreasing along such σ, there hold, for each $\theta \in \langle \alpha, \alpha + \gamma_\sigma \rangle$,

$$V(\sigma(\theta), \theta) \geqq V(\sigma(\alpha + \gamma_\sigma), \alpha + \gamma_\sigma) \geqq a(g(s(\alpha + \gamma_\sigma), \alpha + \gamma_\sigma)) > a(\zeta).$$

Hence it follows

$$V(x, w, \alpha) \geqq V(x, w, \alpha) - V(\sigma(\alpha + \gamma_\sigma), \alpha + \gamma_\sigma) \geqq \int_\alpha^{\alpha+\gamma_\sigma} c(V(\sigma(\theta), \theta))\,\mathrm{d}\theta \geqq$$

$$\geqq \int_\alpha^{\alpha+\gamma_\sigma} c(a(\zeta))\,\mathrm{d}\theta = c(a(\zeta)) \cdot \gamma_\sigma\,,$$

which contradicts (2). Since Ω and σ were arbitrary, this contradiction proves the existence of Ω and T from 3,10. (1) satisfying 3.10. (i). Thus m is asymptotically weakly stable.

3.13. Remark. According to 3.5., condition 3.4. (iv) in both preceding Theorems may be replaced by 3.5. (iv)'.

3.14. Definition. m is said to be *uniformly asymptotically weakly stable* with respect to p iff there exist

(1) $$\psi : I \to I\,, \quad T : I \to \mathsf{R}^+\,, \quad \Omega \in \mathsf{R}^+$$

such that corresponding to each $(x, w, \alpha) \in E$ there exists $s \in S_0(x, w, \alpha)$ with the following properties:

(i) if $g(x, \alpha) \leqq \Omega$ and $\theta \geqq \alpha + T(\zeta)$ then $g(s(\theta), \theta) \leqq \zeta$;
(ii) if $g(x, \alpha) \leqq \psi(\zeta)$ and $\theta \geqq \alpha$ then $g(s(\theta), \theta) \leqq \zeta$.

3.15. Theorem. *m is uniformly asymptotically weakly stable with respect to p iff there exist maps* 3.7. (1) *with properties* 3.7. (i) *to* 3.7. (iii) *and a map*

(1) $$T_0 : I \to \mathsf{R}^+$$

such that

(iv) $(x, w, \alpha) \in$ domain V, $g(x, \alpha) \leqq \Omega$ *imply* $V(\sigma(\theta), \theta) \leqq \zeta$ *for some* $\sigma \in \Sigma_0(x, w, \alpha)$ *and all* $\theta \geqq \alpha + T_0(\zeta)$.

Proof is an easy modification of the proof of Theorem 3.11.

3.16. Remark. Condition 3.7. (iii) in Theorem 3.15. may be replaced by

(iii)' $(x, w, \alpha) \in$ domain V *implies* $a(g(x, \alpha)) \leqq V(x, w, \alpha)$;

so that the assumption of the existence of the map b from 3.7. (1) may be omitted.

3.17. Theorem. *Let there be given a control process p, let there exist maps* 3.7. (1) *with properties* 3.7. (ii) *and* 3.7. (iii), *and let there exist a non-decreasing map*

(1) $$c : I^0 \to I^0$$

such that

(i)′ *corresponding to each* $(x, w, \alpha) \in$ domain V *there exists* $\sigma \in \Sigma_0(x, w, \alpha)$ *such that for each* $\alpha \leqq \beta \leqq \gamma$ *and* $s = \mathrm{proj}_1 \circ \sigma$ *there holds*

$$V(\sigma(\gamma), \gamma) - V(\sigma(\beta), \beta) \leqq - \int_\beta^\gamma c(g(s(\theta), \theta))\, d\theta\,.$$

Then m is uniformly asymptotically weakly stable.

Proof. According to Theorem 3.7. m is uniformly weakly stable. Obviously, Definition 3.6. is satisfied by each solution satisfying (i)′ (see 3.8.). Define

(2) $$\Omega = \delta\,; \quad T : I \to \mathrm{R}^+ : T(\zeta) = \frac{b(\Omega)}{c(\psi(\zeta))}$$

and prove that each solution satisfying (i)′ satisfies also 3.14. (i). Given $\zeta \in I$, $(x, w, \alpha) \in$ $\in$ domain V with $g(x, \alpha) \leqq \Omega$, $\sigma \in \Sigma_0(x, w, \alpha)$ satisfying (i)′, $s = \mathrm{proj}_1 \circ \sigma$, suppose the existence of $\gamma \geqq \alpha + T(\zeta)$ with $g(s(\gamma), \gamma) > \zeta$. Then, according to uniform weak stability, there holds $g(s(\theta), \theta) > \psi(\zeta)$ for each $\theta \in \langle \alpha, \gamma \rangle$. Hence and from (i)′ one obtains

$$V(\sigma(\gamma), \gamma) \leqq V(\sigma(\alpha), \alpha) - \int_\alpha^\gamma c(g(s(\theta), \theta))\, d\theta < b(\Omega) - c(\psi(\zeta))\,.\, T(\zeta) = 0\,,$$

which contradicts the non-negativeness of V and finishes the proof.

3.18. Remark. According to 3.8., condition 3.7. (iii) in 3.16. may be replaced by 2.9. (iii)′ and 2.9. (iv) or 2.10. (iii)″ and 2.9. (iv).

4. SEVERAL SPECIAL STABILITY PROPERTIES

4.1. Notation. In this chapter t will denote again a control system on $P \times U \times$ $\times P \times U = P \times W$ over R, p will denote the corresponding control process, m and g the subset of $P \times R$ and the function $P \times R \to \mathrm{R}^0$, respectively, satisfying the condition $g(x, \alpha) = 0$ iff $(x, \alpha) \in m$. Moreover, p^0 will denote a control process on R^0 over R, $m^0 = \{(0, \theta) \in \mathrm{R}^0 \times R : -\infty < \theta < +\infty\}$, $g^0 : \mathrm{R}^0 \times R \to \mathrm{R}^0 : g^0(r, \theta) =$ $= r$. Instead of E_p, E_{p^0}, D_p, D_{p^0} we shall write E, E_0, D, D_0, respectively. Finally, B_0 will denote the set of all solutions of the control process p^0.

In what follows we shall be concerned with two control processes p and p^0. We formulate several conditions under which certain stability properties of p^0 induce the corresponding stability properties of p. Before setting up the theorems we formulate the following basic conditions.

4.2. Conditions. *There exist maps*

(1) $V : E \to I^0$, $s : E \to B_0$, $a, b : I^0 \to I^0$; *a, b increasing*, $a(r) \to +\infty$ *as* $r \to +\infty$, $b(r) \to 0$ *as* $r \to 0$,

such that

(i) $(\theta, x, w, \alpha) \in D$ *implies* $V(x, w, \alpha) = s(x, w, \alpha)(\alpha)$, $V(y, v, \theta) \leqq s(x, w, \alpha)(\theta)$ *for each* (y, v, θ) *with* $(y, v, \theta)\, t\, (x, w, \alpha)$;

(ii) $(\theta, x, w, \alpha) \in D$ *implies* $V(x, w, \alpha) = s(x, w, \alpha)(\alpha)$, $V(\sigma(\theta), \theta) \leqq s(x, w, \alpha)(\theta)$ *for some* $\sigma \in \Sigma_0(x, w, \alpha)$;

(iii) $(x, w, \alpha) \in E$ *implies* $a(g(x, \alpha)) \leqq V(x, w, \alpha) \leqq b(g(x, \alpha))$.

4.3. Theorem. *Let there exist maps* 4.2. (1) *satisfying* 4.2. (i) *and* 4.2. (iii). *Let* m^0 *be strongly stable with respect to* p^0. *Then* m *is strongly stable with respect to* p.

Proof. According to the assumption there exists map $\omega^0 : R \times I \to I$ such that for each $s' \in B_0$ and $\alpha, \theta \in \text{domain}\, s'$, $\theta \geqq \alpha$, from $s'(\alpha) \leqq \omega^0(\alpha, \zeta^0)$ there follows $s'(\theta) \leqq \zeta^0$. Define a map $\omega : R \times I \to I$ so that $b(\omega(\alpha, \zeta)) \leqq \omega^0(\alpha, a(\zeta))$ holds. Now, given $(\alpha, \zeta) \in R \times I$, $(\theta, x, w, \alpha) \in D$ with $g(x, \alpha) \leqq \omega(\alpha, \zeta)$, $s' = s(x, w, \alpha) \in B_0$ satisfying 4.2. (i), then

$$(1) \qquad V(x, w, \alpha) \leqq b(g(x, \alpha)) \leqq b(\omega(\alpha, \zeta)) \leqq \omega^0(\alpha, a(\zeta)),$$

so that $s'(\alpha) \leqq \omega^0(\alpha, a(\zeta))$. Hence and from the assumption of the strong stability of m^0 there follows $s'(\theta) \leqq a(\zeta)$. Hence, using 4.2. (iii), for arbitrary (y, v, θ) satisfying $(y, v, \theta)\, t\, (x, w, \alpha)$ one obtains

$$a(g(y, \theta)) \leqq V(y, v, \theta) \leqq s'(\theta) \leqq a(\zeta),$$

which gives $g(y, \theta) \leqq \zeta$ and thus m is strongly stable.

4.4. Theorem. *Let there exist maps* 4.2. (1) *satisfying* 4.2. (ii) *and* 4.2. (iii). *Let* m^0 *be strongly stable with respect to* p^0. *Then* m *is weakly stable with respect to* p.

Proof. Define ω as in the preceding proof. Given $(\alpha, \zeta) \in R \times I$ and $(\theta, x, w, \alpha) \in D$ with $g(x, w, \alpha) \leqq \omega(\alpha, \zeta)$, it holds $V(x, w, \alpha) \leqq \omega^0(\alpha, a(\zeta))$. Hence, for $s' = s(x, w, \alpha) \in B_0$ and $\sigma \in \sum_0(x, w, \alpha)$ satisfying 4.2. (ii), there follows $s'(\alpha) \leqq \omega^0(\alpha, a(\zeta))$, $s'(\theta) \leqq a(\zeta)$. Now, denoting $s_1 = \text{proj}_1 \circ \sigma$ and using 4.2. (iii) one obtains

$$a(g(s_1(\theta), \theta)) \leqq V(\sigma(\theta), \theta) \leqq s'(\theta) \leqq a(\zeta),$$

i.e. $g(s_1(\theta), \theta) \leqq \zeta$. Thus the solution σ satisfies the definition of the weak stability of m.

Very simple modifications of the proofs of both preceding Theorems enable one to verify the following two Theorems.

4.5. Theorem. *Let there exist maps* 4.2. (1) *satisfying* 4.2. (i) *and* 4.2. (iii). *Let* m^0 *be uniformly strongly stable with respect to* p^0. *Then* m *is uniformly strongly stable with respect to* p.

4.6. Theorem. *Let there exist maps* 4.2. (1) *satisfying* 4.2. (ii) *and* 4.2. (iii). *Let* m^0 *be uniformly strongly stable with respect to* p^0. *Then* m *is uniformly weakly stable with respect to* p.

4.7. Theorem. *Let there exist maps* 4.2. (1) *satisfying* 4.2. (i) *and* 4.2. (iii). *Let* m^0 *be asymptotically strongly stable with respect to* p^0. *Then* m *is asymptotically strongly stable with respect to* p.

Proof. According to Theorem 4.3. m is strongly stable, so that it remains to prove the existence of the maps $\Omega : R \to I$ and $T : P \times R \times I \to \mathsf{R}^+$ satisfying condition 2.12 (2). Further, according to the assumption, there exist maps $\Omega^0 : R \to I$ and $T^0 : \mathsf{R}^0 \times R \times I \to \mathsf{R}^+$ such that

$$s' \in B_0,\ \alpha, \theta \in \text{domain } s', \quad s'(\alpha) \leqq \Omega^0(\alpha)\,, \quad \theta \geqq \alpha + T^0(s'(\alpha), \alpha, \zeta) \quad \text{imply} \quad s'(\theta) \leqq \zeta\,.$$

It may be easily shown that T^0 may be taken increasing in the first variable (i.e. the map $T_{\alpha,\zeta} : \mathsf{R}^0 \to \mathsf{R}^+ : T_{\alpha,\zeta}(r) = T(r, \alpha, \zeta)$ would be increasing). Define maps Ω and T to satisfy

$$b(\Omega(\alpha)) \leqq \Omega^0(\alpha)\,; \tag{1}$$

$$T(x, \alpha, \zeta) = T^0(b(g(x, \alpha), \alpha, a(\zeta))\,. \tag{2}$$

Let $(\theta, x, w, \alpha) \in D$, $g(x, \alpha) \leqq \Omega(\alpha)$, $\theta \geqq \alpha + T(x, \alpha, \zeta)$ and let $s' = s(x, w, \alpha) \in B_0$ satisfy condition 4.2. (i). From 4.2. (iii) and (1) there follows

$$V(x, w, \alpha) \leqq b(g(x, \alpha)) \leqq b(\Omega(\alpha)) \leqq \Omega^0(\alpha)\,, \tag{3}$$

and thus, using 4.2. (i), also $s(\alpha) \leqq \Omega^0(\alpha)$. Hence, for each $\theta \geqq \alpha + T(s'(\alpha), \alpha, a(\zeta))$ it holds $s'(\theta) \leqq a(\zeta)$. Now, taking $(y, v, \theta) \in E$ with $(y, v, \theta)\, t\, (x, v, \alpha)$ and $\theta \geqq \alpha + T(x, \alpha, \zeta)$, it follows from (2) and the fact that T^0 is increasing in the first variable

$$\theta \geqq \alpha + T^0(b(g(x, \alpha), \alpha, a(\zeta)) \geqq \alpha + T^0(V(x, w, \alpha)), \alpha, a(\zeta)) = \\ = \alpha + T^0(s'(\alpha), \alpha, a(\zeta))\,.$$

Now 4.2. (iii) and 4.2. (i) give

$$a(g(y, \theta)) \leqq V(y, v, \theta) \leqq s'(\theta) \leqq a(\zeta)\,,$$

hence $g(y, \theta) \leqq \zeta$. Thus Ω and T satisfy the definition of the asymptotic strong stability.

4.8. Theorem. *Let there exist maps* 4.2. (1) *satisfying* 4.2. (ii) *and* 4.2. (iii). *Let* m^0 *be asymptotically strongly stable with respect to* p^0. *Then* m *is asymptotically weakly stable with respect to* p.

Proof. According to Theorem 4.4. m is weakly stable, so that it suffices to find maps $\Omega : R \to I$ and $T : P \times R \times I \to I$ satisfying condition 3.10. (i). Define these maps again by 4.7. (1) and 4.7. (2). Now, given $(x, w, \alpha) \in E$ with $g(x, \alpha) \leqq \Omega(\alpha)$ and $\theta \geqq \alpha + T(x, \alpha, \zeta)$, let solutions $s' = s(x, w, \alpha) \in B_0$ and $\sigma \in \Sigma_0(x, w, \alpha)$ satisfy condition 4.2. (ii). Then 4.7. (3) gives $s'(\alpha) \leqq \Omega^0(\alpha)$, hence $s'(\theta) \leqq a(\zeta)$ for each $\theta \geqq \alpha + T^0(s'(\alpha), \alpha, a(\zeta))$. Since from $\theta \geqq \alpha + T(x, \alpha, \zeta)$ it follows $\theta \geqq \alpha + T^0(s'(\alpha), \alpha, a(\zeta))$, one has for $s_1 = \text{proj}_1 \circ \sigma$

$$a(g(s_1(\theta), \theta)) \leqq V(\sigma(\theta), \theta) \leqq s'(\theta) \leqq a(\zeta),$$

from where $g(s_1(\theta), \theta) \leqq \zeta$ follows for each $\theta \geqq \alpha + T(x, \alpha, \zeta)$. Thus, Ω and T satisfy 3.10. (i) and Theorem is proved. Using the stadard technique described above one may easy prove the following two Theorems.

4.9. Theorem. *Let there exist maps* 4.2. (1) *satisfying* 4.2. (i) *and* 4.2. (iii). *Let* m^0 *be uniformly asymptotically strongly stable with respect to* p^0. *Then* m *is uniformly asymptotically strongly stable with respect to* p.

4.10. Theorem. *Let there exist maps* 4.2. (1) *satisfying* 4.2. (ii) *and* 4.2. (iii). *Let* m^0 *be uniformly asymptotically strongly stable with respect to* p^0. *Then* m *is uniformly asymptotically weakly stable with respect to* p.

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