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DEFORMATION OF SURFACES IMMERSED IN UNIMODULAR AFFINE SPACE OF FOUR DIMENSIONS

JINDŘICH KERNDL, Brno

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The local theory of deformations (in Cartan's conception) of subvarieties of so called flat spaces is an important part of classical differential geometry. In this paper, the existence and the properties of surfaces (i.e. varieties of two dimensions) immersed in an affine space of four dimensions, which are in a deformation of second order, are studied.

1. INTRODUCTORY NOTIONS

1.1. Let A_4 be a 4-dimensional affine space and (A) be a surface generated by the point $A = A(u, v)$, being immersed in this space. The admissible couples (u, v) are taken from an open neighborhood of C^2 (C = complex numbers). Let us suppose, (A) to be a surface sustaining conjugate net. To each point of the surface we associate a frame consisting of the point A and of linearly independent vectors I_1, I_2, I_3, I_4 such that

$$(1.1) \qquad [I_1 I_2 I_3 I_4] = 1 .$$

The fundamental equations of the moving frame are

$$(1.2) \qquad \mathrm{d}A = \sum_{j=1}^{4} \omega_j I_j , \quad \mathrm{d}I_j = \sum_{k=1}^{4} \omega_{jk} I_k \quad (j = 1, 2, 3, 4) ,$$

where ω_j, ω_{jk} are linear differential forms in parameters on which the moving frame is depending.

Differentiating (1.1) and using (1.2), we obtain

$$(1.3) \qquad \omega_{11} + \omega_{22} + \omega_{33} + \omega_{44} = 0 .$$

Further, the forms ω fulfil the structure equations of the affine space

$$(1.4)\qquad d\omega_j = \sum_{k=1}^{4} \omega_k \wedge \omega_{kj}, \quad d\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj} \quad (i, j = 1, 2, 3, 4).$$

We can specialize the frame so that the following equations hold

$$(1.5)\qquad \omega_3 = \omega_4 = 0,$$

$$(1.6)\qquad \omega_{13} = \omega_1, \quad \omega_{23} = \omega_{14} = 0, \quad \omega_{24} = \omega_2,$$

$$(1.7)\qquad \omega_{12} = \alpha_1\omega_2, \quad \omega_{21} = \alpha_2\omega_1,$$

$$(1.8)\qquad \omega_{43} = \beta_1\omega_2, \quad \omega_{34} = \beta_2\omega_1,$$

$$(1.9)\qquad 2\omega_{11} - \omega_{33} = \alpha_2\omega_2, \quad 2\omega_{22} - \omega_{44} = \alpha_1\omega_1.$$

There is $\omega_1 \wedge \omega_2 \neq 0$.

Besides other, it means that the vectors I_1, I_2 are touching the conjugate net. Therefore its equation is $\omega_1\omega_2 = 0$. The vectors I_3 and I_4 are parallel with the osculating planes of curves $\omega_2 = 0$ and $\omega_1 = 0$ respectively.

By exterior differentiation of the equations (1.7), (1.8), (1.9), we obtain

$$(1.10)\qquad \begin{aligned} &\omega_1 \wedge \omega_{32} + \omega_2 \wedge (d\alpha_1 - \alpha_1\omega_{11}) - \alpha_1^2\omega_1 \wedge \omega_2 = 0, \\ &\omega_1 \wedge (d\alpha_2 - \alpha_2\omega_{22}) + \omega_2 \wedge \omega_{41} + \alpha_2^2\omega_1 \wedge \omega_2 = 0. \end{aligned}$$

$$(1.11)\qquad \begin{aligned} &\omega_1 \wedge \omega_{41} - \omega_2 \wedge [d\beta_1 - \beta_1(\omega_{22} + \omega_{44} - \omega_{33})] + \alpha_1\beta_1\omega_1 \wedge \omega_2 = 0, \\ &\omega_1 \wedge [d\beta_2 - \beta_2(\omega_{11} + \omega_{33} - \omega_{44})] - \omega_2 \wedge \omega_{32} + \alpha_2\beta_2\omega_1 \wedge \omega_2 = 0. \end{aligned}$$

$$(1.12)\qquad \begin{aligned} &3\omega_1 \wedge \omega_{31} + \omega_2 \wedge (d\alpha_2 - \alpha_2\omega_{22}) - (3\alpha_1\alpha_2 + \beta_1\beta_2)\,\omega_1 \wedge \omega_2 = 0, \\ &\omega_1 \wedge (d\alpha_1 - \alpha_1\omega_{11}) + 3\omega_2 \wedge \omega_{42} + (3\alpha_1\alpha_2 + \beta_1\beta_2)\,\omega_1 \wedge \omega_2 = 0. \end{aligned}$$

Let us denote by δ, as usual, the differentiation such that $\delta u = \delta v = 0$ and let us write $\omega_{ij}(\delta) = e_{ij}$. Then we have from (1.10), (1.11), (1.12)

$$(1.13)\qquad \begin{aligned} &\delta\alpha_1 = \alpha_1 e_{11}, \quad \delta\alpha_2 = \alpha_2 e_{22}, \quad \delta\beta_1 = \beta_1(e_{22} + e_{44} - e_{33}), \\ &\delta\beta_2 \doteq \beta_2(e_{11} + e_{33} - e_{44}). \end{aligned}$$

It results that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are relative invariants. We shall always suppose $\alpha_1\alpha_2\beta_1\beta_2 \neq 0$. (About geometrical signification of vanishing of α_1, α_2, β_1, β_2 see [3].)

With respect to (1.5), (1.6), (1.7), (1.8), the fundamental equations are of the form

$$(1.14)\qquad \begin{aligned} dA &= \omega_1 I_1 + \omega_2 I_2, \\ dI_1 &= \omega_{11} I_1 + \alpha_1\omega_2 I_2 + \omega_1 I_3, \\ dI_2 &= \alpha_2\omega_1 I_1 + \omega_{22} I_2 + \omega_2 I_4, \end{aligned}$$

$$\mathrm{d}I_3 = \omega_{31}I_1 + \omega_{32}I_2 + \omega_{33}I_3 + \beta_2\omega_1 I_4 ,$$
$$\mathrm{d}I_4 = \omega_{41}I_1 + \omega_{42}I_2 + \beta_1\omega_2 I_3 + \omega_{44}I_4 .$$

Moreover, the equations (1.3) and (1.9) hold.

Now, let us consider a surface (B) immersed in a 4-dimensional affine space A'_4 and generated by the point $B = B(u', v')$. Let us take the same suppositions on (B) as those on (A). Let the frame of (B) be consisted of the point B and of the vectors J_1, J_2, J_3, J_4 such that

$$[J_1 J_2 J_3 J_4] = 1 . \tag{1.1'}$$

We denote all expressions connected with (B) by an apostroph. Let the frame associated with (B) be specialized in the same way as that associated with (A).

In particular, $\omega'_1\omega'_2 = 0$ is the equation of conjugate net on (B) and the fundamental equations are of the form

$$\begin{aligned} \mathrm{d}B &= \omega'_1 J_1 + \omega'_2 J_2 , \\ \mathrm{d}J_1 &= \omega'_{11}J_1 + \alpha'_1\omega'_2 J_2 + \omega'_1 J_3 , \\ \mathrm{d}J_2 &= \alpha'_2\omega'_1 J_1 + \omega'_{22}J_2 + \omega'_2 J_4 , \\ \mathrm{d}J_3 &= \omega'_{31}J_1 + \omega'_{32}J_2 + \omega'_{23}J_3 + \beta'_2\omega'_1 J_4 , \\ \mathrm{d}J_4 &= \omega'_{41}J_1 + \omega'_{42}J_2 + \beta'_1\omega'_2 J_3 + \omega'_{44}J_4 . \end{aligned} \tag{1.14'}$$

1.2. Let $C : (A) \to (B)$ be a correspondence such that the point $B = CA$ of the surface (B) corresponds to the point A of the surface (A). Let C be regular. Then it is given by

$$\omega'_1 = \lambda_{11}\omega_1 + \lambda_{12}\omega_2 , \quad \omega'_2 = \lambda_{21}\omega_1 + \lambda_{22}\omega_2 , \tag{1.15}$$

$$\lambda = \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{vmatrix} \neq 0 . \tag{1.16}$$

We shall use the following specification

$$\tau_{ij} = \omega'_{ij} - \omega_{ij} , \quad t_{ij} = e'_{ij} - e_{ij} .$$

By exterior differentiation of (1.15), we obtain

$$\begin{aligned} &\omega_1 \wedge (\mathrm{d}\lambda_{11} + \lambda_{11}\tau_{11}) + \omega_2 \wedge [\mathrm{d}\lambda_{12} + \lambda_{12}(\tau_{11} + \omega_{11} - \omega_{22})] + \\ &\quad + (\lambda_{11}\alpha_2 - \lambda_{11}\lambda_{22}\alpha'_2 - \lambda_{12}\alpha_1 + \lambda_{12}\lambda_{21}\alpha'_2)\,\omega_1 \wedge \omega_2 = 0 , \\ &\omega_1 \wedge [\mathrm{d}\lambda_{21} + \lambda_{21}(\tau_{22} + \omega_{22} - \omega_{11})] + \omega_2 \wedge (\mathrm{d}\lambda_{22} + \lambda_{22}\tau_{22}) + \\ &\quad + (\lambda_{21}\alpha_2 - \lambda_{22}\alpha_1 - \lambda_{12}\lambda_{21}\alpha'_1 + \lambda_{11}\lambda_{22}\alpha'_1)\,\omega_1 \wedge \omega_2 = 0 . \end{aligned} \tag{1.17}$$

Hence we have

$$\begin{aligned}
\delta\lambda_{11} &= -\lambda_{11}t_{11}\,,\\
\delta\lambda_{12} &= \lambda_{12}(e_{22} - t_{11} - e_{11})\,,\\
\delta\lambda_{21} &= \lambda_{21}(e_{11} - t_{22} - e_{22})\,,\\
\delta\lambda_{22} &= -\lambda_{22}t_{22}\,;
\end{aligned}$$

however, $\delta\lambda = 0$ as a consequence of $t_{11} + t_{22} = 0$. The expressions $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$ are relative invariants. With regard to (1.16) we can suppose for example $\lambda_{11} \neq 0$. Now the geometrical signification of $\lambda_{12} = 0$ is that the families of curves $\omega_1 = 0$ and $\omega'_1 = 0$ of the conjugate nets of (A) and (B) respectively correspond mutually. The signification of $\lambda_{21} = 0$ is analogous.

The correspondence $C : (A) \to (B)$ is called *conjugate* in case it is given by relations (we denote $\lambda_{11} = \lambda_1, \lambda_{22} = \lambda_2$)

$$\omega'_1 = \lambda_1\omega_1\,, \quad \omega'_2 = \lambda_2\omega_2\,, \quad \lambda = \lambda_1\lambda_2 \neq 0\,. \tag{1.18}$$

The geometrical characterization of the conjugate correspondences follows from the foregoing consideration. Conjugate nets of both the surfaces (A) and (B) are corresponding. The specification is chosen so that the family $\omega'_1 = 0$ $(\omega'_2 = 0)$ of the surface (B) corresponds to the family $\omega_1 = 0$ $(\omega_2 = 0)$ of the surface (A).

By exterior d fferentiation of (1.18), we get

$$\begin{aligned}
&\omega_1 \wedge (\mathrm{d}\lambda_1 + \lambda_1\tau_{11}) - \lambda_1 r_2\omega_1 \wedge \omega_2 = 0\,,\\
&\omega_2 \wedge (\mathrm{d}\lambda_2 + \lambda_2\tau_{22}) + \lambda_2 r_1\omega_1 \wedge \omega_2 = 0\,,
\end{aligned} \tag{1.19}$$

where $r_1 = \lambda_1\alpha'_1 - \alpha_1$, $r_2 = \lambda_2\alpha'_2 - \alpha_2$. So we have

$$\delta\lambda_1 = -\lambda_1 t_{11}\,, \quad \delta\lambda_2 = -\lambda_2 t_{22}\,, \quad \delta\lambda = 0\,. \tag{1.20}$$

Using Cartan's lemma, the equations (1.19) yield

$$\begin{aligned}
\mathrm{d}\lambda_1 + \lambda_1\tau_{11} &= f_1\omega_1 + \lambda_1 r_2\omega_2\,,\\
\mathrm{d}\lambda_2 + \lambda_2\tau_{22} &= \lambda_2 r_1\omega_1 + f_2\omega_2\,.
\end{aligned} \tag{1.21}$$

We shall consider (A) being a surface immersed in the 4-dimensional projective space P_4 arising from the space A_4 by the projective extension. Then each vector of A_4 is an improper point of P_4. These points generate the 3-dimensional improper space N_3 of the affine space A_4. Without any danger of misunderstanding, we shall speak about the points I_1, I_2 etc. when thinking of the improper points determined by the mentioned vectors. We shall do a similar supposition concerning the surface (B).

The tangent plane $[AI_1I_2]$ of the surface (A) at the point A meets the space N_3 in the straight line $[I_1I_2]$. If the point A is generating the surface (A), then the straight line $[I_1 I_2]$ is generating the line congruence L. In accordance with our suppositions, the congruence L has two different focal surfaces. Similarly by L' the congruence of the straight lines $[J_1 J_2]$ is to be denoted.

To each line of the congruence L we associate a frame consisting of the points I_1, I_2, I_3, I_4 and according to (1.14) the fundamental system of differential equations is of the form

$$
\begin{aligned}
\mathrm{d}I_1 &= \omega_{11} I_1 + \alpha_1 \omega_2 I_2 + \omega_1 I_3 , \\
\mathrm{d}I_2 &= \alpha_2 \omega_1 I_1 + \omega_{22} I_2 + \omega_2 I_4 , \\
\mathrm{d}I_3 &= \omega_{31} I_1 + \omega_{32} I_2 + \omega_{33} I_3 + \beta_2 \omega_1 I_4 , \\
\mathrm{d}I_4 &= \omega_{41} I_1 + \omega_{42} I_2 + \beta_1 \omega_2 I_3 + \omega_{44} I_4 .
\end{aligned} \tag{1.22}
$$

Moreover, the equations (1.3) and (1.9) hold.

Now it can be found that the developable surfaces of the congruence L correspond to the conjugate net of the surface (A) in the following sense: If the point A moves along the curve $\omega_1 = 0$ $(\omega_2 = 0)$ of the surface (A), the corresponding straight line $[I_1 I_2]$ of the tangent plane of the surface (A) at the point A generates the developable surface $\omega_1 = 0$ $(\omega_2 = 0)$ of the congruence L.

Let us suppose that $C : (A) \to (B)$ is a correspondence. By means of C the correspondence $\gamma : L \to L'$ is determined in a natural way so that the improper straight lines of the tangent planes at the points A, $B = CA$ correspond to each other. In particular, if $C : (A) \to (B)$ is conjugate, then $\gamma : L \to L'$ is developable. (About developable correspondences see [2].)

2. AFFINE DEFORMATION

2.1. Let (A) be a surface in an affine space A_4 with the frame specialized so that the fundamental system of differential equations is (1.14). Let us take a similar supposition concerning the surface (B) immersed in the space A'_4. Let us consider the correspondence $C : (A) \to (B)$ given by the relations (1.15), (1.16). Moreover, the equations (1.17) hold.

The correspondence $C : (A) \to (B)$ is called *an affine deformation of order k*, if for each point A of the surface (A) there exists an affinity $T : A_4 \to A'_4$ such that the surfaces (TA), (B) have the analytic contact of order k at the point $B = CA$. We say that T realizes the affine deformation C.

Now, we attend to a deformation of first order. The conditions foı the correspondence C to be an affine deformation of the first order consist in the existence of the affinity T so that it holds

$$
TA = B , \quad T\mathrm{d}A = \mathrm{d}B . \tag{2.1}
$$

Let the affinity T be given by

$$(2.2) \qquad TA = B + \sum_{\nu=1}^{4} a_\nu J_\nu \,, \quad TI_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} J_\nu \,, \quad (\mu = 1, 2, 3, 4) \,.$$

Further, we shall always assume the determinant of the matrix $M = \|a_{\mu\nu}\|$ being equal to one, i.e.

$$(2.3) \qquad \det |M| = 1 \,.$$

Making use of affinity (2.2) and equations (1.14), (1.14′), (1.15), we get from the conditions (2.1) that *any correspondence $C : (A) \to (B)$ is an affine deformation of the first order.* Affinity T (so called tangent affinity) realizing this deformation exists and generally it is of the form

$$(2.4) \qquad TA = B \,, \quad TI_1 = \lambda_{11} J_1 + \lambda_{21} J_2 \,, \quad TI_2 = \lambda_{12} J_1 + \lambda_{22} J_2 \,,$$
$$TI_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} J_\nu \,, \quad \mu = 3, 4 \,;$$

$$(2.5) \qquad \lambda^{-1} = \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \,.$$

2.2. Let us suppose, considering the same specialization of frames of the surfaces (A), (B) as in the previous paragraph, that C is an affine deformation of second order. Then for each point of the surface (A) there exists the affinity $T: A_4 \to A_4'$ so that it holds

$$(2.6) \qquad TA = B \,, \quad T\mathrm{d}A = \mathrm{d}B \,, \quad T\mathrm{d}^2A = \mathrm{d}^2B \,.$$

The correspondence C is given by relations (1.15), (1.16). With regard to the first two equations (2.6) we can suppose that the affinity T is of the form (2.4). Making use of (1.14), we compute

$$(2.7) \qquad \mathrm{d}^2A = (\mathrm{d}\omega_1 + \omega_1\omega_{11} + \alpha_2\omega_1\omega_2) I_1 + \\ + (\mathrm{d}\omega_2 + \omega_2\omega_{22} + \alpha_1\omega_1\omega_2) I_2 + \omega_1^2 I_3 + \omega_2^2 I_4$$

and analogously by using of (1.15)

$$(2.8) \qquad \mathrm{d}^2B = \{\mathrm{d}\lambda_{11}\omega_1 + \lambda_{11}\,\mathrm{d}\omega_1 + \mathrm{d}\lambda_{12}\omega_2 + \lambda_{12}\,\mathrm{d}\omega_2 + (\lambda_{11}\omega_1 + \lambda_{12}\omega_2)\,\omega_{11}' + \\ + \alpha_2'(\lambda_{11}\lambda_{21}\omega_1^2 + \lambda_{21}\lambda_{12}\omega_1\omega_2 + \lambda_{11}\lambda_{22}\omega_1\omega_2 + \lambda_{12}\lambda_{22}\omega_2^2)\}\, J_1 + \\ + \{\mathrm{d}\lambda_{21}\omega_1 + \lambda_{21}\,\mathrm{d}\omega_1 + \mathrm{d}\lambda_{22}\omega_2 + \lambda_{22}\,\mathrm{d}\omega_2 + \\ + (\lambda_{21}\omega_1 + \lambda_{22}\omega_2)\,\omega_{22}' + \\ + \alpha_1'(\lambda_{11}\lambda_{21}\omega_1^2 + \lambda_{12}\lambda_{21}\omega_1\omega_2 + \lambda_{11}\lambda_{22}\omega_1\omega_2 + \lambda_{12}\lambda_{22}\omega_2^2)\}\, J_2 + \\ + (\lambda_{11}^2\omega_1^2 + 2\lambda_{11}\lambda_{12}\omega_1\omega_2 + \lambda_{12}^2\omega_2^2)\, J_3 + \\ + (\lambda_{21}^2\omega_1^2 + 2\lambda_{22}\lambda_{21}\omega_1\omega_2 + \lambda_{22}^2\omega_2^2)\, J_4 \,.$$

Finally, we have from equation (2.7) by means of affinity (2.4)

$$(2.9)\quad T\mathrm{d}^2A = \{\lambda_{11}(\mathrm{d}\omega_1 + \omega_1\omega_{11} + \alpha_2\omega_1\omega_2) + \lambda_{12}(\mathrm{d}\omega_2 + \omega_2\omega_{22} + \alpha_1\omega_1\omega_2) + \\ + a_{31}\omega_1^2 + a_{41}\omega_2^2\}\, J_1 + \{\lambda_{21}(\mathrm{d}\omega_1 + \omega_1\omega_{11} + \alpha_2\omega_1\omega_2) + \\ + \lambda_{22}(\mathrm{d}\omega_2 + \omega_2\omega_{22} + \alpha_1\omega_1\omega_2) + a_{32}\omega_1^2 + a_{42}\omega_2^2\}\, J_2 + \\ + (a_{33}\omega_1^2 + a_{43}\omega_2^2)\, J_3 + (a_{34}\omega_1^2 + a_{44}\omega_2^2)\, J_4\,.$$

With respect to the last equation (2.6), we compare the coefficients of linearly independent vectors J_1, J_2, J_3, J_4 being on the right-hand sides of the expressions (2.8), (2.9). The coefficients of J_3, J_4 are

$$a_{33}\omega_1^2 + a_{43}\omega_2^2 = \lambda_{11}^2\omega_1^2 + 2\lambda_{11}\lambda_{12}\omega_1\omega_2 + \lambda_{12}^2\omega_2^2\,,$$
$$a_{34}\omega_1^2 + a_{44}\omega_2^2 = \lambda_{21}^2\omega_1^2 + 2\lambda_{22}\lambda_{21}\omega_1\omega_2 + \lambda_{22}^2\omega_2^2\,.$$

Hence we have

$$(2.10)\qquad a_{33} = \lambda_{11}^2\,,\quad a_{43} = \lambda_{12}^2\,,\quad a_{34} = \lambda_{21}^2\,,\quad a_{44} = \lambda_{22}^2\,,$$

and also

$$(2.11)\qquad \lambda_{11}\lambda_{12} = 0\,,\quad \lambda_{22}\lambda_{21} = 0\,.$$

Let us attend to the equations (2.11). The condition (1.16) yields that it cannot be simultaneously λ_{11} and λ_{12} or λ_{21} and λ_{22} equal to zero. Let us assume $\lambda_{11} \neq 0$. Then according to the first equation (2.11) there is $\lambda_{12} = 0$. With respect to (1.16) we have $\lambda_{22} \neq 0$ and according to the second equation (2.11) there is $\lambda_{21} = 0$. Therefore it is necessary for C to be a conjugate correspondence.

Now, with the specification used in (1.18), by comparing the coefficients of J_1, J_2, we get

$$(\mathrm{d}\lambda_1 + \lambda_1\tau_{11})\,\omega_1 + \lambda_1 r_2\omega_1\omega_2 - a_{31}\omega_1^2 - a_{41}\omega_2^2 = 0\,,$$
$$(\mathrm{d}\lambda_2 + \lambda_2\tau_{22})\,\omega_2 + \lambda_2 r_1\omega_1\omega_2 - a_{32}\omega_1^2 - a_{42}\omega_2^2 = 0\,.$$

Substituting from (1.21), it results

$$a_{31} = f_1\,,\quad a_{41} = 0\,,\quad r_2 = 0\,,$$
$$a_{42} = f_2\,,\quad a_{32} = 0\,,\quad r_1 = 0\,.$$

The condition (2.5) is $(\lambda_1\lambda_2)^3 = 1$, so $\lambda_1\lambda_2 = 1$ can be chosen.

Cosequently, the necessary conditions for C to be an affine deformation of second order are as follows:

C is conjugate and, moreover, it holds

$$(2.12)\qquad \lambda_1\alpha_1' - \alpha_1 = 0\,,\quad \lambda_2\alpha_2' - \alpha_2 = 0\,,\quad \lambda_1\lambda_2 = 1\,.$$

It is easy to see that the above mentioned conditions are sufficient, too. We formulate this result in

Theorem 1. *Let $C:(A) \to (B)$ be a correspondence determined by relations (1.15), (1.16). The correspondence C is an affine deformation of second order if and only if it is conjugate and the equations (2.12) hold.*

As regards the affinity T realizing affine deformation of second order, we find out

Lemma 1. *The affinity $T: A_4 \to A'_4$ realizing an affine deformation of second order (s.c. osculating affinity) exists and generally is of the form*

$$TA = B\,, \quad TI_1 = \lambda_1 J_1\,, \quad TI_2 = \lambda_2 J_2\,, \quad TI_3 = \lambda_1^2 J_3\,, \quad TI_4 = \lambda_2^2 J_4\,, \quad (\lambda_1 \lambda_2 = 1)\,. \tag{2.13}$$

Further, let us remark that there is

$$\delta\omega_1 = -e_{11}\omega_1\,, \quad \delta\omega_2 = -e_{22}\omega_2\,.$$

With respect to (1.13), we compute that the forms

$$\varphi = \alpha_1 \alpha_2 \omega_1 \omega_2\,, \tag{2.14}$$

$$\psi = \beta_1 \beta_2 \omega_1 \omega_2 \tag{2.15}$$

are invariant (i.e. $\delta\varphi = 0$, $\delta\psi = 0$). We next establish two lemmas declaring the context of an affine deformation of second order with the invariant forms φ, φ' and then by means of these forms the context of the correspondence $C:(A) \to (B)$ with the correspondence $\gamma : L \to L'$.

Lemma 2. *Let $C:(A) \to (B)$ be a correspondence between the surfaces (A) and (B). If C is an affine deformation of second order, it holds*

$$\varphi = \varphi'\,.$$

Actually, let us suppose that C is an affine deformation of second order. Then C is conjugate and the equations (2.12) hold (in accordance with Theorem 1.).

Now, by using of (1.18) we have

$$\varphi' = \alpha'_1 \alpha'_2 \omega'_1 \omega'_2 = \alpha_1 \alpha_2 \omega_1 \omega_2 = \varphi\,.$$

Lemma 3. *Let $C:(A) \to (B)$ be a correspondence. If C is an affine deformation of second order, then the correspondence $\gamma : L \to L'$ is a point deformation.*

It is enough to compare the result of Lemma 2. with Proposition. 2 in [1], p. 16.

2.3. In this section, we shall deal especially with the existence problem. Let us suppose again, $C:(A) \to (B)$ to be an affine deformation of second order, i.e. C is conjugate and (2.12) hold.

From the first equation (1.20), it follows that we can choose

$$\lambda_1 = 1 .$$

That means the frames were specialized by the relation $t_{11} = 0$. With respect to the last equation (2.12), we have also

$$\lambda_2 = 1 ,$$

correspondence C being determined by

$$\omega'_1 = \omega_1 , \quad \omega'_2 = \omega_2 . \tag{2.16}$$

The conditions for C to be an affine deformation of second order are

$$\alpha'_1 = \alpha_1 , \quad \alpha'_2 = \alpha_2 \tag{2.17}$$

and affinity realizing this deformation is

$$TA = B , \quad TI_k = J_k \quad (k = 1, 2, 3, 4) . \tag{2.18}$$

Now, we have in recapitulation

$$\begin{aligned} &\tau_{11} = \tau_{22} = \tau_{33} = \tau_{44} = 0 , \\ &\tau_{12} = \tau_{21} = 0 , \\ &\tau_{43} = \bar{\beta}_1 \omega_2 , \quad \tau_{34} = \bar{\beta}_2 \omega_1 , \end{aligned} \tag{2.19}$$

where $\bar{\beta}_1 = \beta'_1 - \beta_1$, $\bar{\beta}_2 = \beta'_2 - \beta_2$ was denoted.

By exterior differentiation of (2.19), we get

$$\begin{aligned} &\omega_1 \wedge \tau_{31} = 0 , \\ &\omega_2 \wedge \tau_{42} = 0 , \\ &\omega_1 \wedge \tau_{32} = 0 , \\ &\omega_2 \wedge \tau_{41} = 0 , \\ &\omega_1 \wedge \tau_{41} - \omega_2 \wedge [d\bar{\beta}_1 - \bar{\beta}_1(\omega_{22} + \omega_{44} - \omega_{33})] + \alpha_1 \bar{\beta}_1 \omega_1 \wedge \omega_2 = 0 , \\ &\omega_1 \wedge [d\bar{\beta}_2 - \bar{\beta}_2(\omega_{11} + \omega_{33} - \omega_{44})] - \omega_2 \wedge \tau_{32} + \alpha_2 \bar{\beta}_2 \omega_1 \wedge \omega_2 = 0, \end{aligned} \tag{2.20}$$

and moreover the condition

$$\beta'_1 \beta'_2 = \beta_1 \beta_2 . \tag{2.21}$$

Let us take in mind that the equation (2.21) means equality of planar forms ψ and ψ' of the congruences L and L' in the correspondence $\gamma : L \to L'$. (See [1], p. 39.) If we take into consideration Lemma 3, it holds

Theorem 2. *Let $C : (A) \to (B)$ be a correspondence. If C is an affine deformation of second order, then the correspondence $\gamma : L \to L'$ is a point and planar deformation.*

Let us revert to the relations (2.20). We get

$$\delta\bar{\beta}_1 = \bar{\beta}_1(e_{22} + e_{44} - e_{33}), \quad \delta\bar{\beta}_2 = \bar{\beta}_2(e_{11} + e_{33} - e_{44}),$$

$\bar{\beta}_1, \bar{\beta}_2$ being relative invariants. Taking in mind the equation (2.21), two cases are to be distinguished:

$$\text{I.} \quad \bar{\beta}_1 = \bar{\beta}_2 = 0, \qquad \text{II.} \quad \bar{\beta}_1 \neq 0, \quad \bar{\beta}_2 \neq 0.$$

I. $\bar{\beta}_1 = \bar{\beta}_2 = 0.$

We shall next denote this case of affine deformation of second order by $C_0 : (A) \to (B)$ and we shall call it *a special deformation.*

So it holds

$$\beta'_1 = \beta_1, \quad \beta'_2 = \beta_2, \tag{2.22}$$

the equations (2.19) being of the form

$$\tau_{11} = \tau_{22} = \tau_{33} = \tau_{44} = \tau_{12} = \tau_{21} = \tau_{43} = \tau_{34} = 0. \tag{2.23}$$

The corresponding exterior quadratic relations are

$$\begin{aligned} &\omega_1 \wedge \tau_{31} = 0, \quad \omega_2 \wedge \tau_{41} = 0, \\ &\omega_2 \wedge \tau_{42} = 0, \quad \omega_1 \wedge \tau_{41} = 0, \\ &\omega_1 \wedge \tau_{32} = 0, \quad \omega_2 \wedge \tau_{32} = 0. \end{aligned} \tag{2.24}$$

From (2.24), it results that

$$\tau_{31} = m\omega_1, \quad \tau_{42} = n\omega_2, \quad \tau_{32} = \tau_{41} = 0. \tag{2.25}$$

Now, we are in a position to establish directly (see also [1], p. 124).

Theorem 3. *Let $C_0 : (A) \to (B)$ be a special affine deformation of second order. Then the correspondence $\gamma : L \to L'$ is a singular projective deformation of second order.*

The exterior differentiation of (2.25) gives

$$\begin{aligned} &\omega_1 \wedge (\mathrm{d}m - m\omega_{33}) + m\alpha_2\omega_1 \wedge \omega_2 = 0, \\ &\omega_2 \wedge (\mathrm{d}n - n\omega_{44}) - n\alpha_1\omega_1 \wedge \omega_2 = 0 \end{aligned} \tag{2.26}$$

and further

$$\begin{aligned} \alpha_1 m + \beta_2 n &= 0, \\ \beta_1 m + \alpha_2 n &= 0. \end{aligned} \tag{2.27}$$

Let us take notice of the conditions (2.27). If it is $m = 0$, then there is also $n = 0$ (and conversely) and we have $\tau_{31} = \tau_{42} = 0$ as it is seen from (2.25). But now all $\tau_{ij} = 0$ for $i, j = 1, 2, 3, 4$, the surfaces (A) and (B) being equivalent.

When excluding this trivial case, the condition for the existence of a non-zero solution of system (2.27) is

$$\alpha_1\alpha_2 - \beta_1\beta_2 = 0\,, \tag{2.28}$$

being in accordance with the well known fact that if the correspondence $\gamma : L \to L'$ is a singular projective deformation of second order, then the congruences L, L' are W-congruences. (See [1], p. 125.)

From (2.27), taking in mind (2.28), it results

$$\frac{m}{n} = -\frac{\beta_2}{\alpha_1}\,. \tag{2.29}$$

Using (1.13), we have

$$\delta\left(\frac{\beta_2}{\alpha_1}\right) = \frac{\beta_2}{\alpha_1}\left(e_{33} - e_{44}\right)$$

and we are in a position to specialize the frames in such a way that

$$\beta_2 = \alpha_1\,. \tag{2.30}$$

With respect to (2.28), (2.29) there is

$$\beta_1 = \alpha_2\,, \quad m = -n\,. \tag{2.31}$$

From the equations (2.26), we obtain

$$\mathrm{d}m - m\omega_{33} = k_1\omega_1 - m\alpha_2\omega_2\,, \tag{2.32}$$

$$\mathrm{d}n - n\omega_{44} = k_2\omega_2 - n\alpha_1\omega_1 \tag{2.33}$$

and according to the second equation (2.31)

$$m(\omega_{44} - \omega_{33}) = (k_1 + m\alpha_1)\,\omega_1 + (k_2 - m\alpha_2)\,\omega_2\,. \tag{2.34}$$

By exterior differentiation of (2.32), (2.34), we get

$$\begin{aligned} &\omega_1 \wedge (\mathrm{d}k_1 + m\omega_{31}) - m\omega_2 \wedge \mathrm{d}\alpha_2 + (.)\,\omega_1 \wedge \omega_2 = 0\,,\\ &\omega_1 \wedge (\mathrm{d}k_1 + m\omega_{31} + m\,\mathrm{d}\alpha_1) +\\ &+ \omega_2 \wedge (\mathrm{d}k_2 - m\omega_{42} - m\,\mathrm{d}\alpha_2) + (.)\,\omega_1 \wedge \omega_2 = 0\,, \end{aligned} \tag{2.35}$$

where the coefficients at $\omega_1 \wedge \omega_2$ (not being written) do not interest us.

The triplets $[(A), C_0, (B)]$, where $C_0 : (A) \to (B)$ is a special deformation of second order are given by the system (1.3), (1.5), (1.6), (1.7), (1.8), (1.9), (2.16), (2.23), (2.25), (2.32), (2.34) together with the exterior quadratic relations (2.35) and with the relations arising from (1.10), (1.11), (1.12) and having after arrangement the form

$$
\begin{aligned}
&\omega_1 \wedge \omega_{32} + \omega_2 \wedge d\alpha_1 + (.)\,\omega_1 \wedge \omega_2 = 0\,,\\
&\omega_1 \wedge d\alpha_2 + \omega_2 \wedge \omega_{41} + (.)\,\omega_1 \wedge \omega_2 = 0\,,\\
&\omega_1 \wedge \omega_{41} - \omega_2 \wedge d\beta_1 + (.)\,\omega_1 \wedge \omega_2 = 0\,,\\
&\omega_1 \wedge d\beta_2 - \omega_2 \wedge \omega_{32} + (.)\,\omega_1 \wedge \omega_2 = 0\,,\\
&3\omega_1 \wedge \omega_{31} + \omega_2 \wedge d\alpha_2 + (.)\,\omega_1 \wedge \omega_2 = 0\,,\\
&\omega_1 \wedge d\alpha_1 + 3\omega_2 \wedge \omega_{42} + (.)\,\omega_1 \wedge \omega_2 = 0\,.
\end{aligned}
\tag{2.36}
$$

Moreover, we must have in mind that (2.30), (2.31) hold.

The following form of polar matrix corresponds to the Pfaff's forms ω_{32}, $d\alpha_1$, $d\alpha_2$, ω_{41}, ω_{31}, ω_{42}, dk_1, dk_2

$$
\begin{Vmatrix}
-\omega_1 & -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\omega_1 & -\omega_2 & 0 & 0 & 0 & 0\\
0 & 0 & -\omega_2 & -\omega_1 & 0 & 0 & 0 & 0\\
\omega_2 & \omega_1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\omega_2 & 0 & -3\omega_1 & 0 & 0 & 0\\
0 & -\omega_1 & 0 & 0 & 0 & -3\omega_2 & 0 & 0\\
0 & 0 & m\omega_2 & 0 & -m\omega_1 & 0 & -\omega_1 & 0\\
0 & -m\omega_1 & m\omega_2 & 0 & -m\omega_1 & m\omega_2 & -\omega_1 & -\omega_2
\end{Vmatrix},
$$

the rank of which is $h = 8$. The number of Pfaff's forms is $q = 8$, the number of exterior quadratic relations is $s_1 = 8 = h$, the system in our consideration is involutive. Thus we have

Theorem 4. *Let (A) be a surface in A_4, (B) be a surface in A'_4. Let $C_0 : (A) \to (B)$ be a special affine deformation of second order. The triplets $[(A), C_0, (B)]$ exist and depend on eight functions of one argument.*

II. $\bar{\beta}_1 \neq 0$, $\bar{\beta}_2 \neq 0$.

The triplets $[(A), C, (B)]$, $C : (A) \to (B)$ being an affine deformation of second order, are given by the system of equations (1.3), (1.5), (1.6), (1.7), (1.8), (1.9), (2.16), (2.19) together with quadratic relations (1.10), (1.11), (1.12), (2.20); moreover, (2.21) holds. We have

$$
\beta'_2 = \frac{\beta_1\beta_2}{\beta'_1}\,, \quad d\beta'_2 = \frac{1}{\beta'_1}\left(\beta_2\, d\beta_1 + \beta_1\, d\beta_2 - \beta'_2\, d\beta'_1\right).
$$

Therefore the last two equations (2.20) become

$$\begin{gathered}
\omega_1 \wedge \tau_{41} - \omega_2 \wedge [\mathrm{d}\beta'_1 - \beta'_1(\omega_{22} + \omega_{44} - \omega_{33})] + \\
+ \omega_2 \wedge [\mathrm{d}\beta_1 - \beta_1(\omega_{22} + \omega_{44} - \omega_{33})] + \alpha_1\bar{\beta}_1\omega_1 \wedge \omega_2 = 0\,, \\
\beta_2\omega_1 \wedge [\mathrm{d}\beta_1 - \beta_1(\omega_{22} + \omega_{44} - \omega_{33})] - \\
- \bar{\beta}_1\omega_1 \wedge [\mathrm{d}\beta_2 - \beta_2(\omega_{11} + \omega_{33} - \omega_{44})] - \\
- \beta'_2\omega_1 \wedge [\mathrm{d}\beta'_1 - \beta'_1(\omega_{22} + \omega_{44} - \omega_{33})] - \\
- \beta'_1\omega_2 \wedge \tau_{32} + \alpha_2\beta'_1\bar{\beta}_2\omega_1 \wedge \omega_2 = 0\,.
\end{gathered}$$

The following shape of polar matrix corresponds to the Pfaff's forms

$$\begin{gathered}
\omega_{32},\ \mathrm{d}\alpha_1 - \alpha_1\omega_{11},\ \mathrm{d}\alpha_2 - \alpha_2\omega_{22},\ \omega_{41},\ \mathrm{d}\beta_1 - \beta_1(\omega_{22} + \omega_{44} - \omega_{33})\,, \\
\mathrm{d}\beta_2 - \beta_2(\omega_{11} + \omega_{33} - \omega_{44}),\ \omega_{31},\ \omega_{42},\ \tau_{31},\ \tau_{42},\ \tau_{32},\ \tau_{41}\,, \\
\mathrm{d}\beta'_1 - \beta'_1(\omega_{22} + \omega_{44} - \omega_{33})
\end{gathered}$$

$$\left\| \begin{array}{ccccccccccccc}
-\omega_1 & -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\omega_1 & -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\omega_1 & -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\omega_2 & 0 & 0 & 0 & 0 & -\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\omega_2 & 0 & 0 & 0 & -3\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\omega_1 & 0 & 0 & 0 & 0 & 0 & -3\omega_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega_2 & 0 \\
0 & 0 & 0 & 0 & -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega_1 & \omega_2 \\
0 & 0 & 0 & 0 & -\beta_2\omega_1 & \bar{\beta}_1\omega_1 & 0 & 0 & 0 & 0 & \beta'_1\omega_2 & 0 & \beta'_2\omega_1
\end{array} \right\|$$

It may be checked by direct computation that the determinant arising from this matrix by omitting the first column is of the value

$$9\bar{\beta}_2\omega_1^8\omega_2^4 \neq 0\,.$$

The number of Pfaff's forms is $q = 13$, the number of exterior quadratic relations is $s_1 = 12$ and the rank of polar matrix is $h = 12$. It results that the mentioned system is involutive. We obtain

Theorem 5. *Let (A) be a surface in A_4 and (B) be a surface in A'_4. Then the triplets $[(A), C, (B)]$, $C : (A) \to (B)$ being an affine deformation of second order, exist and depend on one function of two arguments.*

Finally, let us remark that if we suppose C to be an affine deformation of third order, we obtain equivalent surfaces.

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Author's address: Brno, Leninova 75 (Katedra matematiky VAAZ).